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Research article

Modular edge irregularity strength of graphs

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Abstract: For a simple graph $G = (V, E)$ with the vertex set $V(G)$ and the edge set $E(G)$, a vertex labeling $\varphi : V(G) \to \{1, 2, \dots, k\}$ is called a k-labeling. The weight of an edge under the vertex labeling φ is the sum of the labels of its end vertices and the modular edge-weight is the remainder of the division of this sum by $|E(G)|$. A vertex k-labeling is called a modular edge irregular if for every two different edges their modular edge-weights are different. The maximal integer k minimized over all modular edge irregular k-labelings is called the modular edge irregularity strength of G. In the paper we estimate the bounds on the modular edge irregularity strength and for caterpillar, cycle, friendship graph and n-sun we determine the precise values of this parameter that prove the sharpness of the lower bound.

Keywords: (modular) irregular labeling; (modular) irregularity strength; (modular) edge irregularity strength; caterpillar; cycle; friendship graph; n-sun
Mathematics Subject Classification: 05C78

1. Introduction

Throughout this paper, $G = (V, E)$ is a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Chartrand et al. in [1] introduced an edge k-labeling $\psi : E(G) \to \{1, 2, \dots, k\}$ of a graph G such that the sum of the labels of edges incident with a vertex is different for all the vertices of G. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the maximal integer k, minimized over all irregular assignments. If no such assignment exists then

$s(G) = \infty$. Obviously, $s(G) < \infty$ if and only if G contains no isolated edge and has at most one isolated vertex. The irregularity strength has attracted much attention [2–8].

Motivated by these papers, Ahmad et al. in [9] defined an edge irregular k-labeling of a graph G to be a vertex labeling $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that the edge-weights $wt_\varphi(uv) = \varphi(u) + \varphi(v)$ are different for all edges, that is $wt_\varphi(uv) \neq wt_\varphi(u'v')$ for all different edges $uv, u'v' \in E(G)$. Furthermore, they defined the *edge irregularity strength*, $es(G)$, of G as the minimum k for which the graph G has an edge irregular k-labeling.

For a simple graph with maximum degree $\Delta(G)$ in [9] is estimated a lower bound of the edge irregularity strength in the form

$$es(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G) \right\}. \tag{1.1}$$

For several families of graphs, the exact value of the edge irregularity strength has been determined, namely for paths, stars, double stars and cartesian product of two paths in [9], for Toeplitz graphs in [10] and for complete m-ary trees in [11]. Other results on the edge irregularity strength and its variations can be found in [12–15].

The *modular irregular labeling* as a modification of the irregular assignment was introduced in [16]. A function $\psi : E(G) \to \{1, 2, \ldots, k\}$ of a graph G of order n is called a *modular irregular labeling* if the weight function $\lambda : V(G) \to \mathbb{Z}_n$ defined by $\lambda(u) = wt_\psi(u) = \sum \psi(uv)$ is bijective and is called as the *modular weight* of the vertex u, where $\mathbb{Z}_n$ is the group of integers modulo n and the sum is over all vertices v adjacent to u. The *modular irregularity strength*, $ms(G)$, is defined as the minimum k for which G has a modular irregular labeling using labels at most k. If there is no such labeling for the graph G then the value of $ms(G)$ is defined as ∞.

Clearly, every modular irregular labeling of a graph with no component of order at most two is also its irregular labeling. This gives a lower bound of the modular irregularity strength, i.e., if G is a graph with no component of order at most two then

$$s(G) \leq ms(G). \tag{1.2}$$

The exact values of the modular irregularity strength have been determined for certain families of graphs, namely for paths, cycles and stars [16], for fan graphs [17], wheels [18] and friendship graphs [19].

Motivated by the modular irregular labeling and the edge irregular labeling, in this paper we study modular edge irregular k-labelings.

For a graph $G = (V, E)$ of size m we define a vertex labeling $\varphi : V(G) \to \{1, 2, \ldots, k\}$ to be a *modular edge irregular k-labeling* if the edge-weight function $\rho : E(G) \to \mathbb{Z}_m$ defined by $\rho(uv) = wt_\varphi(uv) = \varphi(u) + \varphi(v)$ is bijective and is called as the *modular edge-weight* of the edge uv, where $\mathbb{Z}_m$ is the group of integers modulo m. The *modular edge irregularity strength*, $mes(G)$, is defined as the minimum k for which G has a modular edge irregular k-labeling. If there is no such labeling for the graph G, then the value of $mes(G)$ is defined as ∞.

