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Research article

On the Ulam-Hyers-Rassias stability of two structures of discrete fractional three-point boundary value problems: Existence theory

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Abstract: We prove existence and uniqueness of solutions to discrete fractional equations that involve Riemann-Liouville and Caputo fractional derivatives with three-point boundary conditions. The results are obtained by conducting an analysis via the Banach principle and the Brouwer fixed point criterion. Moreover, we prove stability, including Hyers-Ulam and Hyers-Ulam-Rassias type results. Finally, some numerical models are provided to illustrate and validate the theoretical results.

Keywords: discrete fractional operators; stability; existence results; Banach principle
Mathematics Subject Classification: 26A33, 34A08, 34A12

1. Introduction

Fractional difference calculus is a tool used to explain many phenomena in physics, control problems, modeling, chaotic dynamical systems, and various fields of engineering and applied mathematics. In this direction, different kinds of methods and techniques, including numerical and analytical methods, have been utilized by researchers to discuss given fractional discrete and continuous mathematical models and boundary value problems (BVPs) [1–4]. For some recent developments on the existence, uniqueness, and stability of solutions for fractional differential equations, see, for example, [5–23] and the references therein.

Discrete fractional calculus and difference equations open a new study context for mathematicians. For this reason, they have received increasing attention in recent years. Some real-world processes and phenomena are analyzed with the aid of discrete fractional operators, since such operators provide an accurate tool to describe memory. A large number of research articles dealing with difference equations and discrete fractional boundary value problems (FBVPs) can be found in [24–32].

In 2020, Selvam et al. [33] proved the existence of a solution to a discrete fractional difference equation formulated as

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 1 < \varrho \leq 2, \\ \Delta\chi(\varrho-2) = M_1, \qquad\qquad \chi(\varrho+T) = M_2, \end{cases} \tag{1.1}$$

for $\xi \in [0,T]_{\mathbb{N}_0} = [0,1,2,\ldots,T]$, $T \in \mathbb{N}$, $\eta \in [\varrho-1, T+\varrho-1]_{\mathbb{N}_{\varrho-1}}$, M_1 and M_2 constants, $\Phi : [\varrho-2, \varrho+T]_{\mathbb{N}_{\varrho-2}} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuous, and where ${}^{c}\Delta_{\xi}^{\varrho}$ denotes the ϱth-Caputo difference. Here, motivated by the discrete model (1.1), we shall consider two generalized discrete problems.

Our first goal consists to study existence and uniqueness of solutions to the following discrete fractional equation that involves Caputo discrete derivatives:

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 2 < \varrho \leq 3, \\ \Delta\chi(\varrho-3) = A_1, \chi(\varrho+T) = \lambda\Delta^{-\beta}\chi(\eta+\beta), \Delta^2\chi(\varrho-3) = A_2, \end{cases} \tag{1.2}$$

for $0 < \beta \leq 1$, $\xi \in [0,T]_{\mathbb{N}_0} = [0,1,2,\ldots,T]$, $T \in \mathbb{N}$, $\eta \in [\varrho-1, T+\varrho-1]_{\mathbb{N}_{\varrho-1}}$, λ, A_1 and A_2 constants, and where $\Phi : [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

The second goal is to study the stability of solutions to the discrete Riemann-Liouville fractional problem

$$\begin{cases} {}^{RL}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 2 < \varrho \leq 3, \\ \Delta\chi(\varrho-3) = A_1, \chi(\varrho+T) = \lambda\Delta^{-\beta}\chi(\eta+\beta), \Delta^2\chi(\varrho-3) = A_2, \end{cases} \tag{1.3}$$

for $0 < \beta \leq 1$, $\xi \in [0,T]_{\mathbb{N}_0} = [0,1,2,\ldots,T]$ and $\eta \in [\varrho-1, T+\varrho-1]_{\mathbb{N}_{\varrho-1}}$, where ${}^{RL}\Delta_{\xi}^{\varrho}$ is the Riemann-Liouville difference operator.

The organization of the paper is as follows. In Section 2, we collect some fundamental definitions available from the literature. In Section 3, we prove the existence and uniqueness results for the discrete FBVP (1.2). Hyers-Ulam and Hyers-Ulam-Rassias stability of the solution for the FBVP (1.3) is established in Section 4. In Section 5, two examples are given to illustrate the obtained results. We end with Section 6 of conclusions.

2. Preliminaries

We begin by recalling some necessary definitions and essential lemmas that will be used throughout the paper.

Definition 2.1. *(See [27]) Let $\varrho > 0$. The ϱ-order fractional sum of Φ is defined by*

$$\Delta_{\xi}^{-\varrho}\Phi(\xi) = \frac{1}{\Gamma(\varrho)}\sum_{l=a}^{\xi-\varrho}(\xi-l-1)^{(\varrho-1)}\Phi(l), \tag{2.1}$$

where $\xi \in \mathbb{N}_{a+\varrho} := \{a+\varrho, a+\varrho+1, \ldots\}$ *and* $\xi^{(\varrho)} := \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\varrho)}$.

Definition 2.2. *(See [27]) Let* $\varrho > 0$ *and* Φ *be defined on* $\mathbb{N}_a$*. The* ϱ*-order Caputo fractional difference of* Φ *is defined by*

$$ {}_a^C\Delta_\xi^\varrho \Phi(\xi) = \Delta^{-(n-\varrho)}(\Delta^n \Phi(\xi)) = \frac{1}{\Gamma(n-\varrho)} \sum_{l=a}^{\xi-(n-\varrho)} (\xi - l - 1)^{(n-\varrho-1)} \Delta^n \Phi(l), \tag{2.2} $$

while the Riemann-Liouville fractional difference of Φ *is defined by*

$$ {}_a^{RL}\Delta_\xi^\varrho \Phi(\xi) = \Delta^n \Delta^{-(n-\varrho)} \Phi(\xi), \tag{2.3} $$

where $\xi \in \mathbb{N}_{a+n-\varrho}$ *and* $n-1 < \varrho \le n$.

Lemma 2.1. *(See [24, 27]) For* $\varrho > 0$,

$$ \Delta^{-\varrho} {}_a^C\Delta_\xi^\varrho \Phi(\xi) = \Phi(\xi) + C_0 + C_1\xi + \cdots + C_{N-1}\xi^{(N-1)}, \tag{2.4} $$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N-1$, Φ *is defined on* $\mathbb{N}_a$, *and* $0 \le N-1 < \varrho \le N$.

