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Research article

Evolutionary problems driven by variational inequalities with multiple integral functionals

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Abstract: In this paper, the authors focus on extending the well-known weak sharp solutions for variational inequalities to a controlled variational-type inequality governed by convex multiple integral functionals. Simultaneously, some equivalent conditions on weak sharpness associated with solutions of the considered inequality are obtained by using the minimum principle sufficiency property.

Keywords: controlled variational-type inequality; weak sharp solution; convex multiple integral functional; minimum principle sufficiency property; dual gap functional
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1. Introduction

It is very well known that scalar and vector variational-type inequalities are very important in the study of scalar and vector optimization problems. In this regard, in [15], the authors established some connections between generalized variational inequalities and multi-objective optimization problems. In [18], Polyak introduced the concept of a *unique sharp minimizer*. Starting with the research papers [3, 17] and following [13], the variational-type inequalities have been analyzed by using the notion of a *weak sharp solution*. Analogous results have been formulated in Hilbert spaces by Wu and Wu [24]. In [4], Chen et al. constructed the gap functions associated with vector variational inequalities as set-valued maps. In [8], the authors introduced the weak sharp solution set associated with a variational-type inequality problem in a smooth, strictly convex and reflexive Banach space. Alshahrani et al. [1], by considering gap functions, formulated the maximum and minimum principle sufficiency properties for a class of nonsmooth variational inequalities. Also, in terms of its primal gap function, Liu and Wu [11] studied weakly sharp solutions for a class of variational inequalities. An effective algorithm for solving the Poisson-Gaussian total variation model was presented by Pham et al. [16]. Recently, Khazayel and Farajzadeh [9] stated some new vector versions of Takahashi's nonconvex minimization theorem, which involve algebraic notions instead of topological notions. Also, Tavakoli et al. [19] formulated a sufficient condition in order to have the C-pseudomonotone property for multi-functions.

Treanţă [23] and Treanţă and Singh [21] investigated the weak sharp solutions for a class of non-controlled extended variational-type inequalities involving $(\rho, \mathbf{b}, \mathbf{d})$-convex curvilinear/multiple integral functionals. Compared with the above-mentioned research works, the main novelty of this paper is the presence of a control variable in variational inequalities driven by multiple integral functionals. Since the controlled variational inequalities can be converted into variational control problems and, as is well known, the latter often occurs in many applications, all of which have motivated the present study. Concretely, in this paper, by considering several variational techniques presented in Clarke [5], Treanţă [20, 22, 23] and Mititelu and Treanţă [14], we generalize some of the aforementioned results to controlled multidimensional variational-type inequalities involving convex multiple integral functionals and, by using a dual gap functional, several characterization results are formulated. The main results of the paper followed and generalized the ideas for weak sharpness of solutions proposed and exploited in [3, 6, 13] and the references therein. For different but connected ideas on variational inequalities with applications to optimal control problems, the reader is directed to Liu et al. [10] and Antczak [2].

The paper is divided as follows. In Section 2, we give the preliminaries and the problem under study. In order to establish the main results of this work, several auxiliary results are formulated in Section 3. In Section 4, we study weak sharp solutions associated with the considered class of controlled variational-type inequalities involving convex multiple integral functionals. Moreover, a relation between the minimum principle sufficiency property and weak sharpness of solutions is established for the considered controlled variational-type inequality. Section 5 concludes the study.

2. Preliminaries

We start the study with the following working hypotheses and notations:

► the Euclidean space $\mathbb{R}^\ell$, $\ell \geq 1$;

► $K \subset \mathbb{R}^m$ denotes a compact set in $\mathbb{R}^m$, and $K \ni t = (t^\alpha)$, $\alpha = \overline{1, m}$, is a multi-parameter of evolution;

► for $U \subseteq \mathbb{R}^k$ and $\mathcal{P} := K \times \mathbb{R}^n \times U$, we consider the continuously differentiable functions

$$X = \left(X_\alpha^i\right) : \mathcal{P} \to \mathbb{R}^{nm}, \quad i = \overline{1, n},\ \alpha = \overline{1, m},$$

$$Y = \left(Y_\beta\right) : \mathcal{P} \to \mathbb{R}^q, \quad \beta = \overline{1, q};$$

► $dv = dt^1 \cdots dt^m$ represents the element of volume on $\mathbb{R}^m \supset K$;

► let $\overline{\mathcal{S}}$ denote the space of all piecewise smooth *state functions* $b : K \subset \mathbb{R}^m \to \mathbb{R}^n$, having the norm

$$\| b \| = \| b \|_\infty + \sum_{\alpha=1}^{m} \| b_\alpha \|_\infty, \quad \forall b \in \overline{\mathcal{S}},$$

where b_α denotes $\dfrac{\partial b}{\partial t^\alpha}$;

► also, consider $\overline{\mathcal{U}}$ as the space of all piecewise continuous *control functions* $u : K \subset \mathbb{R}^m \to U$, with the uniform norm $\| \cdot \|_\infty$;

► assume the space $\overline{\mathcal{S}} \times \overline{\mathcal{U}}$ is endowed with the inner product

$$\langle (b,u); (e,w) \rangle = \int_K [b(t) \cdot e(t) + u(t) \cdot w(t)] dv, \quad \forall (b,u), (e,w) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$$

and the norm induced by it;

► consider $\mathcal{S} \times \mathcal{U}$ as a nonempty, convex and closed subset of $\overline{\mathcal{S}} \times \overline{\mathcal{U}}$, given by

$$\mathcal{S} \times \mathcal{U} = \left\{ (b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}} : \frac{\partial b^i}{\partial t^\alpha} = X^i_\alpha (t,b,u), \ Y(t,b,u) \leq 0, \ b|_{\partial K} = \varphi = \ given \right\};$$

► in this paper, we use the simplified notations b, u, b_α for $b(t), u(t), b_\alpha(t)$, respectively;

► we assume that the continuously differentiable functions

$$X_\alpha = \left(X^i_\alpha \right) : \mathcal{P} \to \mathbb{R}^n, \quad i = \overline{1,n}, \ \alpha = \overline{1,m}$$

fulfill the complete integrability conditions, that is,

$$D_\zeta X^i_\alpha = D_\alpha X^i_\zeta, \quad \alpha, \zeta = \overline{1,m}, \ \alpha \neq \zeta, \ i = \overline{1,n},$$

where D_ζ denotes the total derivative operator;

► for any two p-tuples $a = \left(a_1, ..., a_p \right)$ and $c = \left(c_1, ..., c_p \right)$ in $\mathbb{R}^p$, the following convention will be used throughout the paper:

$$a = c \Leftrightarrow a_i = c_i, \quad a \leq c \Leftrightarrow a_i \leq c_i,$$

$$a < c \Leftrightarrow a_i < c_i, \quad a \lneq c \Leftrightarrow a \leq c, \ a \neq c, \quad i = \overline{1,p}.$$

Next, we consider the continuously smooth functions $f, g, h : K \times \mathbb{R}^n \times \mathbb{R}^{nm} \times U \to \mathbb{R}$ and, for $(b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$, define the following functionals:

$$\mathcal{F} : \overline{\mathcal{S}} \times \overline{\mathcal{U}} \to \mathbb{R}, \quad \mathcal{F}(b,u) = \int_K f(t,b,b_\alpha,u)\, dv,$$

$$G : \overline{\mathcal{S}} \times \overline{\mathcal{U}} \to \mathbb{R}, \quad G(b,u) = \int_K g(t,b,b_\alpha,u)\, dv,$$

$$\mathcal{H} : \overline{\mathcal{S}} \times \overline{\mathcal{U}} \to \mathbb{R}, \quad \mathcal{H}(b,u) = \int_K h(t,b,b_\alpha,u)\, dv.$$

Definition 2.1. (Treanţă [20]) The functional $\mathcal{F}:\overline{\mathcal{S}}\times\overline{\mathcal{U}}\to\mathbb{R},\ \mathcal{F}(b,u)=\int_K f(t,b,b_\alpha,u)\,dv$, is called *convex* on $\mathcal{S}\times\mathcal{U}$ if the inequality

$$\begin{aligned}&\mathcal{F}(b,u)-\mathcal{F}(b^0,u^0)\\ \geq&\int_K\left[f_b\left(t,b^0,b^0_\alpha,u^0\right)(b-b^0)+f_{b_\alpha}\left(t,b^0,b^0_\alpha,u^0\right)D_\alpha(b-b^0)\right]dv\\ &+\int_K\left[f_u\left(t,b^0,b^0_\alpha,u^0\right)(u-u^0)\right]dv\end{aligned}$$

is satisfied for any $(b,u),(b^0,u^0)\in\mathcal{S}\times\mathcal{U}$.