Note that Muthugurupackiam and Ramya in [20, 21] introduced a definition on even (odd) modular edge irregular labeling, where the set of modular edge-weights contains only even or odd integers.

The main aim of the paper is to show some estimations on the modular edge irregularity strength, investigate the existence of modular edge irregular k-labelings for several families of graphs and determine the precise values of the modular edge irregularity strength that prove the sharpness of the presented lower bound.

2. Results

Directly from the definition it follows that every modular edge irregular k-labeling of a graph is also its edge irregular k-labeling. Thus, for any simple graph G holds

$$\mathrm{es}(G) \leq \mathrm{mes}(G). \tag{2.1}$$

In general, the converse of (2.1) does not hold. However, the validity of the following claim is obvious.

Theorem 2.1. *Let G be a simple graph with $\mathrm{es}(G) = k$. If edge-weights under the corresponding edge irregular k-labeling constitute a set of consecutive integers, then*

$$\mathrm{es}(G) = \mathrm{mes}(G) = k. \tag{2.2}$$

In [9], the precise value of the edge irregularity strength for paths and stars are determined as follows:

Theorem 2.2. *[9] Let P_n be a path on n vertices, $n \geq 2$. Then $\mathrm{es}(P_n) = \left\lceil \frac{n}{2} \right\rceil$.*

Theorem 2.3. *[9] Let $K_{1,n}$ be a star on $n + 1$ vertices, $n \geq 1$. Then $\mathrm{es}(K_{1,n}) = n$.*

The previous two theorems prove that the lower bound of the edge irregularity strength in (1.1) is tight. There is described the existence of the edge irregular $\left\lceil \frac{n}{2} \right\rceil$-labeling (for paths) and the existence of the edge irregular n-labeling (for stars), where the corresponding edge-weights in both cases constitute the set of consecutive integers. According to Theorem 2.1 we have:

Corollary 2.1. *Let P_n be a path on $n \geq 2$ vertices. Then $\mathrm{mes}(P_n) = \left\lceil \frac{n}{2} \right\rceil$.*

Corollary 2.2. *Let $K_{1,n}$ be a star on $n + 1$ vertices, $n \geq 1$. Then $\mathrm{mes}(K_{1,n}) = n$.*

These corollaries prove the tightness of the lower bound of the modular edge irregularity strength given in (2.1).

In the next theorem we characterize the modular edge irregularity strength of cycles.

Theorem 2.4. *Let C_n be a cycle on n vertices, $n \geq 3$. Then*

$$\mathrm{mes}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 1 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \equiv 0, 3 \pmod 4, \\ \infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. Faudree et al. in [22] described irregular assignments f for cycles and proved that

$$s(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 1 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

We define a vertex labeling ψ of C_n such that for $1 \leq i \leq n$

$$\psi(v_i) = f(v_i v_{i+1}).$$

Thus each edge label becomes the vertex label and we get an edge irregular labeling of the cycle. Since for every $n \not\equiv 2 \pmod 4$ in Faudree's irregular assignments of C_n the vertex-weights constitute a set of consecutive integers then according to Theorem 2.1 it implies that

$$\mathrm{mes}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 1 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \equiv 0, 3 \pmod 4. \end{cases}$$

For the remaining case when $n \equiv 2 \pmod 4$, i.e., $n = 4h + 2$ for some positive integer $h \geq 1$, let us suppose that the cycle C_{4h+2} admits a modular edge irregular labeling φ. It means that the sum of all vertex labels used to calculate the edge-weights of C_{4h+2} is congruent to the sum of modular edge-weights. Hence

$$2 \sum_{i=1}^{4h+2} \varphi(v_i) \equiv 0 + 1 + \cdots + (4h+1) = \tfrac{(4h+2)(4h+1)}{2} = (2h+1)(4h+2-1) \equiv (2h+1)(-1)$$
$$\equiv 2h + 1 \pmod{(4h+2)}.$$

