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Research article

On two backward problems with Dzherbashian-Nersesian operator

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Abstract: We investigate the initial-boundary value problems for a fourth-order differential equation within the powerful fractional Dzherbashian-Nersesian operator (FDNO). Boundary conditions considered in this manuscript are of the Samarskii-Ionkin type. The solutions obtained here are based on a series expansion using Riesz basis in a space corresponding to a non-self-adjoint spectral problem. Conditional to some regularity, consistency, alongside orthogonality dependence, the existence and uniqueness of the obtained solutions are exhibited by using Fourier method. Acquired results here are more general than those obtained by making use of conventional fractional operators such as fractional Riemann-Liouville derivative (FRLD), fractional Caputo derivative (FCD) and fractional Hilfer derivative (FHD).

Keywords: Dzherbashian-Nersesian operator; nonlocal boundary conditions; backward problem; Mittag-Leffler function
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1. Introduction

A large number of their applications in real life is the major motivation behind the study of fractional differential equations (FDEs). Many problems dealing with anomalous diffusion [1, 2], heat propagation [3, 4], bioengineering [5, 6], image processing [7], signal processing [8], control theory [9, 10], theory of random walks [11, 12], etc., cause the study of FDEs. Monographs [13–15] may be referred to for profound study of FDEs.

Finding the generalized fractional operators is an important contemporary topic of the agenda of the researchers from the fractional calculus area. Recently it is shown that working with a general class of fractional operators leads us to interesting results from the mathematical view point. However, from the applied point of view the real data plays a crucial role in identifying which particular fractional operator gives better results for a given model.

We examine the following fourth-order FDE

$$\partial^{\varsigma_m}_{0_+,t} v(x,t) + v_{xxxx}(x,t) = H(x,t), \quad (x,t) \in \Omega := [0,1] \times (0,T], \tag{1.1}$$

with initial conditions,

$$\partial^{\varsigma_r}_{0_+,t} v(x,t)\Big|_{t=0} = \omega_r(x), \quad r = 0, ..., m-1, \quad x \in [0,1], \tag{1.2}$$

and the nonlocal Samarskii-Ionkin type boundary conditions

$$v(0,t) = 0 = v_{xx}(1,t),\ v_x(0,t) = v_x(1,t),\ v_{xxx}(0,t) = v_{xxx}(1,t),\ t \in (0,T]. \tag{1.3}$$

Here $\partial^{\varsigma_m}_{0+,t}$, which is defined in Section 2, denotes the FDNO of order ς_m such that $\varsigma_m := \sum_{i=0}^{m} \zeta_i - 1 > 0$, where $\zeta_i \in (0,1]$.

We introduce σ, with dimension in seconds, as an auxiliary parameter to preserve the physical dimensions of fractional temporal operator (see [16])

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\varsigma_1}} \partial^{\varsigma_1}_{0+,t}, \quad 0 < \varsigma_1 \leq 1,$$

and

$$\frac{d^m}{dt^m} \to \frac{1}{\sigma^{m-\varsigma_m}} \partial^{\varsigma_m}_{0+,t}, \quad 0 < \varsigma_m \leq m.$$

Without any loss of generality, we took $\sigma = 1$, which gave rise to Eq (1.1).

The basic objective of this paper is to investigate two backward problems. In the first backward problem, we make the assumption that $H(x,t) := h(x)$, that is, source term depends only on the spatial variable. For the second backward problem, we assume that $H(x,t) := a(t)h(x,t)$, where $h(x,t)$ is known. We are going to determine $\{v(x,t), h(x)\}$ and $\{v(x,t), a(t)\}$ in connection with the following overdetermined conditions

$$v(x,t) = \psi(x), \quad t < T, \tag{1.4}$$

$$\int_0^1 x v(x,t) = \mathcal{E}(t), \quad t \in (0,T], \tag{1.5}$$

respectively.

Originally presented in 1968 [17], the interesting FDNO has been rarely studied. The motivation of this article arises from the resurgence of FDNO of late. The merit goes to article [18] with English translation of the Russian version of [17] in which the aforementioned operator is introduced. In [19], authors have proved how specific fixing of values of parameters in FDNO leads to the recovery of FRLD, FCD and FHD. Moreover, there are some articles in the literature discussing FDEs involving special case of FDNO [20, 21].

In a forward problem, we solve an equation in a region with specific provided data. On the other hand, problem of recovering an unknown input which could be either certain coefficients, initial conditions, boundary conditions, or some source function from provided output is known as a backward problem. In comparison to the study of forward problems, the backward problems are attracting considerable amount of attention of many researchers. For example, see [22] and references therein.

Boundary value problems theory for FDEs is one of the most rapidly growing areas of the theory of differential equations. In works [23, 24] the backward problem related to a fourth-order FDE subject to the Samarskii-Ionkin type nonlocal boundary conditions are investigated. Over the years there has been a constant interest to the study of backward problems of FDEs usually in Caputo and Hilfer sense. See for instance [24–28].

There are many applications of backward problems that involve FDEs. For instance, a stable algorithm using mollification techniques is established in Murio et al. [29] for the backward problem of boundary function for time fractional differential equation (TFDE) from a given noisy temperature distribution. In [30], authors have established a technique associated with regularization and proposed the uniqueness of the unknown terms. Li et al. [30] has derived methods for simultaneous recovery of order of nonlocal integrodifferential operator and a spatial component of coefficient of diffusion for a 1D TFDE proposed to solve a backward source problem for a space FDE in 1D space.

However, backward problems involving FDNO are not researched at a great length. In fact only [19] and [31] investigated backward problems concerning this operator. In our work, nevertheless, we investigate both space dependent and temporal backward problems for a fourth-order FDE with the Samarskii-Ionkin type nonlocal boundary and nonhomogeneous initial conditions.

A regular solution of Eq (1.1) is $v(x,t)$ defined on a domain Ω is continuous together with terms entering the equation. The regular solution of space dependent backward problems (1.1)–(1.4), we refer to the pair $\{v(x,t), h(x)\}$ such that $t^{\zeta_1} v(.,t) \in C^2(0,1)$, $t^{\zeta_1}\partial^{\varsigma_m}_{0_+,t} v(x,.) \in C(0,T]$ and $h(x) \in C(0,1)$. At the same time, for the time dependent backward problem i.e., (1.1)–(1.3) together with (1.5), the regular solution corresponds to pair $\{v(x,t), a(t)\}$ satisfying $t^{\zeta_1} v(.,t) \in C^2(0,1)$, $t^{\zeta_1}\partial^{\varsigma_m}_{0_+,t} v(x,.) \in C(0,T]$ and $a(t) \in C[0,T]$.