Lemma 2.2. *(See ([32]) Let* $0 \le N-1 < \varrho \le N$ *and* Φ *be defined on* $\mathbb{N}_a$*. Then,*

$$ \Delta^{-\varrho} {}_0^{RL}\Delta_\xi^\varrho \Phi(\xi) = \Phi(\xi) + B_1\xi^{\varrho-1} + B_2\xi^{(\varrho-2)} + \cdots + B_N\xi^{(\varrho-N)}, \tag{2.5} $$

for $B_1, \ldots, B_N \in \mathbb{R}$.

Lemma 2.3. *(See [31]) Let* ϱ *and* ξ *be any arbitrary real numbers. Then,*

(1) $\displaystyle\sum_{l=0}^{\xi-\varrho} (\xi - l - 1)^{(\varrho-1)} = \frac{\Gamma(\xi+1)}{\varrho\Gamma(\xi-\varrho+1)}$,

(2) $\displaystyle\sum_{l=0}^{L} (\xi - L - l - 1)^{(\varrho-1)} = \frac{\Gamma(\varrho+L+1)}{\varrho\Gamma(L+1)}$.

Lemma 2.4. *(See [31]) For* $\zeta \in \mathbb{R} \setminus (\mathbb{Z}^- \setminus \{0\})$, *we have*

$$ \Delta^{-\varrho}\xi^{(\zeta)} = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+\varrho+1)}\xi^{(\zeta+\varrho)}. $$

3. Existence and uniqueness of solutions

In this section, we prove the existence and uniqueness of solution for the Caputo three-point discrete fractional problem (1.2). To accomplish this, we denote by $C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ the collection of all continuous functions χ with the norm

$$ \|\chi\| = \max\{|\chi(\xi)| : \xi \in \mathbb{N}_{\varrho-3,\varrho+T}\}. $$

Lemma 3.1. *Let* $2 < \varrho \leq 3$ *and* $\Phi : [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}} \longrightarrow \mathbb{R}$. *A function* $\chi(\xi)$ $(\xi \in [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}})$ *that satisfies the discrete FBVP*

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi + \varrho - 1), \quad 2 < \varrho \leq 3, \\ \Delta\chi(\varrho - 3) = A_1, \chi(\varrho + T) = \lambda\Delta^{-\beta}\chi(\eta + \beta), \Delta^2\chi(\varrho - 3) = A_2, \quad 0 < \beta \leq 1, \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} \chi(\xi) = & \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) \\ & - \frac{\lambda}{K_1\Gamma(\varrho)} \sum_{l=\varrho}^{\eta} \sum_{\xi=0}^{l-\varrho} (\eta + \beta - \rho(l))^{(\beta-1)} (l - \rho(\xi))^{(\varrho-1)} \Phi(\xi + \varrho - 1) \\ & + \frac{\Gamma(\beta)}{K_1\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho + T - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) + K_2 \\ & + [A_1 - A_2(\varrho - 3)]\xi^{(1)} + \frac{A_2}{2}\xi^{(2)}, \end{aligned} \tag{3.2}$$

with

$$K_1 = \lambda \sum_{l=\varrho-3}^{\eta} (\eta + \beta - \rho(l))^{(\beta-1)} - \Gamma(\beta), \tag{3.3}$$

and

$$\begin{aligned} K_2 = & \frac{\lambda}{K_1}[A_2(\varrho - 3) - A_1] \sum_{l=\varrho-3}^{\eta} (\eta + \beta - \rho(l))^{(\beta-1)} l^{(1)} - \frac{A_2}{2K_1}\lambda \sum_{l=\varrho-3}^{\eta} (\eta + \beta - \rho(l))^{(\beta-1)} l^{(2)} \\ & + \frac{\Gamma(\beta)}{K_1}(\varrho + T)[A_1 - A_2(\varrho - 3) + \frac{A_2}{2}(\varrho + T - 1)]. \end{aligned} \tag{3.4}$$

Proof. Let $\chi(\xi)$ be a solution to (3.1). Applying Lemma 2.1 and Definition 2.1, we find that

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) + C_0 + C_1\xi^{(1)} + C_2\xi^{(2)}, \tag{3.5}$$

for $\xi \in [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}}$, where $C_0, C_1, C_2 \in \mathbb{R}$. By using the difference of order 1 for (3.5), we have

$$\Delta\chi(\xi) = \frac{1}{\Gamma(\varrho - 1)} \sum_{l=0}^{\xi-\varrho+1} (\xi - \rho(l))^{(\varrho-2)} \Phi(l + \varrho - 1) + C_1 + 2C_2\xi^{(1)},$$

and

$$\Delta^2\chi(\xi) = \frac{1}{\Gamma(\varrho - 2)} \sum_{l=0}^{\xi-\varrho+2} (\xi - \rho(l))^{(\varrho-3)} \Phi(l + \varrho - 1) + 2C_2.$$

Now, from conditions $\Delta\chi(\varrho-3)=A_1$ and $\Delta^2\chi(\varrho-3)=A_2$, we obtain that

$$C_1 = A_1 - A_2(\varrho-3),$$
$$C_2 = \frac{A_2}{2}.$$

Therefore,

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + C_0 + [A_1 - A_2(\varrho-3)]\xi^{(1)} + \frac{A_2}{2}\xi^{(2)}, \tag{3.6}$$

for $\xi \in [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}}$. By using formula (3.5), one has

$$\begin{aligned}\Delta^{-\beta}\chi(\xi) &= \frac{C_1}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\xi-\beta}(\xi-\rho(l))^{(\beta-1)}l^{(1)} + \frac{C_2}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\xi-\beta}(\xi-\rho(l))^{(\beta-1)}l^{(2)} \\ &+ \frac{C_0}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\xi-\beta}(\xi-\rho(l))^{(\beta-1)} \\ &+ \frac{1}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\xi-\beta}\sum_{\xi=0}^{l-\varrho}(\xi-\rho(l))^{(\beta-1)}(l-\rho(\xi))^{(\varrho-1)}\Phi(\xi+\varrho-1). \end{aligned} \tag{3.7}$$

The other condition of (3.1) gives

$$\begin{aligned}\lambda\Delta^{-\beta}\chi(\eta+\beta) &= \frac{\lambda[A_1 - A_2(\varrho-3)]}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(1)} \\ &+ \frac{\lambda A_2}{2\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(2)} + \frac{\lambda C_0}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)} \\ &+ \frac{\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\eta}\sum_{\xi=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\xi))^{(\varrho-1)}\Phi(\xi+\varrho-1) \\ &= \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + C_0 \\ &+ [A_1 - A_2(\varrho-3)](\varrho+T)^{(1)} + \frac{A_2}{2}(\varrho+T)^{(2)}. \end{aligned}$$

We have

$$\begin{aligned}C_0 &= \frac{\Gamma(\beta)}{K_1\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + K_2 \\ &- \frac{\lambda}{K_1\Gamma(\varrho)}\sum_{l=\varrho}^{\eta}\sum_{\xi=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\xi))^{(\varrho-1)}\Phi(\xi+\varrho-1), \end{aligned}$$

where K_1 and K_2 are defined by (3.3) and (3.4), and one obtains (3.2) by substituting the value of C_0 into (3.6).