A contradiction as $2h + 1$ is odd. □

A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be seen as a sequence of stars $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_r}$, where each K_{1,n_i} is a star with central vertex c_i and n_i leaves for $i = 1, 2, \ldots, r$, and the leaves of K_{1,n_i} include c_{i-1} and c_{i+1}, for $i = 2, 3, \ldots, r-1$. In [23] the authors denote the caterpillar as $S_{n_1,n_2,\ldots,n_r}$, where the vertex set is $V(S_{n_1,n_2,\ldots,n_r}) = \{c_i : 1 \leq i \leq r\} \cup \bigcup_{i=2}^{r-1} \{v_i^j : 2 \leq j \leq n_i - 1\} \cup \{v_1^j : 1 \leq j \leq n_1 - 1\} \cup \{v_r^j : 2 \leq j \leq n_r\}$, and the edge set is $E(S_{n_1,n_2,\ldots n_r}) = \{c_i c_{i+1} : 1 \leq i \leq r-1\} \cup \bigcup_{i=2}^{r-1} \{c_i v_i^j : 2 \leq j \leq n_i - 1\} \cup \{c_1 v_1^j : 1 \leq j \leq n_1 - 1\} \cup \{c_r v_r^j : 2 \leq j \leq n_r\}$, see Figure 1. Thus $|V(S_{n_1,n_2,\ldots,n_r})| = \sum_{i=1}^{r} n_i - r + 2$ and $|E(S_{n_1,n_2,\ldots,n_r})| = \sum_{i=1}^{r} n_i - r + 1$.

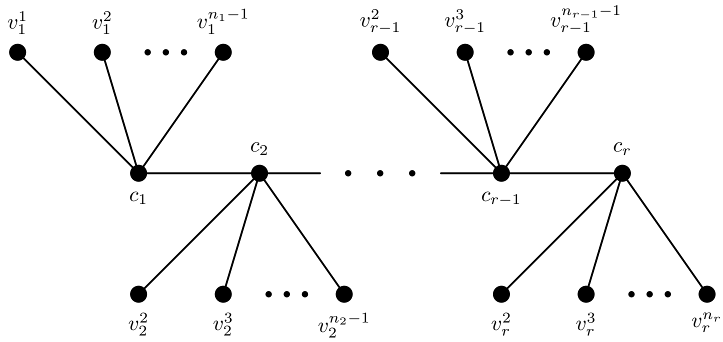

Figure 1. Caterpillar $S_{n_1,n_2,\ldots,n_r}$.

Let $S_{n_1,n_2,\ldots,n_r}$ be a caterpillar and $N_o = \sum_{i=1}^{\lceil \frac{r}{2} \rceil} n_{2i-1}$ and $N_e = \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} n_{2i}$.

Theorem 2.5. *Let $k = \max\left\{N_o - \left\lceil \frac{r}{2} \right\rceil + 1, N_e - \left\lfloor \frac{r}{2} \right\rfloor + 1\right\}$. The caterpillar $S_{n_1,n_2,\ldots,n_r}$ admits a modular edge irregular k-labeling.*

Proof. We are using an idea of Kotzig and Rosa [24] that any caterpillar can be realized in the plane so that its vertices are displaced in two rows, the edges joining these vertices from different rows and no two edges cross. Let $\{A, B\}$ be a bipartition of the vertex set of the caterpillar $S_{n_1,n_2,\dots,n_r}$. Let $a_1, a_2, \dots, a_{N_o-\lceil \frac{r}{2} \rceil+1}$ be the vertices in the partition A, ordered from left to right, and let $b_1, b_2, \dots, b_{N_e-\lfloor \frac{r}{2} \rfloor+1}$ be the vertices in the partition B, ordered from left to right.

Define a vertex labeling φ of $S_{n_1,n_2,\dots,n_r}$ in the following way.

$$\varphi(a_i) = i, \qquad \text{if } 1 \le i \le N_o - \left\lceil \frac{r}{2} \right\rceil + 1,$$
$$\varphi(b_j) = j, \qquad \text{if } 1 \le j \le N_e - \left\lfloor \frac{r}{2} \right\rfloor + 1.$$

It is not complicated to see that the maximal vertex label is $k = \max\left\{N_o - \left\lceil \frac{r}{2} \right\rceil + 1, N_e - \left\lfloor \frac{r}{2} \right\rfloor + 1\right\}$ and the edge-weights create the integer interval from 2 to $|E(S_{n_1,n_2,\dots,n_r})| + 1 = N_o + N_e - r + 2$. Thus, the vertex labeling φ is a modular edge irregular k-labeling. □

Immediately from (1.1) and Theorem 2.5 we obtain the next theorem.