The remaining article is presented as follows. In Section 2, we recall the basic definitions of some fractional operators and Mittag-Leffler function. Furthermore, some properties of FDNO and Mittag-Leffler function are given. Section 3 is dedicated to the study of spectral problem and some useful estimates. Section 4 focuses the major findings concerning the existence and uniqueness of our problems. Section 5 contains the concluding remarks.

2. Preliminaries

This brief section covers some preliminary definitions, related notions and some useful results.

Definition 2.1. *[13, 15] The fractional Riemann-Liouville integral $J^{\alpha}_{0_+,t}$ of order $\alpha > 0$ is defined as*

$$J^{\alpha}_{0_+,t} g(z) := \frac{1}{\Gamma(\alpha)} \int_0^z \frac{g(\tau)}{(z-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0,$$

where $\Gamma(.)$ represents Euler integral of second kind.

Definition 2.2. *[13, 15] The left sided FRLD $D^{\alpha}_{0_+,t}$ of order $\alpha \in (p-1, p)$ is defined as*

$$D^{\alpha}_{0_+,t} g(z) := \frac{d^p}{dz^p} J^{p-\alpha}_{0_+,t} g(z), \quad \Re(\alpha) \geqq 0, \quad p = [\Re(\alpha)] + 1,$$

where $[\Re(\alpha)]$ *means the greatest integer in* $\Re(\alpha)$.

Definition 2.3. *[17] FDNO* $\partial_{0+,t}^{\alpha_p}$ *of order* α_p *is defined as*

$$\partial_{0+,t}^{\alpha_p} g(z) := J_{0+,t}^{1-\beta_p} D_{0+,t}^{\beta_{p-1}} D_{0+,t}^{\beta_{p-2}} ... D_{0+,t}^{\beta_1} D_{0+,t}^{\beta_0} g(z), \quad p \in \mathbb{Z}^+, \quad t > 0, \tag{2.1}$$

where $\alpha_p \in (0, p)$ *is determined by*

$$\alpha_p = \sum_{j=0}^{p} \beta_j - 1 > 0, \quad \beta_j \in (0, 1].$$

Remark 2.1. *[19] For* $\beta_1 = ... = \beta_p = 1$ *and* $\beta_0 = 1 + \alpha - p$*, where* $\beta_0 \in (0, 1)$*, in Eq (2.1), FDNO interpolates FRLD of order* $\alpha \in (p-1, p)$*, i.e.,*

$$\partial_{0+,t}^{\alpha_p} g(z) = \frac{d^p}{dt^p} J_{0+,t}^{p-\alpha} g(z) = D_{0_+,t}^{\alpha} g(z).$$

Remark 2.2. *[19] For* $\beta_0 = ... = \beta_{p-1} = 1$ *and* $\beta_p = 1 + \alpha - p$*, where* $\beta_p \in (0, 1)$*, in Eq (2.1), FDNO operator redcues to FCD of order* $\alpha \in (p-1, p)$*, i.e.,*

$$\partial_{0+,t}^{\alpha_p} g(z) = J_{0+,t}^{p-\alpha} \frac{d^p}{dt^p} g(z) =: {}^{c}D_{0_+,t}^{\alpha} g(z).$$

Remark 2.3. *[19]* $\beta_p = 1 - \beta(p-\alpha)$*,* $\beta_0 = 1 - (p-\alpha)(1-\beta)$*, where* $\beta_0, \beta_p \in (0, 1)$ *and* $\beta_{p-1} = ... = \beta_1 = 1$*, in Eq (2.1), FDNO interploates another famous FHD of order* $\alpha \in (p-1, p)$ *and type* $\beta \in [0, 1]$*, i.e.,*

$$\partial_{0+,t}^{\alpha_p} g(z) = J_{0_+,t}^{\beta(p-\alpha)} \frac{d^p}{dt^p} J_{0_+,t}^{(p-\alpha)(1-\beta)} g(z) =: {}^{H}D_{0_+,t}^{\alpha,\beta} g(z).$$

Lemma 2.1. *[19] Laplace transfom of FDNO represented by (2.1) having order* $\alpha_p \in (0, p)$ *is given as*

$$\mathcal{L}\{\partial_{0+,t}^{\alpha_p} g(z)\} = s^{\alpha_p} \mathcal{L}\{g(z)\} - \sum_{k=1}^{p} s^{\alpha_p - \alpha_{p-k} - 1} \, \partial_{0+,t}^{\alpha_{p-k}} g(z)\Big|_{z=0}.$$

Lemma 2.2. *[19] Let* g_i *denote sequence containing functions with domain* $(0, b]$ *for each* $i \in \mathbb{Z}^+$*, in such a way that the following conditions are satisfied:*

(1) fractional derivatives $D_{0+,t}^{\beta_0} g_i(z)$*,* $D_{0+,t}^{\beta_1} D_{0+,t}^{\beta_0} g_i(z)$*, ...,* $D_{0+,t}^{\beta_{p-1}} ... D_{0+,t}^{\beta_0} g_i(z)$ *for* $p \in \mathbb{Z}^+, t \in (0, b]$ *exist,*

(2) series represented by $\sum_{i=1}^{\infty} g_i(z)$ *and* $\sum_{i=1}^{\infty} D_{0+,t}^{\beta_0} g_i(z)$*,* $\sum_{i=1}^{\infty} D_{0+,t}^{\beta_1} D_{0+,t}^{\zeta_0} g_i(z)$*,...,* $\sum_{i=1}^{\infty} D_{0+,t}^{\beta_{p-1}} ... D_{0+,t}^{\beta_1} D_{0+,t}^{\beta_0} g_i(z)$ *exhibit uniform convergence on the interval* $[\epsilon, b]$ *for arbitrary* $\epsilon > 0$.