Now, let us consider the operator $H : C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R}) \to C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ defined by

$$\begin{aligned}(H\chi)(\xi) = & \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l+\varrho-1, \chi(l+\varrho-1)) \\ & - \frac{\lambda}{K_1\Gamma(\varrho)} \sum_{l=\varrho}^{\eta} \sum_{\tau=0}^{l-\varrho} (\eta+\beta-\rho(l))^{(\beta-1)} (l-\rho(\tau))^{(\varrho-1)} \Phi(l+\varrho-1, \chi(l+\varrho-1)) \\ & + \frac{\Gamma(\beta)}{K_1\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho+T-\rho(l))^{(\varrho-1)} \Phi(l+\varrho-1, \chi(l+\varrho-1)) \\ & + K_2 + [A_1 - A_2(\varrho-3)]\xi^{(1)} + \frac{A_2}{2}\xi^{(2)}.\end{aligned}$$

Theorem 3.1. *Assume that:*
($\mathcal{H}1$) *Function Φ satisfies $|\Phi(\xi,\chi_1) - \Phi(\xi,\chi_2)| \le K|\chi_1 - \chi_2|$, where $K > 0$, $\forall \xi \in \mathbb{N}_{\varrho-3,\varrho+T}$ and $\chi_1,\chi_2 \in C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$. The discrete FBVP (3.1) has a unique solution on $C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ provided*

$$\frac{\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)} \left(1 + \frac{\Gamma(\beta)}{K_1}\right) + \frac{M\lambda}{K_1\Gamma(\varrho)} \le \frac{1}{K} \tag{3.8}$$

with

$$M = \left| \sum_{l=\varrho}^{\eta} \sum_{\tau=0}^{l-\varrho} (\eta+\beta-\rho(l))^{(\beta-1)} (l-\rho(\tau))^{(\varrho-1)} \right|. \tag{3.9}$$

Proof. Let $\chi_1, \chi_2 \in C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$. Then, for each $\xi \in \mathbb{N}_{\varrho-3,\varrho+T}$, we have

$$\begin{aligned}\left|(H\chi_1)(\xi) - (H\chi_2)(\xi)\right| \le & \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi-\rho(l))^{(\varrho-1)} \\ & \times \left|\Phi(l+\varrho-1, \chi_1(l+\varrho-1)) - \Phi(l+\varrho-1, \chi_2(l+\varrho-1))\right| \\ & + \frac{\lambda}{K_1\Gamma(\varrho)} \sum_{l=\varrho}^{\eta} \sum_{\tau=0}^{l-\varrho} (\eta+\beta-\rho(l))^{(\beta-1)} (l-\rho(\tau))^{(\varrho-1)} \\ & \times \left|\Phi(l+\varrho-1, \chi_1(l+\varrho-1)) - \Phi(l+\varrho-1, \chi_2(l+\varrho-1))\right| \\ & + \frac{\Gamma(\beta)}{K_1\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho+T-\rho(l))^{(\varrho-1)} \\ & \times \left|\Phi(l+\varrho-1, \chi_1(l+\varrho-1)) - \Phi(l+\varrho-1, \chi_2(l+\varrho-1))\right|.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|(H\chi_1)(\xi) - (H\chi_2)(\xi)\right\| \le & \frac{K\|\chi_1-\chi_2\|}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi-\rho(l))^{(\varrho-1)} \\ & + \frac{\lambda K\|\chi_1-\chi_2\|}{K_1\Gamma(\varrho)} \sum_{l=\varrho}^{\eta} \sum_{\tau=0}^{l-\varrho} (\eta+\beta-\rho(l))^{(\beta-1)} (l-\rho(\tau))^{(\varrho-1)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(\beta)K\|\chi_1-\chi_2\|}{K_1\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\\
&\leq \frac{K\|\chi_1-\chi_2\|}{\Gamma(\varrho)}\frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)}\\
&+\frac{\lambda K\|\chi_1-\chi_2\|}{K_1\Gamma(\varrho)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}\\
&+\frac{\Gamma(\beta)K\|\chi_1-\chi_2\|}{K_1\Gamma(\varrho)}\frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)}\\
&\leq K\|\chi_1-\chi_2\|\Bigg[\frac{\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}+\frac{M\lambda}{K_1\Gamma(\varrho)}\\
&+\frac{\Gamma(\beta)\Gamma(\varrho+T+1)}{K_1\Gamma(\varrho+1)\Gamma(T+1)}\Bigg].
\end{aligned}
$$

From (3.8), we conclude that H is a contraction. Then, by the Banach contraction principle, the discrete problem (3.1) has a unique solution on $C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$.

Theorem 3.2. *Suppose that* $\Phi:[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}\times\mathbb{R}\longrightarrow\mathbb{R}$ *is a continuous function and*

$$R=\max\{|\Phi(l+\varrho-1,\chi(l+\varrho-1))|,\ \xi\in\mathbb{N}_{\varrho-3,\varrho+T},\chi\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R});\|\chi\|\leq 2|K_2|\}.$$

The discrete problem (3.1) has a solution provided that

$$R\leq\frac{|K_2|-|[A_1-A_2(\varrho-3)]|\,(\varrho+T)^{(1)}-\left|\frac{A_2}{2}\right|(\varrho+T)^{(2)}}{\frac{\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}\left(1+\frac{\Gamma(\beta)}{|K_1|}\right)+\frac{M|\lambda|}{\Gamma(\varrho)|K_1|}}.\tag{3.10}$$