Definition 2.2. (Treanţă [20]) The *variational derivative* $\delta\mathcal{F}(b,u)$ of $\mathcal{F}:\overline{\mathcal{S}}\times\overline{\mathcal{U}}\to\mathbb{R},\ \mathcal{F}(b,u)=\int_K f(t,b,b_\alpha,u)\,dv$, is introduced as

$$\delta\mathcal{F}(b,u)=\frac{\delta\mathcal{F}}{\delta b}+\frac{\delta\mathcal{F}}{\delta u},$$

with (see Einstein summation)

$$\frac{\delta\mathcal{F}}{\delta b}=f_b(t,b,b_\alpha,u)-D_\alpha f_{b_\alpha}(t,b,b_\alpha,u)\in\overline{\mathcal{S}},\quad \frac{\delta\mathcal{F}}{\delta u}=f_u(t,b,b_\alpha,u)\in\overline{\mathcal{U}}$$

and the relation

$$\begin{aligned}\langle(\frac{\delta\mathcal{F}}{\delta b},\frac{\delta\mathcal{F}}{\delta u});(\psi,\Psi)\rangle&=\int_K\left[\frac{\delta\mathcal{F}}{\delta b}(t)\cdot\psi(t)+\frac{\delta\mathcal{F}}{\delta u}(t)\cdot\Psi(t)\right]dv\\ &=\lim_{\varepsilon\to0}\frac{\mathcal{F}(b+\varepsilon\psi,u+\varepsilon\Psi)-\mathcal{F}(b,u)}{\varepsilon}\end{aligned}$$

is satisfied for $(\psi,\Psi)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}},\ \psi|_{\partial K}=0$.

Note. In this paper, by taking into account the above-mentioned definition, we consider the condition $\psi|_{\partial K}=0$.

At this point, we introduce the *controlled multidimensional variational-type inequality problem*: Find $(b^0,u^0)\in\mathcal{S}\times\mathcal{U}$ such that

$$\begin{aligned}(CMVIP)\qquad&\int_K\left[f_b\left(t,b^0,b^0_\alpha,u^0\right)(b-b^0)+f_{b_\alpha}\left(t,b^0,b^0_\alpha,u^0\right)D_\alpha(b-b^0)\right]dv\\ &+\int_K\left[f_u\left(t,b^0,b^0_\alpha,u^0\right)(u-u^0)\right]dv\geq0\end{aligned}$$

for any $(b,u)\in\mathcal{S}\times\mathcal{U}$. The *dual controlled multidimensional variational-type inequality problem* for $(CMVIP)$ is given as follows: Find $(b^0,u^0)\in\mathcal{S}\times\mathcal{U}$ such that

$$\begin{aligned}(DCMVIP)\qquad&\int_K\left[f_b(t,b,b_\alpha,u)(b-b^0)+f_{b_\alpha}(t,b,b_\alpha,u)D_\alpha(b-b^0)\right]dv\\ &+\int_K\left[f_u(t,b,b_\alpha,u)(u-u^0)\right]dv\geq0\end{aligned}$$

for any $(b,u) \in \mathcal{S} \times \mathcal{U}$.

Further, let us denote by $(\mathcal{S} \times \mathcal{U})^*$ and $(\mathcal{S} \times \mathcal{U})_*$ the set of solutions for $(CMVIP)$ and $(DCMVIP)$, respectively. Also, we assume these sets are nonempty.

Remark 2.1. The aforementioned controlled multidimensional variational-type inequality problems can be rewritten as follows: *Find* $(b^0, u^0) \in \mathcal{S} \times \mathcal{U}$ *such that*

$$(CMVIP) \qquad \langle (\frac{\delta \mathcal{F}}{\delta b^0}, \frac{\delta \mathcal{F}}{\delta u^0}); (b - b^0, u - u^0) \rangle \geq 0, \quad \forall (b,u) \in \mathcal{S} \times \mathcal{U},$$

respectively; *Find* $(b^0, u^0) \in \mathcal{S} \times \mathcal{U}$ *such that*

$$(DCMVIP) \qquad \langle (\frac{\delta \mathcal{F}}{\delta b}, \frac{\delta \mathcal{F}}{\delta u}); (b - b^0, u - u^0) \rangle \geq 0, \quad \forall (b,u) \in \mathcal{S} \times \mathcal{U}.$$

In the following, we introduce the gap multiple integral functionals.

Definition 2.3. (Treanţă [20]) The *primal gap functional* for $(CMVIP)$ is given by

$$G(b,u) = \max_{(b^0,u^0) \in \mathcal{S} \times \mathcal{U}} \{ \int_K \left[f_b\,(t,b,b_\alpha,u)\,(b - b^0) + f_{b_\alpha}\,(t,b,b_\alpha,u)\,D_\alpha (b - b^0) \right] dv$$

$$+ \int_K \left[f_u\,(t,b,b_\alpha,u)\,(u - u^0) \right] dv \}$$

for $(b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$. The *dual gap functional* for $(CMVIP)$ is given by

$$\mathcal{H}(b,u) = \max_{(b^0,u^0) \in \mathcal{S} \times \mathcal{U}} \{ \int_K \left[f_b \left(t, b^0, b^0_\alpha, u^0 \right) (b - b^0) \right] dv$$

$$+ \int_K \left[f_{b_\alpha} \left(t, b^0, b^0_\alpha, u^0 \right) D_\alpha (b - b^0) + f_u \left(t, b^0, b^0_\alpha, u^0 \right) (u - u^0) \right] dv \}.$$

Next, consider the following notations:

$$\mathbb{A}(b,u) = \{ (s,v) \in \mathcal{S} \times \mathcal{U} : G(b,u) = \int_K \left[f_b\,(t,b,b_\alpha,u)\,(b - s) \right] dv$$

$$+ \int_K \left[f_{b_\alpha}\,(t,b,b_\alpha,u)\,D_\alpha (b - s) + f_u\,(t,b,b_\alpha,u)\,(u - v) \right] dv \},$$

$$Q(b,u) = \{ (s,v) \in \mathcal{S} \times \mathcal{U} : \mathcal{H}(b,u) = \int_K \left[f_b\,(t,s,s_\alpha,v)\,(b - s) \right] dv$$

$$+ \int_K \left[f_{b_\alpha}\,(t,s,s_\alpha,v)\,D_\alpha (b - s) + f_u\,(t,s,s_\alpha,v)\,(u - v) \right] dv \}$$

for $(b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$.

Remark 2.2. By using the aforementioned notations, we notice the following:

(i)

$$G(b,u) = \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b-b^0, u-u^0)\rangle,$$

$$\mathcal{H}(b,u) = \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b^0}, \frac{\delta\mathcal{F}}{\delta u^0}); (b-b^0, u-u^0)\rangle;$$

(ii)

$$\mathbb{A}(b,u) = \arg \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b-b^0, u-u^0)\rangle$$
$$= \arg \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \left\{-\langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b^0, u^0)\rangle\right\},$$

where $\arg \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b-b^0, u-u^0)\rangle$ denotes the (possibly empty) solution set of $\max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b-b^0, u-u^0)\rangle$;

(iii)

$$Q(b,u) = \arg \max_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b^0}, \frac{\delta\mathcal{F}}{\delta u^0}); (b-b^0, u-u^0)\rangle;$$

(iv) if $\mathbb{A}(b,u) = \emptyset$, then $G(b,u) = \sup_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b}, \frac{\delta\mathcal{F}}{\delta u}); (b-b^0, u-u^0)\rangle$; similarly, if $Q(b,u) = \emptyset$, then $\mathcal{H}(b,u) = \sup_{(b^0,u^0)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta b^0}, \frac{\delta\mathcal{F}}{\delta u^0}); (b-b^0, u-u^0)\rangle$.

In accordance with [13], we formulate the following definitions.

