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Research article

A novel scheme of k-step iterations in digital metric spaces

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Abstract: In computational mathematics, the comparison of convergence rate in different iterative methods is an important concept from theoretical point of view. The importance of this comparison is relevant for researchers who want to discover which one of these iterations converges to the fixed point more rapidly. In this article, we study the different numerical methods to calculate fixed point in digital metric spaces, introduce a new k-step iterative process and conduct an analysis on the strong convergence, stability and data dependence of the mentioned scheme. Some illustrative examples are given to show that this iteration process converges faster.

Keywords: digital metric space; iterative method; fixed point schemes
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1. Introduction

Fixed point theory is considered as one of the most important fields of pure mathematics and is observed in different aspects of applied mathematics, particulary in existence theory and mathematical modeling [1–12]. Recently, the researchers have tried touse ideas of digital images based on the Euclidean topology, along with real analysis in relation to the existing metrics and fixed points. While the fundamental and main motivation of a digital metric space is taken from a Euclidean metric space. A digital metric space is denoted by (E,μ,ρ), where E denotes a family of lattice points, μ is a Euclidean metric, and ρ specifies an adjacency relation on E, so that makes (E,ρ) as a graph.

In 1922, theory of fixed points was started by introducing the Banach contraction principle stated by S. Banach, which guarantees the existence of unique fixed points for a contraction. For digital images, Ege and Karaca [13–16] introduced a digital metric space and stated a well-known Banach contraction principle for the existence of unique fixed points. However, when the existence results are established for fixed points of a function, it is not a simple work to find the value of those fixed points. Therefore, application of iterative processes is a logical method to compute these fixed points. Until now, a vast range of iterative processes has been defined. As we know, the Picard iteration process is used in the famous Banach contraction principle to approximate fixed points of the given contractions. But this iterative method may fail to converge in the case of nonexpansive mappings, even if T has exactly one fixed point. This iterative scheme is defined as

$$Tx_n = x_{n+1}.$$

Krasnosel'skii [17] established and confirmed that the Mann iterative scheme [18] can give an approximation of the fixed points for an existing nonexpansive function. In such an iterative scheme, by considering an arbitrary $x_\circ \in X$, the sequence $\{x_n\}$ is generated as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \qquad \forall\, n \geq 0,$$

where $\alpha_n \in [0, 1]$.

Ishikawa [19] extended an iterative algorithm in 1974 for approximation of the fixed point, in which $\{x_n\}$ is given iteratively by starting from $x_0 \in X$ as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned}$$

for all $n \geq 0$ with $\alpha_n, \beta_n \in [0, 1]$.

These two iterative methods (i.e., the Mann and Ishikawa algorithms) have been investigated by many researchers to approximate the fixed points. Another iterative technique was proposed by Noor [20], for initial point $x_0 \in X$, and $\{x_n\}$ has the form as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{1.1}$$

for all $n \geq 0$ with $\alpha_n, \beta_n, \gamma_n \in [0, 1]$.

In 2007, Agarwal et al. [21] suggested another iterative scheme known as Agarwal iteration algorithm or S-iteration process: for arbitrary $x_0 \in X$, $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)T x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{aligned}$$

with $0 \leq \alpha_n, \beta_n \leq 1$. They showed that, for contraction mappings, the mentioned algorithm converges faster than Mann algorithm.

A few years later, Gursoy and Karakaya in [22] presented a new combined algorithm entitled the Picard S-iteration algorithm, in which by taking $x_0 \in X$ arbitrarily, $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)Tx_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 0, \end{aligned}$$

with $\alpha_n, \beta_n, \gamma_n \in [0, 1]$. These authors proved that this combined iterative method can be used for approximation of fixed points of contractions. Moreover, by providing an example, they guaranteed that the Picard S-iteration algorithm converges faster than all of four previous algorithms, i.e., Mann, Ishikawa, Noor and S schemes.

After that, Thakur et al. [23] dealt with another iteration algorithm entitled Thakur new iteration process, by taking arbitrary $x_0 \in X$ and

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= T((1-\alpha_n)x_n + \alpha_n z_n), \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 0, \end{aligned}$$

with $\alpha_n, \beta_n, \gamma_n \in [0, 1]$. By giving a numerical example in relation to the Suzuki generalized nonexpansive mappings, Thakur et al. confirmed that their iterative algorithm is faster than Picard, Mann, Ishikawa, Agarwal and Noor algorithms.

Again Thakur et al. [24] generalized the following iterative procedure in 2016, in which $\{x_n\}$ is formulated from arbitrary point $x_0 \in X$ by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tz_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n, \quad n \geq 0, \end{aligned}$$

with $\alpha_n, \beta_n, \gamma_n \in [0, 1]$.

Ullah et al. [25] introduced a new kind of iterative algorithm called the $k^\star$-iteration algorithm in 2018. It converges faster than Thakur's combined algorithm. In this scheme, $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= T((1-\alpha_n)z_n + \alpha_n Tz_n), \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 0, \end{aligned}$$

by taking $x_0 \in X$ arbitrarily and $\alpha_n, \beta_n, \gamma_n \in [0, 1]$. With the aid of an example, Ullah and Arshad confirmed that $k^\star$-iteration scheme is faster than S and Picard S-iteration algorithms.

Our main aim in this article is to derive a novel and more rapid iteration algorithm than $k^\star$-iteration scheme in the framework of a digital metric space. Moreover, we analyze the convergence rate for such an iterative scheme in the context of digital metric spaces. Along with these, we establish that our suggested algorithm is stable analytically. Finally, by providing some examples, from a numerical point of view, we shall compare the convergence rate of our iterative scheme with other well-known iterative algorithms.

2. Preliminaries

In this section, to prove the main results, we need some definitions and proposition. The following lemmas will help us to investigate our problem in this specific structure.