Then

$$\partial_{0+,t}^{\alpha_p} \sum_{i=1}^{\infty} g_i(z) = \sum_{i=1}^{\infty} \partial_{0+,t}^{\alpha_p} g_i(z).$$

Definition 2.4. *[32] The classical Mittag-Leffler function (MLF) that was introduced by Magnus Gösta Mittag-Leffler is defined as*

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad Re(\alpha) > 0, \ z \in \mathbb{C}.$$

The generalization of (2.4) was given by Wiman in [33] as follows

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha),\ Re(\beta) > 0,\ z \in \mathbb{C}.$$

In addition, the Mittag-Leffler type function (MLFT) is defined as

$$e_{\alpha,\beta}(t;\lambda) := t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha}), \quad Re(\alpha),\ Re(\beta),\ t,\ \lambda > 0.$$

Lemma 2.3. *[34] If $\alpha < 2$, $\beta \in \mathbb{R}$, μ is such that $\pi\alpha/2 < \mu < min\{\pi, \pi\alpha\}$, $z \in \mathbb{C}$ such that $|z| \geq 0$, $\mu \leq |arg(z)| \leq \pi$ and C_1 is a real constant, then*

$$|E_{\alpha,\beta}(z)| \leq \frac{C_1}{1+|z|}.$$

Lemma 2.4. *[26] For $g(t) \in C[0,T]$, we have*

$$|g(t) * e_{\alpha,\alpha}(t;\lambda)| \leq \frac{C_1}{\lambda} ||g||_t, \quad \alpha, t, \lambda > 0,$$

where $||.||_t$ denotes the Chebyshev norm and

$$||g||_t := \max_{0 \leq t \leq T} |g(t)|.$$

Lemma 2.5. *[25] The following condition is fulfilled for the MLFT $e_{\alpha,\alpha+1}(t;\lambda)$*

$$e_{\alpha,\alpha+1}(t;\lambda) = \frac{1}{\lambda}(1 - e_{\alpha,1}(t;\ \lambda)), \quad t, \lambda > 0.$$

Lemma 2.6. *[25] The MLFT $e_{\alpha,1}(T;\lambda)$ has the following property for $\alpha \in (0,1)$*

$$\frac{1}{1 - e_{\alpha,1}(T;\lambda)} \leq C_2, \quad T, \lambda > 0,$$

where C_2 is a positive constant.

3. Bi-orthogonal system

In the section, a bi-orthogonal system of functions (BOSFs) comprising of eigenfunctions related to spectral problem of (1.1) and (1.3) and its adjoint problem is constructed. Some estimates which are useful in proofs of our main results are also given. However, at the first consideration, we will study the BOSFs.

The spectral problem for (1.1) and (1.3) given below

$$\begin{cases} W^{(iv)}(x) = \lambda W(x), \quad x \in (0,1), \\ W(0) = 0 = W''(1), \quad W'(0) = W'(1), \quad W'''(0) = W'''(1), \end{cases} \tag{3.1}$$

is non-self adjoint (see [23]). The eigenfunctions $\{W_0, W_{1k}\}$ for (3.1) corresponding to eigenvalues $\lambda_0 = 0$ and $\lambda_k = (2\pi k)^4$ and associated eigenfunction W_{2k} (see [35]) are

$$W_0(x) = 2x, \quad W_{2k-1}(x) = 2\sin 2\pi k x,$$

$$W_{2k}(x)=\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}+\cos 2\pi kx,\quad k\in\mathbb{Z}^{+}. \tag{3.2}$$

The adjoint problem for (3.1) is as follows

$$\begin{cases} V^{(iv)}(x)=\lambda V(x), & x\in(0,1),\\ V(0)=V(1), \quad V''(0)=V''(1), & V'(0)=V'''(1)=0. \end{cases} \tag{3.3}$$

The eigenfunctions $\{V_0, V_{1k}\}$ for (3.1) corresponding to eigenvalues $\lambda_0 = 0$ and $\lambda_k = (2\pi k)^4$ and associated eigenfunction V_{2k} (see [35]) are

$$\begin{aligned} &V_0(x)=1, \quad V_{2k-1}(x)=\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}+\sin 2\pi kx,\\ &V_{2k}(x)=2\cos 2\pi kx, \quad k\in\mathbb{Z}^{+}. \end{aligned} \tag{3.4}$$

The systems of functions given by (3.2) and (3.4) form a BOSFs with respect to the one-to-one correspondence as follows [23],

$$\begin{array}{ccc} \{\underbrace{W_0(x)}, & \underbrace{W_{2k-1}(x)}, & \underbrace{W_{2k}(x)}\},\\ \downarrow & \downarrow & \downarrow \\ \{V_0(x), & V_{2k-1}(x), & V_{2k}(x)\}. \end{array}$$

Lemma 3.1. *[23] The sets of functions $\{W_i(x) : i\in\mathbb{Z}^{+}\cup 0\}$ and $\{V_i(x) : i\in\mathbb{Z}^{+}\cup 0\}$, represented by (3.2) and (3.4) respectively constitute Riesz basis for $L^2(0,1)$.*

Lemma 3.2. *[23] Let $g\in L^2(0,1)$ and*

$$a_k=\langle g(x), e^{\mu k(x-1)}\rangle, \quad b_k=\langle g(x), e^{-\mu kx}\rangle,$$

where $\mu\in\mathbb{C}$ such that $\operatorname{Re}\mu>0$ and $\langle .,.\rangle$ is defined as $\langle f,g\rangle := \int_0^1 h(x)g(x)dx$.

Then the series,

$$\sum_{k=1}^{\infty}|a_k|^2, \quad \sum_{k=1}^{\infty}|b_k|^2,$$

are convergent.

Lemma 3.3. *Let $g\in L^2(0,1)$ and $g_k=\langle g(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}\rangle$, then $\sum_{k=1}^{\infty}|g_k|^2$ converges.*

Proof. Consider $g_k=\langle g(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}\rangle$

$$\left|\left\langle g(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}\right\rangle\right|^2=\left|\left\langle g(x), \frac{e^{2\pi kx}}{e^{2\pi k}-1}\right\rangle+\left\langle g(x), \frac{e^{2\pi k(1-x)}}{e^{2\pi k}-1}\right\rangle\right|^2.$$

Taking into the account inequality $(a+b)^2\le 2(a^2+b^2)$, we may write

$$\sum_{k=1}^{\infty}\left|\left\langle g(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}\right\rangle\right|^2\le 2\sum_{k=1}^{\infty}\left|\left\langle g(x), \frac{e^{2\pi kx}}{e^{2\pi k}-1}\right\rangle\right|^2+2\sum_{k=1}^{\infty}\left|\left\langle g(x), \frac{e^{2\pi k(1-x)}}{e^{2\pi k}-1}\right\rangle\right|^2$$

$$= I_1 + I_2. \tag{3.5}$$

Consider I_1, since

$$\begin{aligned}\Big\langle g(x), \frac{e^{2\pi kx}}{e^{2\pi k}-1}\Big\rangle^2 &= \Big\langle g(x), \frac{e^{2\pi k}}{e^{2\pi k}-1} e^{2\pi k(x-1)}\Big\rangle^2, \\ &= \Big\langle g(x), \Big(1+\frac{1}{e^{2\pi k}-1}\Big) e^{2\pi k(x-1)}\Big\rangle^2, \\ &\leq 4\Big\langle g(x), e^{2\pi k(x-1)}\Big\rangle^2.\end{aligned}$$

Hence

$$I_1 \leq 8\sum_{k=1}^{\infty}\Big\langle g(x), e^{2\pi k(x-1)}\Big\rangle^2 = 8\sum_{k=1}^{\infty} b_k^2, \quad \text{where} \quad b_k = \langle g(x), e^{2\pi k(x-1)}\rangle. \tag{3.6}$$

Similarly, we have

$$I_2 \leq 8\sum_{k=1}^{\infty} a_k^2, \quad \text{where} \quad a_k = \langle g(x), e^{-2\pi kx}\rangle. \tag{3.7}$$

In the view of Lemma 3.2 and (3.5)–(3.7), we have the required result. □

Lemma 3.4. *Let $g(x) \in C^2(0,1)$ satisfying $g(0) = 0$ and $g'(0) = g'(1)$. Then the following condition holds:*

$$|g_k| \leq \frac{1}{k^2}\|g''\|_x, \quad k \in \mathbb{Z}^+.$$

Here $\|.\|_x$ represents norm in $L^2(0,1)$ and is defined by $\|g\|_x := \sqrt{\langle g, g\rangle}$.