Theorem 2.6. *Let $S_{n_1,n_2,\dots,n_r}$ be a caterpillar. Then*

$$\max\left\{\left\lceil \frac{N_o+N_e-r+2}{2} \right\rceil, n_i : 1 \le i \le r\right\} \le \text{es}(S_{n_1,n_2,\dots,n_r}) \le \text{mes}(S_{n_1,n_2,\dots,n_r}) \le \max\left\{N_o - \left\lceil \frac{r}{2} \right\rceil + 1, N_e - \left\lfloor \frac{r}{2} \right\rfloor + 1\right\}.$$

Note that if r is even and $N_o = N_e + \alpha$, where $\alpha \in \{-1, 0, 1\}$, or if r is odd and $N_o = N_e + \beta$, where $\beta \in \{0, 1, 2\}$ then

$$\max\left\{\left\lceil \frac{N_o+N_e-r+2}{2} \right\rceil, n_i : 1 \le i \le r\right\} = \max\left\{N_o - \left\lceil \frac{r}{2} \right\rceil + 1, N_e - \left\lfloor \frac{r}{2} \right\rfloor + 1\right\}.$$

Thus we get the following result.

Corollary 2.3. *Let $S_{n_1,n_2,\dots,n_r}$ be a caterpillar. If r is even and $N_o = N_e + \alpha$, where $\alpha \in \{-1, 0, 1\}$, or if r is odd and $N_o = N_e + \beta$, where $\beta \in \{0, 1, 2\}$ then*

$$\text{es}(S_{n_1,n_2,\dots,n_r}) = \text{mes}(S_{n_1,n_2,\dots,n_r}) = \left\lceil \frac{N_o+N_e-r+2}{2} \right\rceil.$$

In compliance with (2.1) the previous corollary proves the sharpness of the lower bound of the modular edge irregularity strength of caterpillars.

Marr and Wallis in their book [25] define an *n-sun* S_n as a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex.

Theorem 2.7. *Let S_n be an n-sun on $2n$ vertices, $n \ge 3$. Then*

$$\text{es}(S_n) = \text{mes}(S_n) = n + 1.$$

Proof. Let $V(S_n) = \{v_i, u_i : 1 \le i \le n\}$ and $E(S_n) = \{v_i v_{i+1}, v_i u_i : 1 \le i \le n\}$, where $v_{n+1} = v_1$. According to (1.1) and (2.1) we have the following lower bound $n + 1 \le \text{es}(S_n) \le \text{mes}(S_n)$. To prove that $n + 1$ is also the upper bound we distinguish two cases according to the parity of n.

Case 1. For $n \ge 3$ odd, we construct a vertex labeling φ as follows

$$\varphi(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd}, \\ \frac{n+1+i}{2}, & \text{if } i \text{ is even}, \end{cases}$$

$$\varphi(u_i) = \begin{cases} 1, & \text{if } i \text{ is odd, } i \neq n, \\ n+1, & \text{if } i \text{ is even and } i = n. \end{cases}$$

The labels of vertices receive the integers from 1 to $n+1$ and for the weights of edges we get

$$wt_\varphi(v_i v_{i+1}) = \begin{cases} \frac{n+3}{2} + i, & \text{if } 1 \leq i \leq n-1, \\ \frac{n+3}{2}, & \text{if } i = n, \end{cases}$$

$$wt_\varphi(v_i u_i) = \begin{cases} \frac{i+3}{2}, & \text{if } i \text{ odd, } i \neq n, \\ \frac{3n+3}{2}, & \text{if } i = n, \\ \frac{3n+3+i}{2}, & \text{if } i \text{ even.} \end{cases}$$

One can easily check that under the vertex labeling φ the edges of the n-sun admit the consecutive weights from 2 to $2n+1$. Thus, the vertex labeling φ is a modular edge irregular $(n+1)$-labeling of S_n for n odd.