Proof. Let $G=\{\chi\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R});\|\chi\|\leq 2|K_2|\}$. For $\chi(\xi)\in G$, we get

$$
\begin{aligned}
\left|(H\chi)(\xi)\right|=&\Bigg|\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1))\\
&+\frac{\lambda}{K_1\Gamma(\varrho)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1))\\
&-\frac{\Gamma(\beta)}{K_1\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1))\\
&+K_2+[A_1-A_2(\varrho-3)]\xi^{(1)}+\frac{A_2}{2}\xi^{(2)}\Bigg|\\
&\leq\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\left|\Phi(l+\varrho-1,\chi(l+\varrho-1))\right|\\
&+\frac{|\lambda|}{|K_1|\Gamma(\varrho)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}\left|\Phi(l+\varrho-1,\chi(l+\varrho-1))\right|
\end{aligned}
$$

$$+\frac{\Gamma(\beta)}{|K_1|\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Big|\Phi(l+\varrho-1,\chi(l+\varrho-1))\Big|$$
$$+|K_2|+\Big|[A_1-A_2(\varrho-3)]\xi^{(1)}\Big|+\Big|\frac{A_2}{2}\xi^{(2)}\Big|$$
$$\le\frac{R}{\Gamma(\varrho)}\Big[\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}+\frac{|\lambda|}{|K_1|}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}$$
$$+\frac{\Gamma(\beta)}{|K_1|}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Big]$$
$$+|K_2|+\Big|[A_1-A_2(\varrho-3)]\Big|(\varrho+T)^{(1)}+\Big|\frac{A_2}{2}\Big|(\varrho+T)^{(2)}$$
$$\le\frac{R}{\Gamma(\varrho)}\Big[\frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)}\Big(1+\frac{\Gamma(\beta)}{|K_1|}\Big)+\frac{M|\lambda|}{|K_1|}\Big]$$
$$+|K_2|+\Big|[A_1-A_2(\varrho-3)]\Big|(\varrho+T)^{(1)}+\Big|\frac{A_2}{2}\Big|(\varrho+T)^{(2)}.$$

From (3.10), we have $\|H\chi\|\le 2|K_2|$, which implies that $H:G\to G$. By Brouwer's fixed point theorem, we know that the discrete problem (3.1) has a solution.

4. Stability analysis

In this section, we study the Hyers-Ulam and Hyers-Ulam-Rassias stability for the solutions of the discrete Riemann-Liouville (RL) FBVP

$$\begin{cases}{}^{RL}\Delta_{\xi}^{\varrho}\chi(\xi)=\Phi(\xi+\varrho-1,\chi(\xi+\varrho-1)), \quad 2<\varrho\le 3,\\ \Delta\chi(\varrho-3)=A_1,\chi(\varrho+T)=\lambda\Delta^{-\beta}\chi(\eta+\beta),\Delta^2\chi(\varrho-3)=A_2, \quad 0<\beta\le 1,\end{cases}\tag{4.1}$$

for $\xi\in[0,1,2,\ldots,T]=[0,T]_{\mathbb{N}_0}$ and $\eta\in[\varrho-1,T+\varrho-1]_{\mathbb{N}_{\varrho-1}}$, where ${}^{RL}\Delta_{\xi}^{\varrho}$ is the RL fractional difference operator. We begin by proving the following lemma.

Lemma 4.1. *Suppose that* $2<\varrho\le 3$ *and* $\Phi:[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}\longrightarrow\mathbb{R}$. *A function* χ *satisfies the discrete problem*

$$\begin{cases}{}^{RL}\Delta_{\xi}^{\varrho}\chi(\xi)=\Phi(\xi+\varrho-1), \quad 2<\varrho\le 3,\\ \Delta\chi(\varrho-3)=A_1, \quad \chi(\varrho+T)=\lambda\Delta^{-\beta}\chi(\eta+\beta), \quad \Delta^2\chi(\varrho-3)=A_2, \quad 0<\beta\le 1,\end{cases}\tag{4.2}$$

if, and only if, $\chi(\xi)$, $\xi\in[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}$, *has the form*

$$\chi(\xi)=\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1)+\Big\{\frac{[A_2-A_1(\varrho-3)]}{\Gamma(\varrho)}+\frac{\frac{f_\Phi+d_\varrho}{h_\varrho}(\varrho-3)}{2(\varrho-1)}\Big\}\xi^{(\varrho-1)}$$
$$+\frac{[A_1-\frac{1}{h_\varrho}[f_\Phi+d_\varrho](\varrho-3)\Gamma(\varrho-2)]}{\Gamma(\varrho-1)}\xi^{(\varrho-2)}+\frac{f_\Phi+d_\varrho}{h_\varrho}\xi^{(\varrho-3)},\tag{4.3}$$

where

$$h_\varrho = \frac{\varrho-3}{\varrho-2}(\varrho+T)^{(\varrho-2)} - \frac{\varrho-3}{2(\varrho-1)}(\varrho+T)^{(\varrho-1)} - (\varrho+T)^{(\varrho-3)} + \frac{\lambda(\varrho-3)}{\Gamma(\beta)2(\varrho-1)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(\varrho-1)} - \frac{\lambda(\varrho-3)}{\Gamma(\beta)(\varrho-2)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(\varrho-2)} + \frac{\lambda}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(\varrho-3)}, \tag{4.4}$$

$$d_\varrho = \frac{A_1(\varrho+T)^{(\varrho-2)}}{\Gamma(\varrho-1)} - \frac{[A_2-A_1(\varrho-3)]\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(\varrho-1)} - \frac{A_1\lambda}{\Gamma(\varrho-1)\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta+\beta-\rho(l))^{(\beta-1)}l^{(\varrho-2)} + \frac{[A_2-A_1(\varrho-3)]}{\Gamma(\varrho)}(\varrho+T)^{(\varrho-1)}, \tag{4.5}$$

$$f_\Phi = -\frac{\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}\Phi(\tau+\varrho-1) + \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1). \tag{4.6}$$

Proof. Let $\chi(\xi)$ be a solution to (4.2). Applying Lemma 2.2 and Definition 2.1, we obtain that the general solution of (4.2) is given by

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + B_1\xi^{(\varrho-1)} + B_2\xi^{(\varrho-2)} + B_3\xi^{(\varrho-3)}, \tag{4.7}$$