Proof. We will prove that g_{2k-1} satisfies the desired relation. The proof of the relation for g_{2k} will follow the same lines.

Consider $g_{2k-1} = \langle g(x), V_{2k-1}(x)\rangle$,

$$g_{2k-1} = \Big\langle g(x), \Big(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1} + \sin 2\pi kx\Big)\Big\rangle.$$

Integration by part gives

$$g_{2k-1} = \frac{-2g(0)}{2\pi k} - \frac{1}{2\pi k}\Big\langle g'(x), \Big(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\Big)\Big\rangle.$$

Using the condition $g(0) = 0$ and integration by parts again yields

$$g_{2k-1} = \frac{g'(0)-g'(1)}{(2\pi k)^2}\Big(\frac{e^{2\pi k}+1}{e^{2\pi k}-1}\Big) + \frac{1}{(2\pi k)^2}\Big\langle g''(x), \Big(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \sin 2\pi kx\Big)\Big\rangle.$$

With the aid of the condition $g'(0) = g'(1)$ and Cauchy-Bunyakovsky-Schwarz Inequality (CBSI), we have

$$|g_{2k-1}| \leq \frac{1}{(2\pi k)^2}\|g''\|_x.$$

□

Lemma 3.5. *Let* $g(x) \in C^5(0,1)$ *satisfying* $g(0) = 0$, $g'(0) = g'(1)$, $g''(1) = 0 = g^{(iv)}(0)$ *and* $g'''(0) = g'''(1)$. *Then the following hold:*

$$|g_{2k-1}| \leq \frac{1}{(2\pi k)^5}\Big|\Big\langle g^{(v)}(x), \frac{e^{2\pi kx} - e^{2\pi k(1-x)}}{e^{2\pi k} - 1} - \cos 2\pi kx\Big\rangle\Big|,$$
$$|g_{2k}| \leq \frac{1}{(2\pi k)^5}\Big|\langle g^{(v)}(x), 2\sin 2\pi kx\rangle\Big|.$$

Proof. The lemma can be proved in the similar manner as Lemma 3.4. □

4. Main results

This section focuses on the key findings of our paper. We develop results concerning the existence and uniqueness of the solutions of backward source problems (1.1)–(1.4) and (1.1)–(1.3) together with (1.5).

4.1. Space dependent backward source problem

In this subsection, we will investigate the backward source problem depending on space (1.1)–(1.4). Solution is represented as infinite series in the form of MLFTs. Under certain conditions related to consistency and regularity on the provided datum, we will prove the existence and uniqueness of the solution of the backward problems (1.1)–(1.4) by using the Weierstrass M-test.

Theorem 4.1. *For* $\varsigma_m \in (0,1)$, *and*

(1) $\omega(x) \in C^2(0,1)$ *satisfying* $\omega(0) = 0$, $\omega'(0) = \omega'(1)$,
(2) $\psi(x) \in C^5(0,1)$ *satisfying* $\psi(0) = 0$, $\psi'(0) = \psi'(1)$, $\psi''(1) = 0 = \psi^{(iv)}(0)$, $\psi'''(0) = \psi'''(1)$,

the backward problems (1.1)–(1.4) possess a unique regular solution.

Proof. At first we build the solution for backward problems (1.1)–(1.4) and then in next steps we will prove its existence and uniqueness.

4.1.1. Solution construction

Due to the reason that the set $\{W_i(x) : i \in \mathbb{Z}^+ \cup \{0\}\}$ forms the Riesz basis for the space $L^2(0,1)$ (see Lemma 3.1), we can expand $v(x,t)$ and $h(x)$ as follows

$$v(x,t) = v_0(t)W_0(x) + \sum_{k=1}^{\infty}\Big(v_{2k-1}(t)W_{2k-1}(x) + v_{2k}(t)W_{2k}(x)\Big), \tag{4.1}$$

$$h(x) = h_0W_0(x) + \sum_{k=1}^{\infty}\Big(h_{2k-1}W_{2k-1}(x) + h_{2k}W_{2k}(x)\Big). \tag{4.2}$$

Substituting Eqs (4.1) and (4.2) in Eq (1.1) and using the fact that the sets $\{W_i(x) : i \in \mathbb{Z}^+ \cup \{0\}\}$ and $\{V_i(x) : i \in \mathbb{Z}^+ \cup \{0\}\}$ establish a BOSFs for the space $L^2(0,1)$, the following system of FDEs is obtained

$$\partial^{\varsigma_m}_{0_+,t}v_0(t) = h_0, \tag{4.3}$$

$$\partial^{\varsigma_m}_{0_+,t} v_{2k-1}(t) + \lambda_k v_{2k-1}(t) = h_{2k-1}, \tag{4.4}$$

$$\partial^{\varsigma_m}_{0_+,t} v_{2k}(t) + \lambda_k v_{2k}(t) = h_{2k}, \tag{4.5}$$

where $v_0(t), v_{2k-1}(t), v_{2k}(t), h_0, h_{2k-1}$, and h_{2k} are unknowns to be determined.

Using Lemma 2.1 in Eqs (4.3)–(4.5), we have

$$v_0(t) = \omega_0 \frac{t^{\zeta_0-1}}{\Gamma(\zeta_0)} + h_0 \frac{t^{\zeta_0+\zeta_1-1}}{\Gamma(\zeta_0+\zeta_1)}, \tag{4.6}$$

$$v_{2k-1}(t) = \omega_{2k-1}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k) + h_{2k-1}\ e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(t;\lambda_k), \tag{4.7}$$

$$v_{2k}(t) = \omega_{2k}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k) + h_{2k}\ e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(t;\lambda_k), \tag{4.8}$$

where $\omega_i := \langle \omega(x), V_i(x)\rangle, \quad i \in \mathbb{Z}^+ \cup \{0\}$.