Definition 2.4. The *polar set* $(\mathcal{S}\times\mathcal{U})^\circ$ of $\mathcal{S}\times\mathcal{U}$ is given by

$$(\mathcal{S}\times\mathcal{U})^\circ = \left\{(e,w) \in \overline{\mathcal{S}}\times\overline{\mathcal{U}} : \langle(e,w);(b,u)\rangle \leq 0,\ \forall (b,u)\in\mathcal{S}\times\mathcal{U}\right\}.$$

Definition 2.5. The *projection* of a point $(b,u)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}}$ onto the set $\mathcal{S}\times\mathcal{U}$ is given by

$$\text{proj}_{\mathcal{S}\times\mathcal{U}}(b,u) = \arg \min_{(e,w)\in\mathcal{S}\times\mathcal{U}} \| (b,u)-(e,w) \| .$$

Definition 2.6. The *normal cone to $\mathcal{S}\times\mathcal{U}$ at* $(b,u)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}}$ is given by

$$N_{\mathcal{S}\times\mathcal{U}}(b,u) = \{(e,w)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}} : \langle(e,w),(s,v)-(b,u)\rangle \leq 0,$$
$$\forall (s,v)\in\mathcal{S}\times\mathcal{U}\},\quad (b,u)\in\mathcal{S}\times\mathcal{U},$$
$$N_{\mathcal{S}\times\mathcal{U}}(b,u) = \emptyset,\quad (b,u)\notin\mathcal{S}\times\mathcal{U}$$

and the *tangent cone to $\mathcal{S}\times\mathcal{U}$ at* $(b,u)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}}$ is $T_{\mathcal{S}\times\mathcal{U}}(b,u) = [N_{\mathcal{S}\times\mathcal{U}}(b,u)]^\circ$.

Remark 2.3. By considering the previous definitions, we notice that $(b^*,u^*)\in(\mathcal{S}\times\mathcal{U})^* \iff \left(-\frac{\delta\mathcal{F}}{\delta b^*}, -\frac{\delta\mathcal{F}}{\delta u^*}\right)\in N_{\mathcal{S}\times\mathcal{U}}(b^*,u^*)$.

3. Some basic results

In this section, some basic results are established.

Proposition 3.1. *Assume* $\mathcal{F}(b,u)=\int_K f(t,b,b_\alpha,u)\,dv$ *is convex on* $\mathcal{S}\times\mathcal{U}$. *Then,*

(i) for any $(b^1,u^1),(b^2,u^2)\in(\mathcal{S}\times\mathcal{U})^*$, *it follows that*

$$\int_K\left[f_b\left(t,b^2,b^2_\alpha,u^2\right)(b^1-b^2)+f_{b_\alpha}\left(t,b^2,b^2_\alpha,u^2\right)D_\alpha(b^1-b^2)\right]dv$$

$$+\int_K\left[f_u\left(t,b^2,b^2_\alpha,u^2\right)(u^1-u^2)\right]dv=0;$$

(ii) the inclusion $(\mathcal{S}\times\mathcal{U})^*\subset(\mathcal{S}\times\mathcal{U})_*$ *is satisfied.*

Proof. (i) By $(b^1,u^1)\in(\mathcal{S}\times\mathcal{U})^*$, we get

$$\int_K\left[f_b\left(t,b^1,b^1_\alpha,u^1\right)(b-b^1)+f_{b_\alpha}\left(t,b^1,b^1_\alpha,u^1\right)D_\alpha(b-b^1)\right]dv$$

$$+\int_K\left[f_u\left(t,b^1,b^1_\alpha,u^1\right)(u-u^1)\right]dv\geq 0,\quad\forall(b,u)\in\mathcal{S}\times\mathcal{U}.$$

Since $(b^2,u^2)\in(\mathcal{S}\times\mathcal{U})^*\subset\mathcal{S}\times\mathcal{U}$, the previous inequality is rewritten as follows:

$$\int_K\left[f_b\left(t,b^1,b^1_\alpha,u^1\right)(b^2-b^1)+f_{b_\alpha}\left(t,b^1,b^1_\alpha,u^1\right)D_\alpha(b^2-b^1)\right]dv$$

$$+\int_K\left[f_u\left(t,b^1,b^1_\alpha,u^1\right)(u^2-u^1)\right]dv\geq 0. \tag{3.1}$$

By hypothesis, the scalar functional $\mathcal{F}(b,u)=\int_K f(t,b,b_\alpha,u)\,dv$ is convex on $\mathcal{S}\times\mathcal{U}$. Consequently, it yields

$$\mathcal{F}(b^1,u^1)-\mathcal{F}(b^2,u^2)$$

$$\geq\int_K\left[f_b\left(t,b^2,b^2_\alpha,u^2\right)(b^1-b^2)+f_{b_\alpha}\left(t,b^2,b^2_\alpha,u^2\right)D_\alpha(b^1-b^2)\right]dv$$

$$+\int_K\left[f_u\left(t,b^2,b^2_\alpha,u^2\right)(u^1-u^2)\right]dv, \tag{3.2}$$

or, equivalently,

$$\mathcal{F}(b^2,u^2)-\mathcal{F}(b^1,u^1)$$

$$\geq\int_K\left[f_b\left(t,b^1,b^1_\alpha,u^1\right)(b^2-b^1)+f_{b_\alpha}\left(t,b^1,b^1_\alpha,u^1\right)D_\alpha(b^2-b^1)\right]dv$$

$$+\int_K\left[f_u\left(t,b^1,b^1_\alpha,u^1\right)(u^2-u^1)\right]dv. \tag{3.3}$$

Combining (3.2) and (3.3) and by considering (3.1), we get

$$\int_K \left[f_b\left(t, b^2, b^2_\alpha, u^2\right)(b^1 - b^2) + f_{b_\alpha}\left(t, b^2, b^2_\alpha, u^2\right) D_\alpha(b^1 - b^2)\right] dv$$

$$+ \int_K \left[f_u\left(t, b^2, b^2_\alpha, u^2\right)(u^1 - u^2)\right] dv \leq 0. \tag{3.4}$$

Similarly as above, by $(b^2, u^2) \in (\mathcal{S} \times \mathcal{U})^*$, we can write

$$\int_K \left[f_b\left(t, b^2, b^2_\alpha, u^2\right)(b^1 - b^2) + f_{b_\alpha}\left(t, b^2, b^2_\alpha, u^2\right) D_\alpha(b^1 - b^2)\right] dv$$

$$+ \int_K \left[f_u\left(t, b^2, b^2_\alpha, u^2\right)(u^1 - u^2)\right] dv \geq 0. \tag{3.5}$$

Now, by considering (3.4) and (3.5), the proof is now completed.

(ii) By $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, it yields

$$\int_K \left[f_b\left(t, b^*, b^*_\alpha, u^*\right)(b - b^*) + f_{b_\alpha}\left(t, b^*, b^*_\alpha, u^*\right) D_\alpha(b - b^*)\right] dv$$

$$+ \int_K \left[f_u\left(t, b^*, b^*_\alpha, u^*\right)(u - u^*)\right] dv \geq 0, \quad \forall (b, u) \in \mathcal{S} \times \mathcal{U}. \tag{3.6}$$

The convexity property on $\mathcal{S} \times \mathcal{U}$ of $\mathcal{F}(b, u)$ (see (3.2) and (3.3)) involves

$$\int_K \left[f_b\left(t, b^1, b^1_\alpha, u^1\right)(b^1 - b^2) + f_{b_\alpha}\left(t, b^1, b^1_\alpha, u^1\right) D_\alpha(b^1 - b^2)\right] dv$$

$$+ \int_K \left[f_u\left(t, b^1, b^1_\alpha, u^1\right)(u^1 - u^2)\right] dv$$

$$\geq \int_K \left[f_b\left(t, b^2, b^2_\alpha, u^2\right)(b^1 - b^2) + f_{b_\alpha}\left(t, b^2, b^2_\alpha, u^2\right) D_\alpha(b^1 - b^2)\right] dv$$

$$+ \int_K \left[f_u\left(t, b^2, b^2_\alpha, u^2\right)(u^1 - u^2)\right] dv, \quad \forall (b^1, u^1), (b^2, u^2) \in \mathcal{S} \times \mathcal{U}. \tag{3.7}$$

Next, by considering (3.6) and (3.7), we obtain

$$\int_K \left[f_b\left(t, b, b_\alpha, u\right)(b - b^*) + f_{b_\alpha}\left(t, b, b_\alpha, u\right) D_\alpha(b - b^*)\right] dv$$

$$+ \int_K \left[f_u\left(t, b, b_\alpha, u\right)(u - u^*)\right] dv \geq 0, \quad \forall (b, u) \in \mathcal{S} \times \mathcal{U}$$

and this completes the proof. □

Remark 3.1. By using the continuity of $\delta\mathcal{F}(b, u)$, we obtain $(\mathcal{S} \times \mathcal{U})_* \subset (\mathcal{S} \times \mathcal{U})^*$. Also, by Proposition 3.1, we obtain $(\mathcal{S} \times \mathcal{U})^* = (\mathcal{S} \times \mathcal{U})_*$. Since the solution set $(\mathcal{S} \times \mathcal{U})_*$ for $(DCMVIP)$ is convex, in consequence, the solution set $(\mathcal{S} \times \mathcal{U})^*$ for $(CMVIP)$ is convex.