$\xi \in [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}}$, where $B_1, B_2, B_3 \in \mathbb{R}$. The first order difference of (4.7) is

$$\Delta\chi(\xi) = \frac{1}{\Gamma(\varrho-1)}\sum_{l=0}^{\xi-\varrho+1}(\xi-\rho(l))^{(\varrho-2)}\Phi(l+\varrho-1) + B_1(\varrho-1)\xi^{(\varrho-2)} + B_2(\varrho-2)\xi^{(\varrho-3)} + B_3(\varrho-3)\xi^{(\varrho-4)},$$

while

$$\Delta^2\chi(\xi) = \frac{1}{\Gamma(\varrho-2)}\sum_{l=0}^{\xi-\varrho+2}(\xi-\rho(l))^{(\varrho-3)}\Phi(l+\varrho-1) + B_1(\varrho-1)(\varrho-2)\xi^{(\varrho-3)} + B_2(\varrho-2)(\varrho-3)\xi^{(\varrho-4)} + B_3(\varrho-3)(\varrho-4)\xi^{(\varrho-5)}.$$

From the conditions $\Delta\chi(\varrho-3) = A_1$ and $\Delta^2\chi(\varrho-3) = A_2$, we obtain that

$$B_2 = \frac{1}{\Gamma(\varrho-1)}[A_1 - B_3(\varrho-3)\Gamma(\varrho-2)],$$

and

$$B_1 = \frac{1}{\Gamma(\varrho)}[A_2 - A_1(\varrho - 3)] + \frac{B_3(\varrho - 3)}{2(\varrho - 1)}.$$

Now, by using the difference of order β for (4.7), it follows that

$$\begin{aligned}\Delta^{-\beta}\chi(\xi) = & \frac{1}{\Gamma(\beta)}\left\{\frac{1}{\Gamma(\varrho)}[A_2 - A_1(\varrho - 3)] + \frac{B_3(\varrho - 3)}{2(\varrho - 1)}\right\}\sum_{l=\varrho-3}^{\xi-\beta}(\xi - \rho(l))^{(\beta-1)}l^{(\varrho-1)} \\ & + \frac{B_3}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\xi-\beta}(\xi - \rho(l))^{(\beta-1)}l^{(\varrho-3)} \\ & + \frac{[A_1 - B_3(\varrho - 3)\Gamma(\varrho - 2)]}{\Gamma(\varrho - 1)\Gamma(\beta)}\sum_{l=\varrho-3}^{\xi-\beta}(\xi - \rho(l))^{(\beta-1)}l^{(\varrho-2)} \\ & + \frac{1}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\xi-\beta}\sum_{\tau=0}^{l-\varrho}(\xi - \rho(l))^{(\beta-1)}(l - \rho(\tau))^{(\varrho-1)}\Phi(\tau + \varrho - 1).\end{aligned} \tag{4.8}$$

Based on condition (4.1), we have

$$\begin{aligned}\lambda\Delta^{-\beta}\chi(\eta + \beta) = & \frac{\lambda}{\Gamma(\beta)}\left\{\frac{1}{\Gamma(\varrho)}[A_2 - A_1(\varrho - 3)] + \frac{B_3(\varrho - 3)}{2(\varrho - 1)}\right\}\sum_{l=\varrho-3}^{\eta}(\eta + \beta - \rho(l))^{(\beta-1)}l^{(\varrho-1)} \\ & + \frac{\lambda}{\Gamma(\varrho - 1)\Gamma(\beta)}[A_1 - B_3(\varrho - 3)\Gamma(\varrho - 2)]\sum_{l=\varrho-3}^{\eta}(\eta + \beta - \rho(l))^{(\beta-1)}l^{(\varrho-2)} \\ & + \frac{B_3\lambda}{\Gamma(\beta)}\sum_{l=\varrho-3}^{\eta}(\eta + \beta - \rho(l))^{(\beta-1)}l^{(\varrho-3)} \\ & + \frac{\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta + \beta - \rho(l))^{(\beta-1)}(l - \rho(\tau))^{(\varrho-1)}\Phi(\tau + \varrho - 1) \\ & = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho + T - \rho(l))^{(\varrho-1)}\Phi(l + \varrho - 1) \\ & + \frac{1}{\Gamma(\varrho - 1)}[A_1 - B_3(\varrho - 3)\Gamma(\varrho - 2)](\varrho + T)^{(\varrho-2)} \\ & + \left\{\frac{1}{\Gamma(\varrho)}[A_2 - A_1(\varrho - 3)] + \frac{B_3(\varrho - 3)}{2(\varrho - 1)}\right\}(\varrho + T)^{(\varrho-1)} + B_3(\varrho + T)^{(\varrho-3)}.\end{aligned}$$

Then,

$$B_3 = \frac{1}{h_\varrho}\Big[f_\Phi + d_\varrho\Big],$$

$$B_2 = \frac{1}{\Gamma(\varrho - 1)}\Big[A_1 - \frac{1}{h_\varrho}\Big[f_\Phi + d_\varrho\Big](\varrho - 3)\Gamma(\varrho - 2)\Big],$$

$$B_1 = \frac{1}{\Gamma(\varrho)}\Big[A_2 - A_1(\varrho-3)\Big] + \frac{\frac{1}{h_\varrho}[f_\Phi + d_\varrho](\varrho-3)}{2(\varrho-1)},$$

where h_ϱ, d_ϱ and f_Φ are defined by (4.4)-(4.6), respectively. Substituting the values of the constants B_1, B_2 and B_3 into (4.7), we obtain (4.3) and our proof is complete.

From Lemma 4.1, the solution of the discrete RL problem (4.1) is given by the formula

$$\begin{aligned}\chi(\xi) &= \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1)) \\ &+ \left\{\frac{[A_2 - A_1(\varrho-3)]}{\Gamma(\varrho)} + \frac{\frac{f_\chi+d_\varrho}{h_\varrho}(\varrho-3)}{2(\varrho-1)}\right\}\xi^{(\varrho-1)} \\ &+ \frac{[A_1 - \frac{1}{h_\varrho}[f_\Phi+d_\varrho](\varrho-3)\Gamma(\varrho-2)]}{\Gamma(\varrho-1)}\xi^{(\varrho-2)} + \frac{f_\chi+d_\varrho}{h_\varrho}\xi^{(\varrho-3)},\end{aligned} \tag{4.9}$$

where d_ρ and h_ρ are defined by (4.4) and (4.5), respectively, and

$$\begin{aligned}f_\chi = &-\frac{\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}\Phi(\tau+\varrho-1,\chi(\tau+\varrho-1)) \\ &+ \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1)).\end{aligned} \tag{4.10}$$