Using the overdetermined condition (1.4), we obtain

$$h_0 = \frac{\Gamma(\zeta_0+\zeta_1)}{T^{\zeta_0+\zeta_1-1}}\Big(\psi_0 - \omega_0 \frac{T^{\zeta_0-1}}{\Gamma(\zeta_0)}\Big), \tag{4.9}$$

$$h_{2k-1} = \frac{\psi_{2k-1} - \omega_{2k-1}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)}, \tag{4.10}$$

$$h_{2k} = \frac{\psi_{2k} - \omega_{2k}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)}, \tag{4.11}$$

where $\psi_i := \langle \psi(x), V_i(x)\rangle, \quad i \in \mathbb{Z}^+ \cup \{0\}$.

Using (4.6)–(4.11) in Eq (4.1), we have

$$\begin{aligned} v(x,t) =& \Big(\omega_0 \frac{t^{\zeta_0-1}}{\Gamma(\zeta_0)} + \frac{t^{\zeta_0+\zeta_1-1}}{T^{\zeta_0+\zeta_1-1}}\Big(\psi_0 - \omega_0 \frac{T^{\zeta_0-1}}{\Gamma(\zeta_0)}\Big)\Big)2x \\ &+ \sum_{k=1}^{\infty}\Big(\omega_{2k-1}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k) \\ &+ \frac{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(t;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)}\Big(\psi_{2k-1} - \omega_{2k-1} e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)\Big)\Big) 2\sin 2\pi k x \\ &+ \sum_{k=1}^{\infty}\Big(\omega_{2k}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k) + \frac{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(t;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)} \\ &\Big(\psi_{2k} - \omega_{2k} e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)\Big)\Big)\Big(\frac{e^{2\pi k x} - e^{2\pi k(1-x)}}{e^{2\pi k}-1} + \cos 2\pi k x\Big). \end{aligned} \tag{4.12}$$

Using (4.9)–(4.11) in (4.2), we obtain

$$\begin{aligned} h(x) =& \Big(\frac{\Gamma(\zeta_0+\zeta_1)}{T^{\zeta_0+\zeta_1-1}}\Big(\psi_0 - \omega_0 \frac{T^{\zeta_0-1}}{\Gamma(\zeta_0)}\Big)\Big)2x \\ &+ \sum_{k=1}^{\infty}\Big(\frac{\psi_{2k-1} - \omega_{2k-1}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)}\Big)2\sin 2\pi k x \\ &+ \sum_{k=1}^{\infty}\Big(\frac{\psi_{2k} - \omega_{2k}\ e_{\zeta_0+\zeta_1-1,\zeta_0}(T;\lambda_k)}{e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1}(T;\lambda_k)}\Big)\Big(\frac{e^{2\pi k x} - e^{2\pi k(1-x)}}{e^{2\pi k}-1} + \cos 2\pi k x\Big). \end{aligned} \tag{4.13}$$

4.1.2. Solution existence

With a focus to prove that the solution exists and is regular, we demonstrate that $t^{\zeta_1}v(x,t)$, $t^{\zeta_1}v_{xxxx}(x,t)$, $t^{\zeta_1}\partial^{\varsigma_m}_{0_+,t}v(x,t)$, and $h(x)$ represent continuous functions.

On using Eq (4.9), the CBSI and the fact that $|V_0(x)| = 1$, we have

$$|h_0| \leq \frac{\Gamma(\zeta_0+\zeta_1)}{T^{\zeta_0+\zeta_1-1}}\left(\frac{T^{\zeta_0-1}}{\Gamma(\zeta_0)}\|\omega\|_x + \|\psi\|_x\right). \tag{4.14}$$

Making use of the Lemmas 2.3, 2.5, 2.6 and Eq (4.10), we obtain

$$|h_{2k-1}| \leq \frac{C_1C_2}{T^{\zeta_1}}|\omega_{2k-1}| + C_2\lambda_k|\psi_{2k-1}|. \tag{4.15}$$

Using Lemmas 3.4 and 3.5, we get

$$|h_{2k-1}| \leq \frac{C_1C_2}{(2\pi k)^2T^{\zeta_1}}\|\omega''\|_x + C_2\lambda_k\left|\frac{1}{(2\pi k)^5}\left\langle\psi^{(v)}(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\right\rangle\right|.$$

Using the fact that $ab \leq a^2 + b^2$, we have

$$\begin{aligned}|h_{2k-1}| \leq{}& \frac{C_1C_2}{k^2T^{\zeta_1}}\|\omega''\|_x + C_2\Big(\frac{1}{(2\pi k)^2} + \Big|\Big\langle\psi^{(v)}(x), \frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}\Big\rangle\Big|^2\Big)\\ &+ C_2\Big(\frac{1}{(2\pi k)^2} + \frac{1}{2}\Big|\langle\psi^{(v)}(x), \sqrt{2}\cos 2\pi kx\rangle\Big|^2\Big).\end{aligned}$$

In the view of Lemma 3.3, there exists some finite $C_3 > 0$. Thus we can write

$$\begin{aligned}|h_{2k-1}| \leq{}& \frac{C_1C_2}{k^2T^{\zeta_1}}|\omega''|_x + C_2\Big(\frac{1}{(2\pi k)^2} + C_3\Big) + C_2\Big(\frac{1}{(2\pi k)^2}\\ &+ \frac{1}{2}\Big|\langle\psi^{(v)}(x), \sqrt{2}\cos 2\pi kx\rangle\Big|^2\Big).\end{aligned}$$

Since $\{\sqrt{2}\cos 2\pi kx\}_{k=1}^{\infty}$ forms an orthonormal basis for $L^2(0,1)$, therefore, by using Bessel's inequality, we have

$$\sum_{k=1}^{\infty}|h_{2k-1}| \leq \sum_{k=1}^{\infty}\frac{C_1C_2}{k^2T^{\zeta_1}}\|\omega''\|_x + \sum_{k=1}^{\infty}\frac{C_2}{k^2} + C_2C_3 + C_2\|\psi^{(v)}\|_x^2. \tag{4.16}$$

Likewise, from (4.11), we obtain

$$\sum_{k=1}^{\infty}|h_{2k}| \leq \sum_{k=1}^{\infty}\frac{C_1C_2}{k^2T^{\zeta_1}}\|\omega''\|_x + \sum_{k=1}^{\infty}\frac{C_2}{k^2} + C_2\|\psi^{(v)}\|_x^2. \tag{4.17}$$

By means of Eqs (4.2), (4.14), (4.16) and (4.17), sum of the series majorizing $h(x)$ converges. Therefore, with the aid of Weierstrass M-test, $h(x)$ exhibits a continuous function.