Proposition 3.2. *Let $\mathcal{H}(b,u)$ be differentiable on $\overline{\mathcal{S}} \times \overline{\mathcal{U}}$. Then, for any $(b,u),(v,\mu) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$ and $(e,w) \in Q(b,u)$, the inequality*

$$\langle(\frac{\delta\mathcal{H}}{\delta b}, \frac{\delta\mathcal{H}}{\delta u}); (v,\mu)\rangle \geq \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (v,\mu)\rangle$$

is satisfied.

Proof. By considering Definition 2.3, it follows that

$$\mathcal{H}(b,u) = \max_{(e,w)\in\mathcal{S}\times\mathcal{U}} \int_K \left[f_b\,(t,e,e_\alpha,w)\,(b-e) + f_{b_\alpha}\,(t,e,e_\alpha,w)\,D_\alpha(b-e)\right] dv$$
$$+ \int_K \left[f_u\,(t,e,e_\alpha,w)\,(u-w)\right] dv$$

for $(b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$, and (see Remark 2.2) we get

$$\mathcal{H}(b,u) = \max_{(e,w)\in\mathcal{S}\times\mathcal{U}} \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (b-e, u-w)\rangle, \quad \forall (b,u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}},$$

or,

$$\mathcal{H}(b,u) = \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (b-e, u-w)\rangle, \quad \forall (e,w) \in Q(b,u). \tag{3.8}$$

Also, the inequality

$$\mathcal{H}(s,v) \geq \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (s-e, v-w)\rangle \tag{3.9}$$

is true for any $(e,w) \in \mathcal{S} \times \mathcal{U}$ and $(s,v) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$, and, by using (3.8) and (3.9), it yields

$$\mathcal{H}(s,v) - \mathcal{H}(b,u) \geq \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (s-b, v-u)\rangle, \quad \forall (e,w) \in Q(b,u)$$

for any $(b,u),(s,v) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$. For $(s,v) = (b,u) + \lambda(v,\mu) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$, with $\lambda > 0$, the above inequality can be rewritten as

$$\mathcal{H}(b+\lambda v, u+\lambda\mu) - \mathcal{H}(b,u) \geq \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (\lambda v, \lambda\mu)\rangle,$$
$$\forall (e,w) \in Q(b,u), \quad \forall (b,u),(v,\mu) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}},$$

or, by dividing with $\lambda > 0$, we obtain

$$\frac{\mathcal{H}(b+\lambda v, u+\lambda\mu) - \mathcal{H}(b,u)}{\lambda} \geq \langle(\frac{\delta\mathcal{F}}{\delta e}, \frac{\delta\mathcal{F}}{\delta w}); (v,\mu)\rangle,$$
$$\forall (e,w) \in Q(b,u), \quad \forall (b,u),(v,\mu) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}.$$

Next, by taking the limit for $\lambda \to 0$ and by Definition 2.2, the proof is now completed. □

Proposition 3.3. *Let $\mathcal{H}(b,u)$ be differentiable on $(\mathcal{S}\times\mathcal{U})^*$ and $\mathcal{F}(b,u)$ be convex on $\mathcal{S}\times\mathcal{U}$. In addition, suppose the implication*

$$\langle(\frac{\delta\mathcal{H}}{\delta b^*}, \frac{\delta\mathcal{H}}{\delta u^*}); (v,\mu)\rangle \geq \langle(\frac{\delta\mathcal{F}}{\delta s}, \frac{\delta\mathcal{F}}{\delta v}); (v,\mu)\rangle \Longrightarrow (\frac{\delta\mathcal{H}}{\delta b^*}, \frac{\delta\mathcal{H}}{\delta u^*}) = (\frac{\delta\mathcal{F}}{\delta s}, \frac{\delta\mathcal{F}}{\delta v})$$

is satisfied for any $(b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*$, $(v,\mu) \in \overline{\mathcal{S}}\times\overline{\mathcal{U}}$ *and* $(s, v) \in Q(b^*, u^*)$*. Then, we have the equality* $Q(b^*, u^*) = (\mathcal{S}\times\mathcal{U})^*$, $\forall(b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*$.

Proof. "⊂" Let us consider $(s, v) \in Q(b^*, u^*)$. It yields

$$\mathcal{H}(b^*, u^*) = \int_K [f_b\,(t, s, s_\alpha, v)\,(b^* - s) + f_{b_\alpha}\,(t, s, s_\alpha, v)\,D_\alpha(b^* - s)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, v)\,(u^* - v)]\,dv, \quad (b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*. \tag{3.10}$$

The functional $\mathcal{F}(b, u)$ is convex on $\mathcal{S}\times\mathcal{U}$ (by hypothesis) and $(b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*$. By using Remark 3.1 and Proposition 3.1, we obtain $(b^*, u^*) \in (\mathcal{S}\times\mathcal{U})_*$, that is

$$\int_K [f_b\,(t, b, b_\alpha, u)\,(b - b^*) + f_{b_\alpha}\,(t, b, b_\alpha, u)\,D_\alpha(b - b^*)]\,dv$$

$$+ \int_K [f_u\,(t, b, b_\alpha, u)\,(u - u^*)]\,dv \geq 0 \tag{3.11}$$

for any $(b, u) \in \mathcal{S}\times\mathcal{U}$. By (3.10) and (3.11), it yields $\mathcal{H}(b^*, u^*) = 0$, $\forall(b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*$, or equivalently,

$$\int_K [f_b\,(t, s, s_\alpha, v)\,(b^* - s) + f_{b_\alpha}\,(t, s, s_\alpha, v)\,D_\alpha(b^* - s)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, v)\,(u^* - v)]\,dv = 0, \quad (b^*, u^*) \in (\mathcal{S}\times\mathcal{U})^*. \tag{3.12}$$

By (3.12), for any $(b, u) \in \mathcal{S}\times\mathcal{U}$, we obtain

$$\int_K [f_b\,(t, s, s_\alpha, v)\,(b - s) + f_{b_\alpha}\,(t, s, s_\alpha, v)\,D_\alpha(b - s)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, v)\,(u - v)]\,dv$$

$$= \int_K [f_b\,(t, s, s_\alpha, v)\,(b - b^*) + f_{b_\alpha}\,(t, s, s_\alpha, v)\,D_\alpha(b - b^*)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, v)\,(u - u^*)]\,dv. \tag{3.13}$$

In the following, by using the definition of the dual gap functional $\mathcal{H}(b, u)$ of $(CMVIP)$, we can write

$$\frac{\mathcal{H}(b^* + \lambda(b - b^*), u^* + \lambda(u - u^*)) - \mathcal{H}(b^*, u^*)}{\lambda}$$

$$\geq \int_K [f_b\,(t, b^*, b^*_\alpha, u^*)\,(b - b^*) + f_{b_\alpha}\,(t, b^*, b^*_\alpha, u^*)\,D_\alpha(b - b^*)]\,dv$$

$$+ \int_K [f_u\,(t, b^*, b^*_\alpha, u^*)\,(u - u^*)]\,dv$$

for any $\lambda \in [0,1]$ and $(b,u) \in \mathcal{S} \times \mathcal{U}$. Taking the limit for $\lambda \to 0$ and using Definition 2.2, we obtain

$$\langle (\frac{\delta \mathcal{H}}{\delta b^*}, \frac{\delta \mathcal{H}}{\delta u^*}); (b - b^*, u - u^*) \rangle$$

$$\geq \int_K \left[f_b\left(t, b^*, b^*_\alpha, u^*\right)(b - b^*) + f_{b_\alpha}\left(t, b^*, b^*_\alpha, u^*\right) D_\alpha(b - b^*) \right] dv$$

$$+ \int_K \left[f_u\left(t, b^*, b^*_\alpha, u^*\right)(u - u^*) \right] dv. \tag{3.14}$$