Case 2. For $n \geq 4$ even, we consider a vertex labeling ψ defined such that

$$\psi(v_i) = \begin{cases} i, & \text{if } 1 \leq i \leq n-1, \\ n+1, & \text{if } i = n, \end{cases}$$

$$\psi(u_i) = \begin{cases} i, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ i+2, & \text{if } \frac{n}{2}+1 \leq i \leq n-3, \\ n, & \text{if } i = n-2, n-1, n. \end{cases}$$

The maximal vertex labels is $n+1$ and the edge-weights are the following

$$wt_\psi(v_i v_{i+1}) = \begin{cases} 2i+1, & \text{if } 1 \leq i \leq n-2, \\ 2n, & \text{if } i = n-1, \\ n+2, & \text{if } i = n, \end{cases}$$

$$wt_\psi(v_i u_i) = \begin{cases} 2i, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ 2i+2, & \text{if } \frac{n}{2}+1 \leq i \leq n-3, \\ i+n, & \text{if } i = n-2, n-1, \\ 2n+1, & \text{if } i = n. \end{cases}$$

Thus the weights of edges are consecutive numbers from 2 to $2n+1$. This means that the vertex labeling ψ is a modular edge irregular $(n+1)$-labeling of S_n for n even.

Thus

$$\text{mes}(S_n) = n+1$$

for every $n \geq 3$. □

A friendship graph f_n, $n \geq 1$, is a set of n triangles having a common central vertex, and otherwise disjoint. Let w denote the central vertex. For the ith triangle, $1 \leq i \leq n$, let u_i and v_i denote the other two vertices. Thus f_n contains $2n+1$ vertices w, u_i, v_i, $1 \leq i \leq n$ and $3n$ edges wu_i, wv_i, u_iv_i, $1 \leq i \leq n$.

As $|E(f_n)| = 3n$ and the maximum degree $\Delta(f_n) = 2n$ then (1.1) implies that $\text{es}(f_n) \geq 2n$. However, if under a vertex labeling φ of f_n there exist two vertices $u, v \in V(f_n)$ such that $\varphi(u) = \varphi(v)$ then using the fact that u and v have just one common neighbor, say $z \in V(f_n)$, we obtain $wt_\varphi(uz) = \varphi(u) + \varphi(z) = \varphi(v) + \varphi(z) = wt_\varphi(vz)$. It means that under any edge irregular labeling of the friendship graph f_n all the vertex values must be different. That way

$$\text{es}(f_n) \geq 2n + 1 \tag{2.3}$$

and according to (2.1) we also have a lower bound of the modular edge irregularity strength for f_n.

The following theorem shows that the lower bound of the edge irregularity strength of f_n in (2.3) is acquired just for a few values of n. A similar idea of the proving was used in [26] for showing the edge-antimagicness of friendship graphs with difference $d = 1$.

Theorem 2.8. *The friendship graph f_n of order $2n + 1$ admits an edge irregular $(2n + 1)$-labeling with consecutive edge-weights if and only if $n \in \{1, 3, 4, 5, 7\}$.*

Proof. Suppose that there exists a vertex labeling $\varphi : V(G) \to \{1, 2, \dots, 2n + 1\}$ such that the edge-weights of f_n successively attain consecutive values $x, x+1, \dots, x+3n-1$. If $\varphi(w) = t$, $1 \leq t \leq 2n+1$, then the set of vertex labels under the vertex labeling φ can be partitioned into three subsets $A = \{1, 2, \dots, t - 1\}$, $B = \{t\}$ and $C = \{t + 1, t + 2, \dots, 2n + 1\}$. Thus $\varphi(V(f_n)) = A \cup B \cup C$.

We are able to see that weights of edges wu_i and wv_i, $1 \leq i \leq n$, constitute the set $W = \{t + 1, t + 2, \dots, 2t - 1, 2t + 1, 2t + 2, \dots, 2n + t + 1\}$. It is not difficult to see that the set of edge-weights W_A=$\{x, x + 1, \dots, t\}$ can only be created as sums of two distinct values in the set A and the set of edge-weights $W_B = \{2n + t + 2, 2n + t + 3, \dots, x + 3n - 1\}$ can only be created as sums of two distinct values in the set B. The sets W_A and W_B contain consecutive integers each while the set W has a gap. The missing edge-weight $2t$ can be obtained only as sum of a value from the set A, say a, and a value from the set B, say b. Thus in the set $A - \{a\}$ we have $t - 2$ numbers and the corresponding set of edge-weights $W_A = \{x, x + 1, \dots, t\}$ has the cardinality $|W_A| = \frac{t-2}{2}$, i.e., t must be even. This also implies that $x = \frac{t}{2} + 2$.