By CBSI, (4.14) and the fact that $|V_0(x)| = 1$, we get

$$t^{\zeta_1}|v_0(t)| \leq \frac{t^{\zeta_0+\zeta_1-1}}{\Gamma(\zeta_0)}\|\omega\|_x + \frac{t^{\zeta_0+\zeta_1-1}}{\Gamma(\zeta_0)}\|\omega\|_x + \frac{t^{\zeta_0+\zeta_1-1}}{T^{\zeta_0-1}}\|\psi\|_x. \tag{4.18}$$

On using Lemma 2.3, Eqs (4.7) and (4.15), we have

$$|v_{2k-1}(t)| \le \frac{C_1}{\lambda_k t^{\zeta_1}}|\omega_{2k-1}| + \frac{C_1^2 C_2}{\lambda_k T^{\zeta_1}}|\omega_{2k-1}| + C_1 C_2 |\psi_{2k-1}|.$$

Using Lemma 3.4, CBSI and the fract $\frac{1}{\lambda_k} \le \frac{1}{k^4}$, we have

$$t^{\zeta_1}\sum_{k=1}^{\infty}|v_{2k-1}(t)| \le \sum_{k=1}^{\infty}\frac{C_1}{k^4}\|\omega\|_x + \sum_{k=1}^{\infty}\frac{C_1^2 C_2}{k^4}\|\omega\|_x + \sum_{k=1}^{\infty}\frac{C_1 C_2 t^{\zeta_1}}{k^2}\|\psi''\|_x. \tag{4.19}$$

Similarly,

$$t^{\zeta_1}\sum_{k=1}^{\infty}|v_{2k}(t)| \le \sum_{k=1}^{\infty}\frac{C_1}{k^4}\|\omega\|_x + \sum_{k=1}^{\infty}\frac{C_1^2 C_2}{k^4}\|\omega\|_x + \sum_{k=1}^{\infty}\frac{C_1 C_2 t^{\zeta_1}}{k^2}\|\psi''\|_x. \tag{4.20}$$

On using Eqs (4.1), (4.18)–(4.20), Lemma 3.3 and the fact $|W_k(x)| \le 2$ for $k \in \mathbb{N}$, we see that $t^{\zeta_1}v(x,t)$ is bounded above by a convergent sequence and thus is a continuous function.

Similarly for $v_{xxxx}(x,t)$, we have

$$\begin{aligned} t^{\zeta_1}|v_{xxxx}(x,t)| \le & \sum_{k=1}^{\infty}\frac{4C_1}{k^2}\|\omega''\|_x + \sum_{k=1}^{\infty}\frac{4C_1^2 C_2}{k^2}\|\omega''\|_x + \sum_{k=1}^{\infty}\frac{4C_1 C_2 t^{\zeta_1}}{k^2} + 2C_1 C_3 t^{\zeta_1} \\ & + 4C_1 t^{\zeta_1}\|\psi^{(v)}\|_x^2, \end{aligned}$$

which on using Weierstrass M-test converges.

In order to establish the continuity of $\partial_{0_+,t}^{\varsigma_m} v(x,t)$, according to the Lemma 2.2, we first need the series representations of $v(x,t)$ and $D_{0_+,t}^{\zeta_0} v(x,t)$ to be uniformly convergent. In view of (4.18)–(4.20), $v(x,t)$ is already continuous. Therefore, we only need to show the contiuity of $D_{0_+,t}^{\zeta_0} v(x,t)$.

On using the Lemma 15.2 of [13], Lemmas 2.3, 2.5, 2.6, 3.4, 3.5 and the CBSI, we have

$$\begin{aligned} t|D_{0_+,t}^{\zeta_0} v(x,t)| \le & \frac{2t^{\zeta_0}}{\Gamma(\zeta_0)}\|\omega\|_x + \frac{2T^{\zeta_0}}{\Gamma(\zeta_0)}\|\omega\|_x + 2t\|\psi\|_x + 4C_2 t\|\psi^{(v)}\|_x^2 \\ & + \sum_{k=1}^{\infty}\Big(\frac{4C_1}{k^2}\|\omega''\|_x + \frac{4C_1^2 C_2}{k^2}\|\omega''\|_x + \frac{4C_1 C_2 t}{k^2} + \frac{2C_2 C_3 t}{k^2}\Big). \end{aligned} \tag{4.21}$$

Obviously, $D_{0_+,t}^{\zeta_0} v(x,t)$ is majorized by series which are convergent and therefore it corresponds to a continuous function.

Moreover, from Eqs (4.3)–(4.5), we have the following estimates

$$t^{\zeta_1}|\partial_{0_+,t}^{\varsigma_m} v_0(t)| \le \frac{\Gamma(\zeta_0+\zeta_1)}{\Gamma(\zeta_0)}\|\omega\|_x + \frac{\Gamma(\zeta_0+\zeta_1)}{T^{\zeta_0-1}}\|\psi\|_x, \tag{4.22}$$

$$\begin{aligned} t^{\zeta_1}\Big|\sum_{k=1}^{\infty}\partial_{0_+,t}^{\varsigma_m} v_{2k-1}(t)\Big| \le & \sum_{k=1}^{\infty}\frac{C_1}{k^2}\|\omega''\|_x + (1+C_1)\Big(\sum_{k=1}^{\infty}\frac{C_1 C_2}{k^2 T^{\zeta_1}}\|\omega''\|_x \\ & + \sum_{k=1}^{\infty}\frac{C_2}{k^2} + C_2 C_3\Big)\Big) + (1+C_1)C_2\|\psi^{(v)}\|_x^2, \end{aligned} \tag{4.23}$$

$$t^{\zeta_1}\Big|\sum_{k=1}^{\infty}\partial_{0+,t}^{\varsigma_m}v_{2k}(t)\Big| \leq \sum_{k=1}^{\infty}\frac{C_1}{k^2}\|\omega''\|_x + (1+C_1)\Big(\sum_{k=1}^{\infty}\frac{C_1C_2}{k^2T^{\zeta_1}}\|\omega''\|_x + \sum_{k=1}^{\infty}\frac{C_2}{k^2}\Big) + (1+C_1)C_2\|\psi^{(v)}\|_x^2. \tag{4.24}$$

By the virtue of Lemma 2.2, i.e., $\partial_{0+,t}^{\varsigma_m}\sum_{k=i}^{\infty}g_i(t) = \sum_{k=i}^{\infty}\partial_{0+,t}^{\varsigma_m}g_i(t)$, Lemma 3.3 and (4.22)–(4.24), we notice that $t^{\zeta_1}\partial_{0+,t}^{\varsigma_m}v(x,t)$ is continuous. □

Uniqueness. Uniqueness of the pair $\{u(x,t), h(x)\}$ can be achieved by supposing two different solutions of the system (1.1)–(1.4) and along the same lines as in Theorem 4.1 of [36].
Remark. By plugging $\zeta_0 = 1-(1-\alpha)(1-\beta)$ and $\zeta_1 = 1-\beta(1-\alpha)$, where $\zeta_0, \zeta_1 \in (0,1)$, into Eqs (4.12) and (4.13), results for space dependent backward source problem of Aziz et al. [24] are recovered.