By Proposition 3.2 and the hypothesis, we obtain $(\frac{\delta \mathcal{H}}{\delta b^*}, \frac{\delta \mathcal{H}}{\delta u^*}) = (\frac{\delta \mathcal{F}}{\delta s}, \frac{\delta \mathcal{F}}{\delta v})$. Therefore, (3.14) becomes

$$\langle (\frac{\delta \mathcal{F}}{\delta s}, \frac{\delta \mathcal{F}}{\delta v}); (b - b^*, u - u^*) \rangle$$

$$\geq \int_K \left[f_b\left(t, b^*, b^*_\alpha, u^*\right)(b - b^*) + f_{b_\alpha}\left(t, b^*, b^*_\alpha, u^*\right) D_\alpha(b - b^*) \right] dv$$

$$+ \int_K \left[f_u\left(t, b^*, b^*_\alpha, u^*\right)(u - u^*) \right] dv,$$

or, equivalently,

$$\int_K \left[f_b\left(t, s, s_\alpha, v\right)(b - b^*) + f_{b_\alpha}\left(t, s, s_\alpha, v\right) D_\alpha(b - b^*) \right] dv$$

$$\int_K \left[f_u\left(t, s, s_\alpha, v\right)(u - u^*) \right] dv$$

$$\geq \int_K \left[f_b\left(t, b^*, b^*_\alpha, u^*\right)(b - b^*) + f_{b_\alpha}\left(t, b^*, b^*_\alpha, u^*\right) D_\alpha(b - b^*) \right] dv$$

$$+ \int_K \left[f_u\left(t, b^*, b^*_\alpha, u^*\right)(u - u^*) \right] dv. \tag{3.15}$$

By considering (3.13) and (3.15), it yields

$$\int_K \left[f_b\left(t, s, s_\alpha, v\right)(b - s) + f_{b_\alpha}\left(t, s, s_\alpha, v\right) D_\alpha(b - s) \right] dv$$

$$+ \int_K \left[f_u\left(t, s, s_\alpha, v\right)(u - v) \right] dv$$

$$\geq \int_K \left[f_b\left(t, b^*, b^*_\alpha, u^*\right)(b - b^*) + f_{b_\alpha}\left(t, b^*, b^*_\alpha, u^*\right) D_\alpha(b - b^*) \right] dv$$

$$+ \int_K \left[f_u\left(t, b^*, b^*_\alpha, u^*\right)(u - u^*) \right] dv.$$

Since $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, the previous inequality implies

$$\int_K \left[f_b\left(t, s, s_\alpha, v\right)(b - s) + f_{b_\alpha}\left(t, s, s_\alpha, v\right) D_\alpha(b - s) \right] dv$$

$$+ \int_K \left[f_u\left(t, s, s_\alpha, v\right)(u - v) \right] dv \geq 0, \quad \forall (b,u) \in \mathcal{S} \times \mathcal{U},$$

involving $(s, v) \in (\mathcal{S} \times \mathcal{U})^*$.

"⊃" Let us consider $(s, \nu), (b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$. By using Proposition 3.1, we obtain

$$\int_K [f_b\,(t, s, s_\alpha, \nu)\,(b^* - s) + f_{b_\alpha}\,(t, s, s_\alpha, \nu)\,D_\alpha(b^* - s)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, \nu)\,(u^* - \nu)]\,dv = 0.$$

Since $\mathcal{H}(b^*, u^*) = 0,\ \forall(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, it yields

$$\mathcal{H}(b^*, u^*) = \int_K [f_b\,(t, s, s_\alpha, \nu)\,(b^* - s) + f_{b_\alpha}\,(t, s, s_\alpha, \nu)\,D_\alpha(b^* - s)]\,dv$$

$$+ \int_K [f_u\,(t, s, s_\alpha, \nu)\,(u^* - \nu)]\,dv,$$

implying $(s, \nu) \in Q(b^*, u^*)$. This completes the proof. □

4. Main results

In this section, we study weak sharp solutions for the considered controlled multidimensional variational-type inequality involving a convex multiple integral functional.

Definition 4.1. The set of solutions $(\mathcal{S} \times \mathcal{U})^*$ for $(CMVIP)$ is *weakly sharp* if

$$\left(-\frac{\delta \mathcal{F}}{\delta b^*}, -\frac{\delta \mathcal{F}}{\delta u^*}\right) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*} \left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b}, \overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b}, \overline{u})\right]^\circ\right)$$

for all $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, or, in an equivalent form, there exists a positive number $\gamma > 0$ such that

$$\gamma B \subset \left(\frac{\delta \mathcal{F}}{\delta b^*}, \frac{\delta \mathcal{F}}{\delta u^*}\right) + [T_{\mathcal{S}\times\mathcal{U}}(b^*, u^*) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(b^*, u^*)]^\circ, \quad \forall(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*,$$

where $int(M)$ represents the interior of the set M and B stands for the open unit ball in $\overline{\mathcal{S}} \times \overline{\mathcal{U}}$.

Lemma 4.1. *There exists* $\gamma > 0$ *satisfying*

$$\gamma B \subset \left(\frac{\delta \mathcal{F}}{\delta e}, \frac{\delta \mathcal{F}}{\delta w}\right) + [T_{\mathcal{S}\times\mathcal{U}}(e, w) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(e, w)]^\circ, \quad \forall(e, w) \in (\mathcal{S} \times \mathcal{U})^* \tag{4.1}$$

if and only if

$$\langle(\frac{\delta \mathcal{F}}{\delta e}, \frac{\delta \mathcal{F}}{\delta w}); (s, \nu)\rangle \geq \gamma \parallel (s, \nu) \parallel, \quad \forall(s, \nu) \in T_{\mathcal{S}\times\mathcal{U}}(e, w) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(e, w). \tag{4.2}$$

Proof. The equivalent form of (4.1) is

$$\gamma(b, K) - (\frac{\delta \mathcal{F}}{\delta e}, \frac{\delta \mathcal{F}}{\delta w}) \in [T_{\mathcal{S}\times\mathcal{U}}(e, w) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(e, w)]^\circ,$$

$$\forall(e, w) \in (\mathcal{S} \times \mathcal{U})^*,\ \forall(b, \upsilon) \in B,$$

or

$$\langle\gamma(b,\upsilon)-(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w});(s,\nu)\rangle\leq 0,$$
$$\forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*,\ \forall(b,\upsilon)\in B,\ \forall(s,\nu)\in T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w).$$

Considering $B\ni(b,\upsilon)=\dfrac{(s,\nu)}{\|(s,\nu)\|}$, $(s,\nu)\neq(0,0)$, the above inequality is (4.2).