Since the sum of all the values in the set $A - \{a\}$ is equal to the sum of all the edge-weights in the set $W_A = \{\frac{t}{2} + 2, \frac{t}{2} + 3, \dots, t\}$, then

$$\frac{t(t-1)}{2} - a = \frac{(3t+4)(t-2)}{8}. \tag{2.4}$$

As $a \in A$ then $1 \leq a \leq t - 1$ and from (2.4) we get

$$8 \leq t^2 - 2t + 8 \leq 8t - 8,$$

which is equivalent to

$$t^2 - 2t \geq 0 \quad \text{and} \quad t^2 - 10t + 16 \leq 0.$$

As t must be even the previous inequalities give

$$t \in \{2, 4, 6, 8\}. \tag{2.5}$$

In the computation of the edge-weights of f_n, the label t of the vertex w is used $2n$ times and the labels of the vertices u_i and v_i, $1 \leq i \leq n$, are used twice each and the sum of the all edge-weights of f_n is equal to the sum of all the vertex labels, used to calculate the edge-weights. Then we get the following equation

$$2\sum_{i=1}^{n}(\varphi(u_i)+\varphi(v_i))+2n\varphi(w)=\sum_{i=1}^{n}\Big(wt_\varphi(wu_i)+wt_\varphi(wv_i)\Big)+\sum_{i=1}^{n}wt_\varphi(u_iv_i),$$

which gives

$$(2n+2)(2n+1)+2t(n-1)=\frac{3n(3n+t+3)}{2},$$

and it immediately follows that

$$0=n^2-n(3+t)+4(t-1). \tag{2.6}$$

According to (2.5) from the Eq (2.6) we receive all the possible integer values of the parameters n, x and t as follows:

$$(n,x,t)\in\{(1,3,2),(3,4,4),(4,3,2),(4,4,4),(4,5,6),(4,6,8),(5,5,6),(7,6,8)\}. \tag{2.7}$$

For the converse, it is not difficult to find the corresponding edge irregular $(2n+1)$-labelings of f_n for parameters (n,x,t) from (2.7). Figures 2–4 illustrate searched $(2n+1)$-labelings of f_n, where integers in italic font represent the edge-weights. This concludes the proof. □

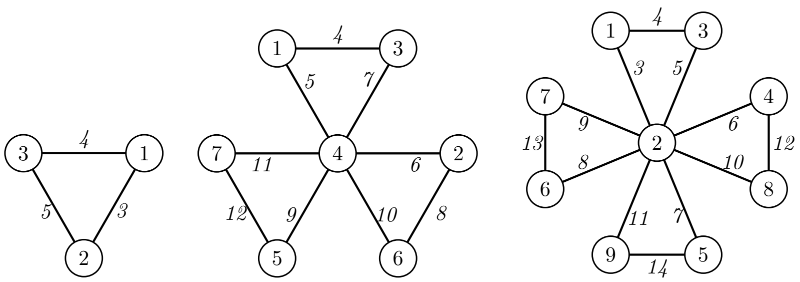

Figure 2. An edge irregular 3-labeling of f_1 for $(n,x,t)=(1,3,2)$, an edge irregular 7-labeling of f_3 for $(n,x,t)=(3,4,4)$ and an edge irregular 9-labeling of f_4 for $(n,x,t)=(4,3,2)$.