4.2. *Time dependent backward source problem*

This subsection is devoted to the study of backward source problem (1.1)–(1.3) together with integral overdetermined condition (1.5). Existence and uniqueness of solution, which like that in space dependent backward sorce problem, is represented as infinite series in the form of MLFTs, is proved using M-test proposed by Weierstrass along with fixed point theorem named after under specific consistency and regularity conditions on the provided datum.

Theorem 4.2. *For $\varsigma_m \in (0,1)$, and*

(1) $\omega(x) \in C^2(0,1)$ satisfying $\omega(0) = 0$, $\omega'(0) = \omega'(1)$.
(2) $h(\cdot,t) \in C^2(0,1)$ satisfying $h(0,t) = 0$, $h_x(0,t) = h_x(1,t)$. Furthermore

$$0 < \frac{1}{M_1} \leq \Big|\int_0^1 xh(x,t)\,dx\Big|, \quad where M_1 > 0,$$

then the backward problem (1.1)–(1.3) with the condition (1.5) has a unique regular solution.

Proof. We construct the solution of backward source problem (1.1)–(1.3) alongside integral overdetermined condition (1.5) and afterwards we demonstrate that the solution exists and is unique.

4.2.1. Solution construction

Due to the reason that the set $\{W_i(x) : i \in \mathbb{Z}^+ \cup \{0\}$ forms the Riesz basis for the space $L^2(0,1)$ (see Lemma 3.1), we can expand $v(x,t)$ and $h(x,t)$ as follows

$$v(x,t) = v_0(t)W_0(x) + \sum_{k=1}^{\infty}\Big(v_{2k-1}(t)W_{2k-1}(x) + v_{2k}(t)W_{2k}(x)\Big), \tag{4.25}$$

$$h(x,t) = h_0(t)W_0(x) + \sum_{k=1}^{\infty}\Big(h_{2k-1}(t)W_{2k-1}(x) + h_{2k}(t)W_{2k}(x)\Big). \tag{4.26}$$

Substituting Eqs (4.25) and (4.26) in Eq (1.1) and due to the reason that the sets $\{W_i(x) : i \in \mathbb{Z}^+ \cup \{0\}$ and $\{V_i(x) : i \in \mathbb{Z}^+ \cup \{0\}\}$ constitute a BOSFs for the space $L^2(0,1)$, the following system of FDEs is obtained

$$\partial_{0+,t}^{\varsigma_m}v_0(t) = a(t)h_0(t), \tag{4.27}$$

$$\partial^{\varsigma_m}_{0_+,t} v_{2k-1}(t) + \lambda_k v_{2k-1}(t) = a(t)h_{2k-1}(t), \tag{4.28}$$

$$\partial^{\varsigma_m}_{0_+,t} v_{2k}(t) + \lambda_k v_{2k}(t) = a(t)h_{2k}(t). \tag{4.29}$$

Using Lemma 2.1 in Eqs (4.27)–(4.29), we get

$$v_0(t) = \frac{t^{\zeta_0-1}}{\Gamma(\zeta_0)}\omega_0 + \frac{t^{\zeta_0+\zeta_1-2}}{\Gamma(\zeta_0+\zeta_1-1)} * a(t)h_0(t), \tag{4.30}$$

$$v_{2k-1}(t) = e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\omega_{2k-1} + e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1-1}(t;\lambda_k) * a(t)h_{2k-1}(t), \tag{4.31}$$

$$v_{2k}(t) = e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\omega_{2k} + e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1-1}(t;\lambda_k) * a(t)h_{2k}(t), \tag{4.32}$$

where $*$ denotes the Laplace convolution.

Using (4.30)–(4.32) in (4.25), we obtain

$$\begin{aligned} v(x,t) =& \Big(\frac{t^{\zeta_0-1}}{\Gamma(\zeta_0)}\omega_0 + \frac{t^{\zeta_0+\zeta_1-2}}{\Gamma(\zeta_0+\zeta_1-1)} * a(t)h_0(t)\Big)2x \\ &+ \sum_{k=1}^{\infty}\Big(e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\omega_{2k-1} \\ &+ e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1-1}(t;\lambda_k) * a(t)h_{2k-1}(t)\Big)2\sin 2\pi kx \\ &+ \sum_{k=1}^{\infty}\Big(e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\omega_{2k} + e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1-1}(t;\lambda_k) * a(t)h_{2k}(t)\Big) \\ &\Big(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1} + \cos 2\pi kx\Big). \end{aligned} \tag{4.33}$$

Using (1.5) and Eq (1.1) results in the following Volterra integral equation

$$a(t) = \Big(\int_0^1 xh(x,t)dx\Big)^{-1}\Big(\partial^{\varsigma_m}_{0+,t}\mathcal{E}(t) + \Lambda(t) + \int_0^t \Upsilon(t,\tau)a(\tau)d\tau\Big), \tag{4.34}$$

where

$$\Lambda(t) = \sum_{k=1}^{\infty}\lambda_k\ e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\Big(-\frac{1}{\pi k}\omega_{2k-1} + \Big(-\frac{1}{2\pi^2k^2} + \frac{e^{2\pi k}+1}{2\pi k(e^{2\pi k}-1)}\Big)\omega_{2k}\Big),$$

$$\Upsilon(t,\tau) = \sum_{k=1}^{\infty}\lambda_k\ e_{\zeta_0+\zeta_1-1,\zeta_0+\zeta_1-1}(t-\tau;\lambda_k)\Big(-\frac{1}{\pi k}h_{2k-1}(\tau) + \Big(-\frac{1}{2\pi^2k^2} + \frac{e^{2\pi k}+1}{2\pi k(e^{2\pi k}-1)}\Big)h_{2k}(\tau)\Big).$$

We introduce $\wp(a(t)) := a(t)$, where

$$\wp(a(t)) = \Big(\int_0^1 xh(x,t)dx\Big)^{-1}\Big(\partial^{\varsigma_m}_{0+,t}\mathcal{E}(t) + \Lambda(t) + \int_0^t \Upsilon(t,\tau)a(\tau)d\tau\Big).$$

We aim to show that $\wp : C([0,T]) \to C([0,T])$ is a contraction map. In the first place, we will show that $\wp(a(t)) \in C([0,T])$ for $a(t) \in C([0,T])$, i.e., the series involved in $\Lambda(t)$ and $\Upsilon(t,\tau)$ exhibit the uniform convergence.