Conversely, let us consider that the relation (4.2) is fulfilled. Then, there exists $\gamma>0$ satisfying

$$\langle\gamma(b,\upsilon)-(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w});(s,\nu)\rangle=\langle\gamma(b,\upsilon);(s,\nu)\rangle-\langle(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w});(s,\nu)\rangle$$
$$\leq\gamma\|(s,\nu)\|-\gamma\|(s,\nu)\|=0,$$
$$\forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*,\ \forall(b,\upsilon)\in B,\ \forall(s,\nu)\in T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w),$$

that is,

$$\langle\gamma(b,\upsilon)-(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w});(s,\nu)\rangle\leq 0,$$
$$\forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*,\ \forall(b,\upsilon)\in B,\ \forall(s,\nu)\in T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w),$$

or, equivalently,

$$\gamma(b,\upsilon)-(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w})\in\left[T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)\right]^{\circ}$$

for $\forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*$, $\forall(b,\upsilon)\in B$. The above relation implies (4.1) and this completes the proof. □

Theorem 4.1. *Let $\mathcal{H}(b,u)$ be differentiable on $(\mathcal{S}\times\mathcal{U})^*$ and $\mathcal{F}(b,u)$ be convex on $\mathcal{S}\times\mathcal{U}$. In addition, suppose the implication*

$$\langle(\frac{\delta\mathcal{H}}{\delta b^*},\frac{\delta\mathcal{H}}{\delta u^*});(\nu,\mu)\rangle\geq\langle(\frac{\delta\mathcal{F}}{\delta s},\frac{\delta\mathcal{F}}{\delta \nu});(\nu,\mu)\rangle\Longrightarrow\left(\frac{\delta\mathcal{H}}{\delta b^*},\frac{\delta\mathcal{H}}{\delta u^*}\right)=\left(\frac{\delta\mathcal{F}}{\delta s},\frac{\delta\mathcal{F}}{\delta \nu}\right)$$

is satisfied for any $(b^*,u^*)\in(\mathcal{S}\times\mathcal{U})^*$, $(\nu,\mu)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}}$ and $(s,\nu)\in Q(b^*,u^*)$, and $\left(\dfrac{\delta\mathcal{F}}{\delta b^*},\dfrac{\delta\mathcal{F}}{\delta u^*}\right)$ is constant on $(\mathcal{S}\times\mathcal{U})^*$. Then, $(\mathcal{S}\times\mathcal{U})^*$ is weakly sharp if and only if there exists $\gamma>0$ so that

$$\mathcal{H}(b,u)\geq\gamma d\left((b,u),(\mathcal{S}\times\mathcal{U})^*\right),\quad\forall(b,u)\in\mathcal{S}\times\mathcal{U},$$

where $d\left((b,u),(\mathcal{S}\times\mathcal{U})^*\right)=\min\limits_{(e,w)\in(\mathcal{S}\times\mathcal{U})^*}\|(b,u)-(e,w)\|$.

Proof. "$\Longrightarrow$" Let $(\mathcal{S}\times\mathcal{U})^*$ be weakly sharp. Consequently, by Definition 4.1, we get

$$\left(-\frac{\delta\mathcal{F}}{\delta e},-\frac{\delta\mathcal{F}}{\delta w}\right)\in int\left(\bigcap_{(\bar{b},\bar{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\bar{b},\bar{u})\cap N_{(\mathcal{S}\times\mathcal{U})^*}(\bar{b},\bar{u})\right]^{\circ}\right)$$

for any $(e,w)\in(\mathcal{S}\times\mathcal{U})^*$. Equivalently, by using Lemma 4.1, there exists $\gamma>0$ satisfying (4.1) (or (4.2)).

Next, by considering the convexity property of $(\mathcal{S}\times\mathcal{U})^*$, it follows that

$$\mathrm{proj}_{(\mathcal{S}\times\mathcal{U})^*}(b,u)=(\hat{e},\hat{w})\in(\mathcal{S}\times\mathcal{U})^*,\quad\forall(b,u)\in\mathcal{S}\times\mathcal{U}$$

and, in accordance with [7], we get $(b,u)-(\hat{e},\hat{w}) \in T_{\mathcal{S}\times\mathcal{U}}(\hat{e},\hat{w}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\hat{e},\hat{w})$. By considering the hypothesis and by using Lemma 4.1, we obtain

$$\langle(\frac{\delta\mathcal{F}}{\delta\hat{e}},\frac{\delta\mathcal{F}}{\delta\hat{w}});(b-\hat{e},u-\hat{w})\rangle \geq \gamma \parallel (b,u)-(\hat{e},\hat{w}) \parallel = \gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*),$$

or,

$$\begin{aligned}\int_K \left[f_b\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,(b-\hat{e}) + f_{b_\alpha}\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,D_\alpha(b-\hat{e})\right] dv \\ + \int_K \left[f_u\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,(u-\hat{w})\right] dv \\ \geq \gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*), \quad \forall (b,u)\in\mathcal{S}\times\mathcal{U}.\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}\mathcal{H}(b,u) \geq \int_K \left[f_b\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,(b-\hat{e}) + f_{b_\alpha}\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,D_\alpha(b-\hat{e})\right] dv \\ + \int_K \left[f_u\,(t,\hat{e},\hat{e}_\alpha,\hat{w})\,(u-\hat{w})\right] dv, \quad \forall (b,u)\in\mathcal{S}\times\mathcal{U},\end{aligned}$$

by (4.3), we get

$$\mathcal{H}(b,u) \geq \gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*), \quad \forall (b,u)\in\mathcal{S}\times\mathcal{U}.$$

"$\Longleftarrow$" Let us consider that there exists a positive number $\gamma > 0$ such that

$$\mathcal{H}(b,u) \geq \gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*), \quad \forall (b,u)\in\mathcal{S}\times\mathcal{U}.$$

For any $(e,w)\in(\mathcal{S}\times\mathcal{U})^*$, the situation $T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)=\{(0,0)\}$ implies

$$[T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)]^\circ = \overline{\mathcal{S}}\times\overline{\mathcal{U}},$$

and

$$\gamma B \subset \left(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w}\right) + [T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)]^\circ, \quad \forall (e,w)\in(\mathcal{S}\times\mathcal{U})^*$$

is obviously. Let $(0,0)\neq(\overline{b},\overline{u})\in T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)$. This fact means that there exists a sequence $(\overline{b}^k,\overline{u}^k)$ converging to $(\overline{b},\overline{u})$ with $(e,w)+t_k(\overline{b}^k,\overline{u}^k)\in\mathcal{S}\times\mathcal{U}$, so that

$$\begin{aligned}d((e,w)+t_k(\overline{b}^k,\overline{u}^k),(\mathcal{S}\times\mathcal{U})^*) \geq d((e,w)+t_k(\overline{b}^k,\overline{u}^k),\mathcal{H}_{\overline{b},\overline{u}}) \\ = \frac{t_k\langle(\overline{b},\overline{u});(\overline{b}^k,\overline{u}^k)\rangle}{\|(\overline{b},\overline{u})\|}.\end{aligned} \tag{4.4}$$

Here, $\mathcal{H}_{\overline{b},\overline{u}} = \left\{(b,u)\in\overline{\mathcal{S}}\times\overline{\mathcal{U}} : \langle(\overline{b},\overline{u});(b,u)-(e,w)\rangle = 0\right\}$ is a hyperplane orthogonal to $(\overline{b},\overline{u})$ and passing through (e,w). By the hypothesis and (4.4), it follows that

$$\mathcal{H}((e,w)+t_k(\overline{b}^k,\overline{u}^k)) \geq \gamma\frac{t_k\langle(\overline{b},\overline{u});(\overline{b}^k,\overline{u}^k)\rangle}{\|(\overline{b},\overline{u})\|},$$

or ($\mathcal{H}(e,w)=0,\ \forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*$),

$$\frac{\mathcal{H}((e,w)+t_k(\overline{b}^k,\overline{u}^k))-\mathcal{H}(e,w)}{t_k}\geq\gamma\frac{\langle(\overline{b},\overline{u});(\overline{b}^k,\overline{u}^k)\rangle}{\|(\overline{b},\overline{u})\|}. \tag{4.5}$$

By taking the limit for $k\to\infty$ in (4.5) (using a classical result of functional analysis), we obtain

$$\lim_{\lambda\to 0}\frac{\mathcal{H}((e,w)+\lambda(\overline{b},\overline{u}))-\mathcal{H}(e,w)}{\lambda}\geq\gamma\|(\overline{b},\overline{u})\|, \tag{4.6}$$

where $\lambda>0$. The inequality (4.6) can be formulated as

$$\langle(\frac{\delta\mathcal{H}}{\delta e},\frac{\delta\mathcal{H}}{\delta w});(\overline{b},\overline{u})\rangle\geq\gamma\|(\overline{b},\overline{u})\|. \tag{4.7}$$

Next, by the hypothesis and (4.7), it yields

$$\langle\gamma(b,\upsilon)-(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w});(\overline{b},\overline{u})\rangle=\langle\gamma(b,\upsilon);(\overline{b},\overline{u})\rangle-\langle(\frac{\delta\mathcal{H}}{\delta e},\frac{\delta\mathcal{H}}{\delta w});(\overline{b},\overline{u})\rangle$$

$$\leq\gamma\|(\overline{b},\overline{u})\|-\gamma\|(\overline{b},\overline{u})\|=0$$

for any $(b,\upsilon)\in B$, and

$$\gamma B\subset\left(\frac{\delta\mathcal{F}}{\delta e},\frac{\delta\mathcal{F}}{\delta w}\right)+\left[T_{\mathcal{S}\times\mathcal{U}}(e,w)\cap N_{(\mathcal{S}\times\mathcal{U})^*}(e,w)\right]^\circ,\quad\forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*.$$

This completes the proof. □

Remark 4.1. (i) The *weak sharpness property* of the solution set for the variational problem

$$\min_{(b,u)\in\mathcal{S}\times\mathcal{U}}\mathcal{H}(b,u)$$

is described by the inequality (recall that $\mathcal{H}(e,w)=0,\ \forall(e,w)\in(\mathcal{S}\times\mathcal{U})^*$)

$$\mathcal{H}(b,u)-\mathcal{H}(b^*,u^*)\geq\gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*),\quad\forall(b,u)\in\mathcal{S}\times\mathcal{U},\ (b^*,u^*)\in(\mathcal{S}\times\mathcal{U})^*$$

formulated in Theorem 4.1.