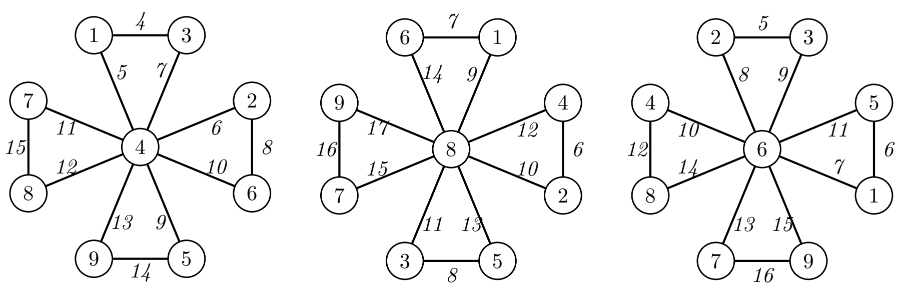

Figure 3. The edge irregular 9-labelings of f_4 for $(n,x,t)=(4,4,4)$, $(n,x,t)=(4,6,8)$ and $(n,x,t)=(4,5,6)$.

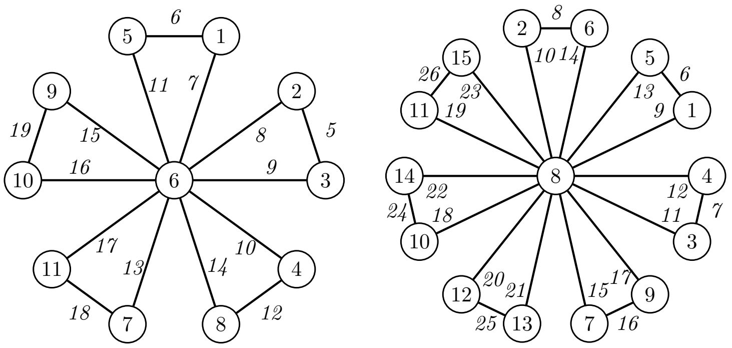

Figure 4. An edge irregular 11-labeling of f_5 for $(n, x, t) = (5, 5, 6)$ and an edge irregular 15-labeling of f_7 for $(n, x, t) = (7, 6, 8)$.

Applying Theorem 2.1 to Theorem 2.8 we achieve the following corollary.

Corollary 2.4. *Let f_n be a friendship graph on $2n+1$ vertices. If $n \in \{1, 3, 4, 5, 7\}$ then* $\mathrm{mes}(f_n) = 2n+1$.

The next theorem gives a condition when no modular edge irregular labeling of f_n exists.

Theorem 2.9. *If f_n is a friendship graph on $2n+1$ vertices and $n \equiv 2 \pmod 4$, then f_n has no modular edge irregular labeling and* $\mathrm{mes}(f_n) = \infty$.

Proof. Assume that the friendship graph f_n on $2n+1$ vertices admits a modular edge irregular labeling φ. Then the sum of all vertex labels used to calculate the edge-weights of f_n is congruent to the sum of modular edge-weights. It means if $D = 2\sum_{i=1}^{n}(\varphi(u_i) + \varphi(v_i)) + 2n\varphi(w)$ then

$$D \equiv \sum_{s=0}^{3n-1} s \pmod{3n},$$

where $\sum_{s=0}^{3n-1} s = \frac{3n(3n-1)}{2}$.

If $n \equiv 2 \pmod 4$, i.e., $n = 4h + 2$ for some positive integer $h \geq 1$, then using properties of congruence we get

$$D \equiv \frac{(12h+6)(12h+5)}{2} = (6h+3)(12h+6-1) \equiv (6h+3)(-1) \equiv 6h+3 \pmod{(12h+6)}.$$

This contradicts the fact that D is even. □

The next theorem provides lower and upper bounds on the parameter $\mathrm{mes}(f_n)$ for n odd.

Theorem 2.10. *For the friendship graph f_n of order $2n+1$, $n \geq 9$ odd, we have*

$$2n + 1 \leq \mathrm{mes}(f_n) \leq \tfrac{5n+1}{2}.$$

Proof. The lower bound follows from (2.1) and (2.3). To see the upper bound let us define a vertex labeling φ of f_n, for $n \geq 9$ odd, in the following way

$$\varphi(u_i) = i, \qquad \text{if } 1 \leq i \leq n,$$

$$\varphi(v_i) = \begin{cases} \frac{3n+2-i}{2}, & \text{if } i \text{ is odd,} \\ \frac{4n+2-i}{2}, & \text{if } i \text{ is even,} \end{cases}$$
$$\varphi(w) = \frac{5n+1}{2}.$$