Lemma 4.2. *If χ is a solution of (4.3), then*

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + Q(\xi) + f_\chi K(\xi), \tag{4.11}$$

where

$$\begin{aligned}Q(\xi) = &\left\{\frac{[A_2 - A_1(\varrho-3)]}{\Gamma(\varrho)} + \frac{d_\varrho(\varrho-3)}{2h_\varrho(\varrho-1)}\right\}\xi^{(\varrho-1)} \\ &+ \frac{[A_1 - \frac{d_\varrho}{h_\varrho}(\varrho-3)\Gamma(\varrho-2)]}{\Gamma(\varrho-1)}\xi^{(\varrho-2)} + \frac{d_\varrho}{h_\varrho}\xi^{(\varrho-3)},\end{aligned}$$

and

$$K(\xi) = \frac{(\varrho-3)}{2h_\varrho(\varrho-1)}\xi^{(\varrho-1)} - \frac{(\varrho-3)}{h_\varrho(\varrho-2)}\xi^{(\varrho-2)} + \frac{\xi^{(\varrho-3)}}{h_\varrho}.$$

Proof. Take χ as a solution of (4.2). For $\xi \in [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}}$, then

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + \left\{\frac{[A_2 - A_1(\varrho-3)]}{\Gamma(\varrho)} \quad + \frac{\frac{f_\chi+d_\varrho}{h_\varrho}(\varrho-3)}{2(\varrho-1)}\right\}\xi^{(\varrho-1)}$$

$$+\frac{[A_1-\frac{1}{h_\varrho}[f_\chi+d_\varrho](\varrho-3)\Gamma(\varrho-2)]}{\Gamma(\varrho-1)}\xi^{(\varrho-2)}+\frac{f_\chi+d_\varrho}{h_\varrho}\xi^{(\varrho-3)}$$
$$=\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1)+\left\{\frac{[A_2-A_1(\varrho-3)]}{\Gamma(\varrho)}+\frac{d_\varrho(\varrho-3)}{2h_\varrho(\varrho-1)}\right\}\xi^{(\varrho-1)}$$
$$+\frac{[A_1-\frac{d_\varrho}{h_\varrho}(\varrho-3)\Gamma(\varrho-2)]}{\Gamma(\varrho-1)}\xi^{(\varrho-2)}$$
$$+\frac{d_\varrho}{h_\varrho}\xi^{(\varrho-3)}+f_\chi\left[\frac{(\varrho-3)}{2h_\varrho(\varrho-1)}\xi^{(\varrho-1)}-\frac{(\varrho-3)}{h_\varrho(\varrho-2)}\xi^{(\varrho-2)}+\frac{\xi^{(\varrho-1)}}{h_\varrho}\right]$$
$$=\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1)+Q(\xi)+f_\chi K(\xi).$$

The proof is complete.

Definition 4.1. *We say that the discrete RL problem (4.1) is Hyers-Ulam stable if for each function* $\nu\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *of*

$$\left|{}^{RL}\Delta_\xi^\varrho\nu(\xi)-\Phi(\xi+\varrho-1,\nu(\xi+\varrho-1))\right|\le\epsilon,\quad \xi\in[0,T]_{\mathbb{N}_0},\tag{4.12}$$

and $\epsilon>0$*, there exists* $\chi\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *solution of (4.1) and* $\delta>0$ *such that*

$$\left|\nu(\xi)-\chi(\xi)\right|\le\delta\epsilon,\quad \xi\in[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}.\tag{4.13}$$

Definition 4.2. *We say that the discrete RL problem (4.1) is Hyers-UlamRassias stable if for each function* $\nu\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *of*

$$\left|{}^{RL}\Delta_\xi^\varrho\nu(\xi)-\Phi(\xi+\varrho-1,\nu(\xi+\varrho-1))\right|\le\epsilon\theta(\xi+\varrho-1),\quad \xi\in[0,T]_{\mathbb{N}_0},\tag{4.14}$$

and $\epsilon>0$*, there exists* $\chi\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *solution of (4.1) and* $\delta_2>0$ *such that*

$$\left|\nu(\xi)-\chi(\xi)\right|\le\delta_2\epsilon\theta(\xi+\varrho-1),\quad \xi\in[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}.\tag{4.15}$$

Remark 4.1. *A function* $\chi(\xi)\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *is a solution of (4.12) if, and only if, there exists* $\mu:[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}\longrightarrow\mathbb{R}$ *satisfying:*

$(\mathcal{H}2)$ $\left|\mu(\xi+\varrho-1)\right|\le\epsilon,\quad \xi\in[0,T]_{\mathbb{N}_0}$;

$(\mathcal{H}3)$ ${}^{RL}\Delta_\xi^\varrho\nu(\xi)=\Phi(\xi+\varrho-1,\nu(\xi+\varrho-1))+\mu(\xi+\varrho-1),\quad \xi\in[0,T]_{\mathbb{N}_0}$.

Remark 4.2. *A function* $\chi(\xi)\in C(\mathbb{N}_{\varrho-3,\varrho+T},\mathbb{R})$ *is a solution of (4.14) if, and only if, there exists* $\mu:[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}\longrightarrow\mathbb{R}$ *satisfying*

$(\mathcal{H}4)$ $\left|\mu(\xi+\varrho-1)\right|\le\epsilon\theta(\xi+\varrho-1),\quad \xi\in[0,T]_{\mathbb{N}_0}$,

$(\mathcal{H}5)$ ${}^{RL}\Delta_\xi^\varrho\nu(\xi)=\Phi(\xi+\varrho-1,\nu(\xi+\varrho-1))+\mu(\xi+\varrho-1),\quad \xi\in[0,T]_{\mathbb{N}_0}$.

Lemma 4.3. *If ν satisfies (4.12), then*

$$\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))-Q(\xi)-f_{\nu}K(\xi)\right|\leq\frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}.$$

Proof. Using our hypothesis, and based on Remark 4.1, the solution to ($\mathcal{H}3$) satisfies

$$\begin{aligned}\nu(\xi)&=\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))+Q(\xi)+f_{\nu}K(\xi)\\&\quad+\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\mu(l+\varrho-1).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))-Q(\xi)-f_{\nu}K(\xi)\right|\\&=\left|\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\mu(l+\varrho-1)\right|\\&\leq\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}|\mu(l+\varrho-1)|\\&\leq\frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)},\end{aligned}$$

and the desired inequality is derived.

Theorem 4.1. *If condition ($\mathcal{H}1$) holds and (4.13) is satisfied, then the discrete RL problem (4.1) is Hyers-Ulam stable under the condition*

$$K\leq\frac{\Gamma(\beta)\Gamma(T+1)\Gamma(\varrho+1)}{2M_1\Gamma(\varrho+T+1)\Gamma(\beta)+MM_1\lambda\varrho\Gamma(T+1)},\tag{4.16}$$

where $M_1=\max(1,|K(\xi)|)$.