(ii) If

$$\mathcal{H}(b,u)\geq\gamma d((b,u),(\mathcal{S}\times\mathcal{U})^*),\quad\forall(b,u)\in\mathcal{S}\times\mathcal{U}$$

is fulfilled, the function $\mathcal{H}$ provides an error bound for the distance from a feasible point and the solution set $(\mathcal{S}\times\mathcal{U})^*$. The supremum of the positive constant γ is called the *modulus of sharpness* for the solution set $(\mathcal{S}\times\mathcal{U})^*$.

The second characterization result of weak sharpness for $(\mathcal{S}\times\mathcal{U})^*$ implies the notion of a *minimum principle sufficiency property*, introduced by Ferris and Mangasarian [6].

Definition 4.2. The controlled variational-type inequality ($CMVIP$) satisfies the *minimum principle sufficiency property* if $\mathbb{A}(b^*,u^*)=(\mathcal{S}\times\mathcal{U})^*$ for any $(b^*,u^*)\in(\mathcal{S}\times\mathcal{U})^*$.

Lemma 4.2. *The following inclusion* $\arg\max_{(e,w)\in\mathcal{S}\times\mathcal{U}}\langle(b,u);(e,w)\rangle \subset (\mathcal{S}\times\mathcal{U})^*$ *is fulfilled for any* $(b,u) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right) \neq \emptyset.$

Proof. Consider $(e,w) \in (\mathcal{S}\times\mathcal{U}) \setminus (\mathcal{S}\times\mathcal{U})^*$. By using the convexity property of $(\mathcal{S}\times\mathcal{U})^*$, it follows that

$$\text{proj}_{(\mathcal{S}\times\mathcal{U})^*}(e,w) = (\hat{e},\hat{w}) \in (\mathcal{S}\times\mathcal{U})^*,$$

and (see [7]) we get $(e,w)-(\hat{e},\hat{w}) \in T_{\mathcal{S}\times\mathcal{U}}(\hat{e},\hat{w}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\hat{e},\hat{w})$. There exists $\alpha > 0$ such that

$$\langle(b,u)+(v,\mu);(e,w)-(\hat{e},\hat{w})\rangle < 0, \quad \forall(v,\mu) \in \alpha B,$$

and any $(b,u) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right)$, or, equivalently,

$$\langle(b,u);(e,w)\rangle < \langle(b,u);(\hat{e},\hat{w})\rangle - \langle(v,\mu);(e,w)-(\hat{e},\hat{w})\rangle, \quad \forall(v,\mu) \in \alpha B,$$

and any $(b,u) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right)$. For

$$(v,\mu) = \alpha\frac{(e,w)-(\hat{e},\hat{w})}{\| (e,w)-(\hat{e},\hat{w}) \|} \in \alpha B,$$

the previous inequality becomes

$$\langle(b,u);(e,w)\rangle < \langle(b,u);(\hat{e},\hat{w})\rangle - \alpha \| (e,w)-(\hat{e},\hat{w}) \| \tag{4.8}$$

for $(b,u) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right)$. By (4.8), we conclude that

$$(e,w) \notin \arg\max_{(e,w)\in\mathcal{S}\times\mathcal{U}}\langle(b,u);(e,w)\rangle,$$

that is,

$$\arg\max_{(e,w)\in\mathcal{S}\times\mathcal{U}}\langle(b,u);(e,w)\rangle \subset (\mathcal{S}\times\mathcal{U})^*$$

for $(b,u) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right)$. The proof is complete. □

Theorem 4.2. *Consider that the set of solutions* $(\mathcal{S}\times\mathcal{U})^*$ *for* $(CMVIP)$ *is weakly sharp and* $\mathcal{F}(b,u)$ *is convex on* $\mathcal{S}\times\mathcal{U}$*. Then,* $(CMVIP)$ *satisfies the minimum principle sufficiency property.*

Proof. By using Definition 4.2, if $\mathbb{A}(b^*,u^*) = (\mathcal{S}\times\mathcal{U})^*$ for any $(b^*,u^*) \in (\mathcal{S}\times\mathcal{U})^*$, then $(CMVIP)$ satisfies the minimum principle sufficiency property. Since $(\mathcal{S}\times\mathcal{U})^*$ is weakly sharp, we obtain

$$\left(-\frac{\delta\mathcal{F}}{\delta b^*}, -\frac{\delta\mathcal{F}}{\delta u^*}\right) \in int\left(\bigcap_{(\overline{b},\overline{u})\in(\mathcal{S}\times\mathcal{U})^*}\left[T_{\mathcal{S}\times\mathcal{U}}(\overline{b},\overline{u}) \cap N_{(\mathcal{S}\times\mathcal{U})^*}(\overline{b},\overline{u})\right]^\circ\right)$$

for any $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$ and, by Lemma 4.2, it follows that

$$\arg \max_{(e,w)\in\mathcal{S}\times\mathcal{U}} \langle(-\frac{\delta\mathcal{F}}{\delta b^*}, -\frac{\delta\mathcal{F}}{\delta u^*}); (e, w)\rangle \subset (\mathcal{S} \times \mathcal{U})^* \Longleftrightarrow \mathbb{A}(b^*, u^*) \subset (\mathcal{S} \times \mathcal{U})^*. \tag{4.9}$$

Further, let $(s, \nu) \in (\mathcal{S} \times \mathcal{U})^*$. For $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, by Proposition 3.1, we get

$$\int_K [f_b(t, b^*, b^*_\alpha, u^*)(s - b^*) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(s - b^*)]\, dv$$
$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(\nu - u^*)]\, dv = 0. \tag{4.10}$$

By (4.10), for any $(e, w) \in \mathcal{S} \times \mathcal{U}$, it yields

$$\int_K [f_b(t, b^*, b^*_\alpha, u^*)(s - e) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(s - e)]\, dv$$
$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(\nu - w)]\, dv$$
$$= \int_K [f_b(t, b^*, b^*_\alpha, u^*)(b^* - e) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(b^* - e)]\, dv$$
$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(u^* - w)]\, dv. \tag{4.11}$$

Since $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, the relation (4.11) provides

$$\int_K [f_b(t, b^*, b^*_\alpha, u^*)(s - e) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(s - e)]\, dv$$
$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(\nu - w)]\, dv \le 0, \quad \forall (e, w) \in \mathcal{S} \times \mathcal{U},$$

that is, $(s, \nu) \in \mathbb{A}(b^*, u^*)$ and, consequently,

$$(\mathcal{S} \times \mathcal{U})^* \subset \mathbb{A}(b^*, u^*). \tag{4.12}$$

The proof is completed by considering (4.9) and (4.12). □

Theorem 4.3. *Let $\mathcal{H}(b, u)$ be differentiable on $(\mathcal{S}\times\mathcal{U})^*$ and $\mathcal{F}(b, u)$ be convex on $\mathcal{S}\times\mathcal{U}$. Also, suppose the implication*

$$\langle(\frac{\delta\mathcal{H}}{\delta b^*}, \frac{\delta\mathcal{H}}{\delta u^*}); (\nu, \mu)\rangle \ge \langle(\frac{\delta\mathcal{F}}{\delta s}, \frac{\delta\mathcal{F}}{\delta \nu}); (\nu, \mu)\rangle \Longrightarrow \left(\frac{\delta\mathcal{H}}{\delta b^*}, \frac{\delta\mathcal{H}}{\delta u^*}\right) = \left(\frac{\delta\mathcal{F}}{\delta s}, \frac{\delta\mathcal{F}}{\delta \nu}\right)$$

is true for any $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, $(\nu, \mu) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$ and $(s, \nu) \in Q(b^*, u^*)$, and $\left(\frac{\delta\mathcal{F}}{\delta b^*}, \frac{\delta\mathcal{F}}{\delta u^*}\right)$ is constant on $(\mathcal{S} \times \mathcal{U})^*$. Then, $(CMVIP)$ satisfies the minimum principle sufficiency property if and only if $(\mathcal{S} \times \mathcal{U})^*$ is weakly sharp.