Thus the labels of vertices u_i, v_i, $1 \le i \le n$ are the consecutive integers from 1 to $2n$ and the vertex w receives the maximum label $\frac{5n+1}{2}$. Then for the weights of the edges u_iv_i, $1 \le i \le n$, we have

$$wt_\varphi(u_iv_i) = \begin{cases} \frac{3n+2+i}{2}, & \text{if } i \text{ is odd,} \\ \frac{4n+2+i}{2}, & \text{if } i \text{ is even,} \end{cases}$$

and for the weights of the edges wu_i and wv_i, $1 \le i \le n$, we get

$$wt_\varphi(wu_i) = \frac{5n+1}{2} + i,$$
$$wt_\varphi(wv_i) = \begin{cases} \frac{8n+3-i}{2}, & \text{if } i \text{ is odd,} \\ \frac{9n+3-i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

We can observe that the edge-weights of f_n, under the vertex labeling φ, form the sequence of consecutive integers $\frac{3n+3}{2}, \frac{3n+5}{2}, \dots, \frac{9n+1}{2}$.

In the light of Theorem 2.1, it follows that the vertex labeling φ is a modular edge irregular $\frac{5n+1}{2}$-labeling of f_n and it proves that $\text{mes}(f_n) \le \frac{5n+1}{2}$ for $n \ge 9$ odd. Thus, we arrive at the desired result.

□

3. Conclusions

In this paper is introduced a new graph invariant, namely the modular edge irregularity strength and estimated its lower bound. For several families of graphs (paths, stars, cycles and n-suns) are determined the precise values of the modular edge irregularity strength that prove the sharpness of this lower bound. For caterpillars $S_{n_1,n_2,\dots,n_r}$ realized in the plane as a balanced (respectively almost balanced) bipartite graph we proved that if r is even and $N_o = N_e + \alpha$, where $\alpha \in \{-1, 0, 1\}$, or if r is odd and $N_o = N_e + \beta$, where $\beta \in \{0, 1, 2\}$ then $\text{mes}(S_{n_1,n_2,\dots,n_r}) = \left\lceil \frac{N_o+N_e-r+2}{2} \right\rceil$. For the other cases we proved only an upper bound for the modular edge irregularity strength. Therefore we propose the following open problem.

Open Problem 3.1. *For the caterpillar $S_{n_1,n_2,\dots,n_r}$ determine the exact value of the modular edge irregularity strength for $N_o = N_e + \gamma$ for every integer γ.*

For the friendship graph f_n of order $2n + 1$ was proved that

$$\text{mes}(f_n) \begin{cases} = 2n + 1, & \text{if } n \in \{1, 3, 4, 5, 7\}, \\ = \infty, & \text{if } n \equiv 2 \pmod 4, \\ \le \frac{5n+1}{2}, & \text{if } n \ge 9 \text{ odd.} \end{cases}$$

For further research, we suggest the following open problems.

Open Problem 3.2. *For the friendship graph f_n of order $2n + 1$ and $n \geq 9$ odd, determine the exact value of the modular edge irregularity strength.*

It is a matter of algebraic argumentation to show that there does not exist a modular edge irregular 17-labeling of f_8. Thus according to Figure 5 we get that $\text{mes}(f_8) = 18$. The remaining open case for the existence of a modular edge irregular labeling of f_n is $n \equiv 0 \pmod 4$ for $n \geq 12$. Therefore we propose

Open Problem 3.3. *For the friendship graph f_n of order $2n+1$, $n \geq 12$ and $n \equiv 0 \pmod 4$, determine the exact value of the modular edge irregularity strength.*

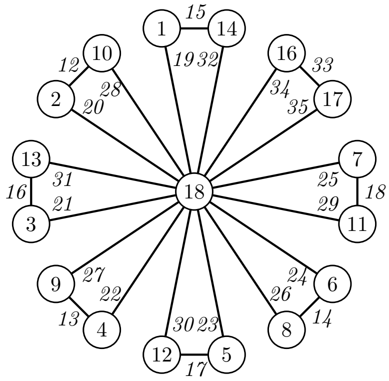

Figure 5. An edge irregular 18-labeling of f_8.

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Conflict of interest

In this article, all authors disclaim any conflict of interest.

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