Proof. Let $\xi\in[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}}$. From Lemma 4.3, we have

$$\begin{aligned}|\nu(\xi)-\chi(\xi)|&\leq\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\chi(l+\varrho-1))-Q(\xi)-f_{\chi}K(\xi)\right|\\&\leq\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))-Q(\xi)-f_{\nu}K(\xi)\right|\\&\quad+\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}|\Phi(l+\varrho-1,\chi(l+\varrho-1))-\Phi(l+\varrho-1,\nu(l+\varrho-1))|\\&\quad+|K(\xi)||f_{\nu}-f_{\chi}|.\end{aligned}$$

It follows that

$$
\begin{aligned}
|\nu(\xi)-\chi(\xi)| \leq & \frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}+\frac{K}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}|\chi(l+\varrho-1)-\nu(l+\varrho-1)| \\
& +|K(\xi)|K\Big\{\frac{\lambda}{\Gamma(\varrho)\Gamma(\beta)}\sum_{l=\varrho}^{\eta}\sum_{\tau=0}^{l-\varrho}(\eta+\beta-\rho(l))^{(\beta-1)}(l-\rho(\tau))^{(\varrho-1)}|\nu(l+\varrho-1)-\chi(l+\varrho-1)| \\
& +\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}|\chi(l+\varrho-1)-\nu(l+\varrho-1)|\Big\} \\
\leq & \frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}+\frac{K\|\chi-\nu\|}{\Gamma(\varrho)}\frac{\Gamma(\xi+1)}{\varrho\Gamma(\xi+1-\varrho)} \\
& +|K(\xi)|K\|\chi-\nu\|\Big[\frac{M\lambda}{\Gamma(\varrho)\Gamma(\beta)}+\frac{1}{\Gamma(\varrho)}\frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)}\Big] \\
\leq & \frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}+\frac{2KM_1}{\Gamma(\varrho+1)}\frac{\Gamma(\varrho+T+1)}{\Gamma(T+1)}\|\chi-\nu\|+\|\chi-\nu\|\frac{KMM_1\lambda}{\Gamma(\varrho)\Gamma(\beta)}.
\end{aligned}
$$

Therefore,

$$
\|\nu-\chi\| \leq \frac{\epsilon\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)}+\|\nu-\chi\|\Big[\frac{2KM_1}{\Gamma(\varrho+1)}\frac{\Gamma(\varrho+T+1)}{\Gamma(T+1)}+\frac{KMM_1\lambda}{\Gamma(\varrho)\Gamma(\beta)}\Big].
$$

Moreover, $\|\nu-\chi\| \leq \epsilon\delta$, where

$$
\delta=\frac{\Gamma(\beta)\Gamma(\varrho+T+1)}{\Gamma(\beta)\Gamma(T+1)\Gamma(\varrho+1)-2KM_1\Gamma(\varrho+T+1)\Gamma(\beta)-KMM_1\lambda\varrho\Gamma(T+1)}>0.
$$

Thus, the discrete RL problem (4.1) is Hyers-Ulam stable.

Lemma 4.4. *If ν solves (4.14) under the condition:*
($\mathcal{H}6$) *The function $\theta:[\varrho-3,\varrho+T]_{\mathbb{N}_{\varrho-3}} \longrightarrow \mathbb{R}$ is increasing and there exists a constant $\gamma>0$ such that*

$$
\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\theta(l+\varrho-1) \leq \gamma\theta(\xi+\varrho-1), \quad \xi \in [0,T]_{\mathbb{N}_0},
$$

then

$$
\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))-Q(\xi)-f_{\phi_\nu}K(\xi)\right| \leq \gamma\theta(\xi+\varrho-1).
$$

Proof. Let ν satisfy (4.14). From Remark 4.2, the solution to ($\mathcal{H}5$) satisfies

$$
\begin{aligned}
\nu(\xi)= & \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))+Q(\xi)+f_\nu K(\xi) \\
& +\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\mu(l+\varrho-1).
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\left|\nu(\xi)-\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1,\nu(l+\varrho-1))-Q(\xi)-f_\nu K(\xi)\right|\\
&=\left|\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\mu(l+\varrho-1)\right|\\
&\le\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}|\mu(l+\varrho-1)|\\
&\le\frac{\epsilon}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}|\theta(l+\varrho-1)|\\
&\le\gamma\epsilon\theta(l+\varrho-1),
\end{aligned}$$

and the desired inequality is derived.

Remark 4.3. *About the restrictiveness of hypotheses ($\mathcal{H}1$)–($\mathcal{H}6$), and the bounds imposed on the family of discrete systems that satisfy them, one should note that such hypotheses are the usual conditions for proving existence, uniqueness or stability of solutions. In fact, the conditions ($\mathcal{H}2$)–($\mathcal{H}6$) are considered as the fundamental conditions in the definition of Hyers-Ulam-Rassias stability. Condition ($\mathcal{H}1$) is a standard Lipschitz condition while other constants are computed based on the given fractional system. Therefore, these conditions are natural and, in real systems, with specified numerical data, their expressions are reduced to numerical bounds.*

Theorem 4.2. *If the inequality (4.16) and the hypotheses ($\mathcal{H}1$) and ($\mathcal{H}6$) are satisfied, then the discrete RL problem (4.1) is Hyers-Ulam-Rassias stable.*

Proof. From Lemmas 4.4 and 2.3, we obtain that

$$\|\nu-\chi\|\le\delta_2\epsilon\theta(\xi+\varrho-1),$$

where

$$\delta_2=\frac{\Gamma(\varrho+1)\Gamma(\beta)\Gamma(T+1)}{\Gamma(\varrho+1)\Gamma(\beta)\Gamma(T+1)-2KM_1\Gamma(\beta)\Gamma(\varrho+T+1)-KMM_1\lambda\varrho\Gamma(T+1)}>0.$$

Thus, the discrete RL problem (4.1) is Hyers-Ulam-Rassias stable.

5. Illustrative examples

In this section, we consider two examples to illustrate the obtained results.