Proof. Let $(CMVIP)$ satisfy the minimum principle sufficiency property. Therefore, for any $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$, we have $\mathbb{A}(b^*, u^*) = (\mathcal{S} \times \mathcal{U})^*$. For $(b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*$ and $(b, u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}}$, we obtain

$$\mathcal{H}(b, u) \geq \int_K [f_b(t, b^*, b^*_\alpha, u^*)(b - b^*) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(b - b^*)] \, dv$$

$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(u - u^*)] \, dv. \tag{4.13}$$

In the following, considering $P(b, u) = \langle (\frac{\delta \mathcal{F}}{\delta b^*}, \frac{\delta \mathcal{F}}{\delta u^*}); (b, u) \rangle$, $(b, u) \in \mathcal{S} \times \mathcal{U}$, we have $\mathbb{A}(b^*, u^*)$ as the solution set for $\min_{(b,u) \in \mathcal{S} \times \mathcal{U}} P(b, u)$. For other related investigations, we refer the readers to Mangasarian and Meyer [12]. In accordance with Remark 4.1, we have

$$P(b, u) - P(\tilde{b}, \tilde{u}) \geq \gamma d((b, u), \mathbb{A}(b^*, u^*)), \quad \forall (b, u) \in \mathcal{S} \times \mathcal{U}, \ (\tilde{b}, \tilde{u}) \in \mathbb{A}(b^*, u^*),$$

or,

$$\langle (\frac{\delta \mathcal{F}}{\delta b^*}, \frac{\delta \mathcal{F}}{\delta u^*}); (b, u) - (b^*, u^*) \rangle \geq \gamma d((b, u), (\mathcal{S} \times \mathcal{U})^*), \quad \forall (b, u) \in \mathcal{S} \times \mathcal{U},$$

or, equivalently,

$$\int_K [f_b(t, b^*, b^*_\alpha, u^*)(b - b^*) + f_{b_\alpha}(t, b^*, b^*_\alpha, u^*) D_\alpha(b - b^*)] \, dv$$

$$+ \int_K [f_u(t, b^*, b^*_\alpha, u^*)(u - u^*)] \, dv \geq \gamma d((b, u), (\mathcal{S} \times \mathcal{U})^*), \quad \forall (b, u) \in \mathcal{S} \times \mathcal{U}. \tag{4.14}$$

By considering Theorem 4.1 and (4.13) and (4.14), we obtain that $(\mathcal{S} \times \mathcal{U})^*$ is weakly sharp.

"$\Longleftarrow$" This implication is an immediate consequence of Theorem 4.2. □

Now, let us illustrate the effectiveness of the main results established in this section with the following application.

Application 4.1. Denote by K a square fixed by the diagonally opposite points $t_1 = (0, 0)$ and $t_2 = (2, 2)$ in $\mathbb{R}^2$. Also, let

$$\overline{\mathcal{S}} \times \overline{\mathcal{U}} = \{(b, u) \mid b : K \to [-1, 4], b = \text{piecewise smooth function};$$

$$u : K \to \mathbb{R}, u = \text{piecewise continuous function}\},$$

and let it be equipped with the standard Euclidean inner product and the induced norm

$$\mathcal{S} \times \mathcal{U} = \{(b, u) \in \overline{\mathcal{S}} \times \overline{\mathcal{U}} \mid \frac{\partial b}{\partial t^1} = \frac{\partial b}{\partial t^2} = u(t), \ 0 \leq b(t) \leq 1,$$

$$b(0, 0) = b(2, 2) = 0\},$$

and the real-valued continuously differentiable function

$$f : J^1(\mathbb{R}^2, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}, \quad f(t, b, b_\vartheta, u) = b^2 + 4b.$$

Now, let us consider the following bi-dimensional controlled variational inequality problem: *Find* $(y, w) \in \mathcal{S} \times \mathcal{U}$ *such that*

$$(BCVIP) \qquad \int_K \{\frac{\partial f}{\partial b}(t, y, y_\vartheta, w)(b - y) + \frac{\partial f}{\partial b_\vartheta}(t, y, y_\vartheta, w) D_\vartheta(b - y)$$

$$+\frac{\partial f}{\partial u}(t, y, y_\vartheta, w)(u - w)\}dt^1dt^2 \geq 0$$

for any $(b, u) \in \mathcal{S} \times \mathcal{U}$.

By direct computation, the dual gap-type multiple integral functional

$$\mathcal{H} : \overline{\mathcal{S}} \times \overline{\mathcal{U}} \to \mathbb{R}, \quad \mathcal{H}(b, u) = \int_K h(t, b, b_\vartheta, u)\, dt^1dt^2$$

is as follows

$$\mathcal{H}(b, u) = \max_{(y,w)\in\mathcal{S}\times\mathcal{U}} \int_K \{\frac{\partial f}{\partial b}(t, y, y_\vartheta, w)(b - y) + \frac{\partial f}{\partial b_\vartheta}(t, y, y_\vartheta, w) D_\vartheta(b - y)$$

$$+\frac{\partial f}{\partial u}(t, y, y_\vartheta, w)(u - w)\}dt^1dt^2$$

$$= \max_{(y,w)\in\mathcal{S}\times\mathcal{U}} \int_K (2y + 4)(b - y)dt^1dt^2 = \begin{cases} \displaystyle\int_K 4b\,dt^1dt^2, & -1 \leq b < 2 \\ \displaystyle\int_K \frac{(b + 2)^2}{2} dt^1dt^2, & 2 \leq b \leq 4. \end{cases}$$

As well, the mutiple integral functional

$$\mathcal{F} : \overline{\mathcal{S}} \times \overline{\mathcal{U}} \to \mathbb{R}, \quad \mathcal{F}(b, u) = \int_K f(t, b, b_\vartheta, u)\, dt^1dt^2$$

$$= \int_K \left(b^2 + 4b\right) dt^1dt^2,$$

is convex on $\mathcal{S} \times \mathcal{U}$.

As it can easily be seen, we obtain

$$(\mathcal{S} \times \mathcal{U})^* = \{(y, w) \mid y : K \to [0, 1], y(t) = 0;\ w : K \to \mathbb{R}, w(t) = 0,\ \forall t \in K\},$$

$$\mathbb{A}(b^*, u^*) = (\mathcal{S} \times \mathcal{U})^*,\ \forall (b^*, u^*) \in (\mathcal{S} \times \mathcal{U})^*; \quad \delta_\beta \mathcal{F}(b, u) = 2b + 4.$$

Obviously, the dual gap-type multiple integral functional $\mathcal{H}(b, u)$ is differentiable on $(\mathcal{S} \times \mathcal{U})^*$ and, for any $(b, u) \in \mathcal{S} \times \mathcal{U}$, there exists $\gamma > 0$ such that

$$\mathcal{H}(b, u) = \int_K 4b\,dt^1dt^2 \geq \gamma d((b, u), (\mathcal{S} \times \mathcal{U})^*).$$

Following the same steps as in Theorem 4.1, it results that $(\mathcal{S} \times \mathcal{U})^*$ is weakly sharp with the positive modulus γ. Also, by applying Theorems 4.2 and 4.3, it follows that $(BCVIP)$ satisfies the minimum principle sufficiency property.

5. Conclusions

In this paper, we have extended the well-known weak sharp solutions for variational inequalities to a controlled variational-type inequality governed by convex multiple integral functionals. Simultaneously, by using the minimum principle sufficiency property, some equivalent conditions on weak sharpness associated with solutions of the considered inequality have been obtained.

Conflict of interest

The authors declare that they have no competing interests.

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