Consider

$$|\Lambda(t)| = \Big|\sum_{k=1}^{\infty} \lambda_k \; e_{\zeta_0+\zeta_1-1,\zeta_0}(t;\lambda_k)\Big(-\frac{1}{\pi k}\omega_{2k-1} + \Big(-\frac{1}{2\pi^2 k^2} + \frac{e^{2\pi k}+1}{2\pi k(e^{2\pi k}-1)}\Big)\omega_{2k}\Big)\Big|.$$

Using Lemma 2.3 and triangular inequality, we have

$$|\Lambda(t)| \leq \sum_{k=1}^{\infty}\Big(\frac{C_1}{\pi k t^{\zeta_1}}|\omega_{2k-1}| + \Big(\frac{1}{2\pi^2 k^2} + \frac{1}{\pi k}\Big)\frac{C_1}{t^{\zeta_1}}|\omega_{2k}|\Big).$$

Using Lemma 3.4, we have

$$t^{\zeta_1}|\Lambda(t)| \leq \sum_{k=1}^{\infty}\Big(\frac{2C_1}{k^3}\|\omega''\|_x + \frac{C_1}{k^2}\|\omega\|_x\Big). \tag{4.35}$$

Similarly,

$$(t-\tau)|\Upsilon(t,\tau)| \leq \sum_{k=1}^{\infty}\Big(\frac{2C_1}{k^3}\|h_{xx}(x,t)\|_{x,t} + \frac{C_1}{k^2}\|h(x,t)\|_t\Big). \tag{4.36}$$

With aid of the M-Test proposed by Weierstrass, the series in (4.35) and (4.36) are convergent. Therefore, we have $\wp(a(t)) \in C([0,T])$ for $a(t) \in C([0,T])$. Therefore,

$$\|\Upsilon(t,\tau)\|_{t\times t} \leq K, \quad K > 0. \tag{4.37}$$

Now,

$$|\wp(a) - \wp(b)| \leq \Big(\int_0^1 xh(x,t)dx\Big)^{-1}\int_0^t |a(\tau) - b(\tau)|\; |\Upsilon(t,\tau)|d\tau,$$

on the basis of inequality (4.37), we have

$$|\wp(a) - \wp(b)| \leq TKM_1 \max_{0\leq t\leq T}|a(\tau) - b(\tau)|.$$

Accordingly, we have

$$\|\wp(a) - \wp(b)\|_t \leq TKM_1\|a-b\|_t.$$

Making use of the fixed-point theorem, we see that $\wp(.)$ is a contraction for $T < 1/KM_1$. This guarantees that $a(.) \in C([0,T])$ is uniquely determined.

4.2.2. Solution existence

In order to demonstrate that the solution of backward source problems (1.1)–(1.3) with integral overdetermined data (1.5) exists, we need demonstrate that the series representations of $a(t)$, $t^{\zeta_1}v(x,t)$, $t^{\zeta_1}v_{xxxx}(x,t)$ and $t^{\zeta_1}\partial_{0_+,t}^{\varsigma_m}v(x,t)$ are continuous functions. Since $a(t)$ already being proved to be continuous using Banach fixed point theorem, therefore, we only need to prove that the rest of the series represent continuous functions.

As $a(t) \in C([0,T])$, we can find M_2 such that $|a(t)| \leq M_2$. Therefore, by the virtue of Lemmas 2.3, 3.4, 3.5, CBSI and the fact $\frac{1}{\lambda_k} \leq \frac{1}{k^4}$, estimates for $t^{\zeta_1} v(x,t)$, $t^{\zeta_1} v_{xxxx}(x,t)$, $t^{\zeta_1}\partial^{\varsigma_m}_{0_+,t} v(x,t)$ are as under

$$\begin{aligned}
t^{\zeta_1}|v(x,t)| \leq & \frac{2t^{\zeta_0+\zeta_1-1}}{\Gamma(\zeta_0)}\|\omega\|_x + \frac{2M_2 t^{\zeta_0+2\zeta_1-1}}{\Gamma(\zeta_0+\zeta_1)}\|h(x,t)\|_{x,t} \\
& + \sum_{k=1}^{\infty}\Big(\frac{4C_1}{k^4}\|\omega\|_x + \frac{4C_1 M_2 t^{\zeta_1}}{k^4}\|h(x,t)\|_{x,t}\Big), \\
t^{\zeta_1}|v_{xxxx}(x,t)| \leq & \sum_{k=1}^{\infty}\Big(\frac{4C_1}{k^2}\|\omega''\|_x + \frac{4C_1 M_2 t^{\zeta_1}}{k^2}\|h_{xx}(x,t)\|_{x,t}\Big), \\
t^{\zeta_1}|\partial^{\varsigma_m}_{0_+,t} v(x,t)| \leq & 2M_2 t^{\zeta_1}\|h(x,t)\|_{x,t} + \sum_{k=1}^{\infty}\Big(\frac{4C_1}{k^2}\|\omega''\|_x \\
& + \frac{4(1+C_1)M_2 t^{\zeta_1}}{k^2}\|h_{xx}(x,t)\|_{x,t}\Big),
\end{aligned}$$

respectively, which on using Weierstrass M-test represent the continuous functions. □

Uniqueness. It has been already proved that the source term $a(t)$ is unique through the use of fixed-point theorem. The uniqueness of $u(x,t)$ of the system can be achieved in the same way as in Ali et al. [27].

Remark. The results for time dependent backward source problem of Aziz et al. [24] can be recovered by substituting $\zeta_0 = 1-(1-\alpha)(1-\beta)$ and $\zeta_1 = 1-\beta(1-\alpha)$, where $\zeta_0, \zeta_1 \in (0,1)$, in Eqs (4.33) and (4.34).

5. Concluding remarks

Samarskii-Ionkin type problems for a fourth-order FDE involving FDNO have been examined. Backward problems of recovering source terms depending on space and time dependent from final temperature distribution and integral type overdetermination datum respectively, have been the highlights of the paper. Solutions of both the backward source problems are expressed as infinite series in the form of MLFTs. Under certain conditions of consistency and regularity on the given data, existence and uniqueness of the backward source problems are demonstrated by using Laplace transform method, methods of the theory of integral equations and spectral method. Furthermore the general results of this paper contains as a particular case the ones from Aziz et al. [24] related to Hilfer operator.

Conflict of interest

The authors declare no conflict of interest.

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