Example 5.1. *Let*

$$\begin{cases}{}^{*}\Delta_{\xi}^{\varrho}\chi(\xi)=\Phi(\xi+\varrho-1,\chi(\xi+\varrho-1)),\quad \xi\in\mathbb{N}_{0,4},\\[2ex] \Delta\chi(\varrho-3)=1,\quad \chi(\varrho+4)=0.3\Delta^{-0.5}\chi(\eta+0.5),\quad \Delta^2\chi(\varrho-3)=0,\end{cases}\tag{5.1}$$

where ${}^{*}\Delta_{\xi}^{\varrho}$ *denotes the operator* ${}^{c}\Delta_{\xi}^{\varrho}$ *or* ${}^{RL}\Delta_{\xi}^{\varrho}$. *Set* $\beta = 0.5$, $T = 4$, $\lambda = 0.7$, $A_1 = 1$, $A_2 = 0$, *and*

$$\Phi(\xi + 1.5, \chi(\xi + 1.5)) = \frac{137}{10^5} \sin(\chi(\xi + 1.5)).$$

- *If* ${}^{*}\Delta_{\xi}^{\varrho}\chi(\xi) = {}^{c}\Delta_{\xi}^{\frac{5}{2}}\chi(\xi)$ *and* $\eta = \frac{5}{2}$, *then we obtain that*

$$\begin{aligned} K_1 &= \lambda \sum_{l=\varrho-3}^{\eta} (\eta + \beta - \rho(l))^{(\beta-1)} - \Gamma(\beta) \\ &= \frac{\lambda\Gamma(\eta + \beta - \varrho + 4)}{\beta\Gamma(\eta - \varrho + 4)} - \Gamma(\beta) \\ &= 0.9416, \end{aligned}$$

$$M = \left| \sum_{l=\varrho}^{\eta} \sum_{\xi=0}^{l-\varrho} (\eta + \beta - \rho(l))^{(\beta-1)} (l - \rho(\xi))^{(\varrho-1)} \right| = 2.3562. \tag{5.2}$$

Hence, the inequality (3.8) *takes the form*

$$\frac{\Gamma(\varrho + T + 1)}{\Gamma(\varrho + 1)\Gamma(T + 1)}\left(1 + \frac{\Gamma(\beta)}{K_1}\right) + \frac{M\lambda}{K_1\Gamma(\varrho)} \approx 68.9403 \leq \frac{1}{K} \approx 729.9270,$$

such that

$$K = \frac{137}{10^5}.$$

From Theorem 3.1, the discrete problem (5.1) *has a unique solution.*

- *In the case* ${}^{*}\Delta_{\xi}^{\varrho}\chi(\xi) = {}^{RL}\Delta_{\xi}^{3}\chi(\xi)$ *and* $\eta = 3$, *we obtain*

$$\begin{aligned} M &= 3.5449, \\ M_1 &= 1.8824, \\ h_{\varrho} &= \frac{\lambda\Gamma(\eta + \beta - \varrho + 4)}{\Gamma(\beta + 1)\Gamma(\eta - \varrho + 4)} - 1 = 0.5313, \\ K_1 &= \frac{\lambda\Gamma(\eta + \beta - \varrho + 4)}{\beta\Gamma(\eta - \varrho + 4)} - \Gamma(\beta) = 0.9416. \end{aligned}$$

Also,

$$\frac{\Gamma(\beta)\Gamma(T + 1)\Gamma(\varrho + 1)}{2M_1\Gamma(\varrho + T + 1)\Gamma(\beta) + MM_1\lambda\varrho\Gamma(T + 1)} \approx 0.0075. \tag{5.3}$$

If $K = 0.0014 < 0.0075$ *and*

$$|{}^{RL}\Delta_{\xi}^{3}\nu(\xi) - \Phi(\xi + 2, \nu(\xi + 2))| \leq \epsilon, \quad \xi \in [0, 4]_{\mathbb{N}_0}, \tag{5.4}$$

holds, then, by Theorem 4.1, the discrete RL problem (5.1) *is Hyers-Ulam stable.*

Example 5.2. *Let*

$$\begin{cases} {}^{c}\Delta_{\xi}^{2.4}\chi(\xi) = \dfrac{1}{10^7}\chi^4(\xi + 1.4), \quad \xi \in \mathbb{N}_{0,2}, \\ \Delta\chi(-0.4) = 2, \quad \chi(3.4) = 0.8\Delta^{-1/3}\chi(1/3 + 2.4), \quad \Delta^2\chi(-0.6) = 0. \end{cases} \tag{5.5}$$

After some calculations, we find that

$$M = 3.277,$$

$$K_1 = \lambda \sum_{l=\varrho-3}^{\eta} (\eta + \beta - \rho(l))^{(\beta-1)} - \Gamma(\beta) = 1.0253,$$

$$K_2 = \frac{A_1\Gamma(\beta)}{K_1}(\varrho + T) - \frac{\lambda A_1}{K_1}\frac{(\eta - \beta(3 - \varrho))\Gamma(\eta - \varrho + \beta + 4)}{\beta(\beta + 1)\Gamma(\eta - \varrho + 4)} = 16.2963.$$

We define the following Banach space:

$$C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R}) = \left\{\chi(t)|[-0.5, 6.5]_{\mathbb{N}_{-0.5}} \to \mathbb{R}, \ \|\chi\| \le 2|K_2| = 32.5927\right\}.$$

Note that

$$\frac{|K_2| - \big|[A_1 - A_2(\varrho - 3)]\big|(\varrho + T)^{(1)} - \left|\dfrac{A_2}{2}\right|(\varrho + T)^{(2)}}{\dfrac{\Gamma(\varrho + T + 1)}{\Gamma(\varrho + 1)\Gamma(T + 1)}\left(1 + \dfrac{\Gamma(\beta)}{|K_1|}\right) + \dfrac{M|\lambda|}{\Gamma(\varrho)|K_1|}} = 0.3262.$$

It is clear that $|\Phi(t,\chi)| \le 0.0586 \le 0.3262$ *whenever* $\chi \in [-32.5927, 32.5927]$. *Therefore, by Theorem 3.2, we find out that the discrete FBVP* (5.5) *has a solution.*

6. Conclusions

We proved existence and uniqueness of solution to discrete fractional boundary value problems (FBVPs) involving fractional difference operators via the Brouwer fixed point theorem and the Banach contraction principle. Different versions of stability criteria were obtained for a discrete FBVP involving Riemann-Liouville difference operators. The results were illustrated by suitable examples. The approach of this paper is new and can be a beginning method for discussing different real-world models in the context of discrete behavior structures. In particular, our results can contribute for the development of discrete fractional boundary value problems describing discrete dynamics of some physical applications. In future works, we plan to extend our approach to other types of discrete differential inclusions or fully-hybrid discrete fractional differential equations.

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Conflict of interest

The authors declare no conflict of interest.

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