Lemma 2.1. *[26] Let $c \in \mathbb{R}$ such that $0 \leq c < 1$, and $\{\zeta_n\}$ be a sequence of positive numbers with $\lim\limits_{n \longrightarrow \infty} \zeta_n = 0$ for all $n \geq 0$. Then for every sequence $\{\kappa_n\}$ of positive numbers satisfying*

$$\kappa_{n+1} \leq c(\kappa_n) + \zeta_n,$$

we have $\lim\limits_{n \to \infty} \kappa_n = 0$.

Definition 2.2. *[26] Let $\{a_n\} \subseteq \mathbb{R}$ and $\{b_n\} \subseteq \mathbb{R}$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then $\{a_n\}$ converges faster than $\{b_n\}$ whenever*

$$\lim_{n \to \infty} |\frac{a_n - a}{b_n - b}| = 0.$$

Definition 2.3. *[26] Consider $\{\kappa_n\}$ and $\{\nu_n\}$ as two iterations of fixed points that tend to some fixed point p on a metric space (X, d) so that we have the following error estimates*

$$d(\kappa_n, p) \leq a_n, \qquad d(\nu_n, p) \leq b_n,$$

where $\{a_n\} \subseteq \mathbb{N}$ and $\{b_n\} \subseteq \mathbb{N}$ converge to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that $\{\kappa_n\}$ tends to p more rapid than $\{\nu_n\}$.

Definition 2.4. *Let (E, μ, ρ) be a digital metric space and $D : E \times [0, 1] \to E$ be a map such that $D(p, \alpha + \beta) = D(p, \alpha) + D(p, \beta)$ and $D(p, 1) = p$. We say that a digital metric space have linear digital structure if for all $p, q, r, s \in E$ and $\alpha, \beta \in [0, 1]$,*

$$\mu(D(p, \alpha) + D(q, \beta), D(r, \alpha) + D(s, \beta)) \leq \alpha\mu(p, r) + \beta\mu(q, s). \tag{2.1}$$

Definition 2.5. *Let (E, μ, ρ) be a digital metric space, T a self-map on E, and $F_T = \{p \in E | T(p) = p\}$ a set of fixed points of T. Moreover, consider $\{x_n\}$ as a sequence produced by an iteration scheme under T given as*

$$x_{n+1} = f_{T,\alpha_n}(x_n), \tag{2.2}$$

where $x_0 \in X$ is considered as the initial approximation, $\alpha_n \in [0, 1]$ and f is a function involving the digital structure. Let $\{x_n\}$ tends to the fixed point p of T and $\epsilon_n = \mu(x_{n+1}, f_{T,\alpha_n}(x_n))$, $n \geq 0$. In this case, we say that the above iteration algorithm is T-stable (or stable with respect to T) if and only if $\lim\limits_{n \longrightarrow \infty} \epsilon_n = 0$ gives $\lim\limits_{n \longrightarrow \infty} x_n = 0$.

Now, in a digital metric space, we can rewrite the Mann iteration as:

$$x_{n+1} = D(x_n, (1 - \alpha_n)) + D(T(x_n), \alpha_n), \alpha_n \in [0, 1].$$

and the Ishikawa iteration as:

$$x_{n+1} = D(x_n, (1 - \alpha_n)) + D(T(y_n), \alpha_n), \alpha_n \in [0, 1],$$

$$y_n = D(x_n, (1-\beta_n)) + D(T(x_n), \beta_n), \beta_n \in [0,1].$$

Also, we can write the S-iteration process as follows:

$$x_{n+1} = D(Tx_n, (1-\alpha_n)) + D(T(y_n), \alpha_n), \alpha_n \in [0,1],$$

$$y_n = D(x_n, (1-\beta_n)) + D(T(x_n), \beta_n), \beta_n \in [0,1],$$

and, the Picard S-iteration process as follows

$$x_{n+1} = T(y_n),$$

$$y_n = D(T\ x_n, (1-\alpha_n)) + D(T(z_n), \alpha_n), \alpha_n \in [0,1],$$

$$z_n = D(x_n, (1-\beta_n)) + D(T(x_n), \beta_n), \beta_n \in [0,1].$$

Also, in a digital metric space, the $k^\star$-iteration algorithm introduced by Ullah et al. can be written as:

$$x_{n+1} = T(y_n),$$

$$y_n = T\ (D(z_n, (1-\alpha_n)) + D(T(z_n), \alpha_n)), \alpha_n \in [0,1],$$

$$z_n = D(x_n, (1-\beta_n)) + D(T(x_n), \beta_n), \beta_n \in [0,1].$$

3. Gerneralized $k^\star$-iteration process

Motivated by this fact that three-step iterative schemes give better approximation than two-step ones and also two-step iterative schemes give better approximations than one-step schemes, we present a generalized k-step iterative algorithm. Let $K \neq \emptyset$ be a set in the digital metric space (E, μ, ρ) with linear digital structure so that $E \subset Z^n$, where $n \in \mathbb{N}$ and ρ stands for an adjacency relation between the members of E. Moreover, let $T : K \to K$ be an arbitrary map. For the real sequences $\{\alpha_n^{(1)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(2)}\}_{n=0}^{\infty}$, $\{\alpha_n^{(3)}\}_{n=0}^{\infty}, \ldots, \{\alpha_n^{(k)}\}_{n=0}^{\infty} \in [0,1]$, we define a generalized k-step iteration ($k^\star$) as

$$\begin{aligned}
x_{n+1}^{(1)} &= T\ x_n^{(2)},\\
x_n^{(2)} &= T((1-\alpha_n^{(1)})x_n^{(3)} + \alpha_n^{(1)}T\ x_n^{(3)}),\\
x_n^{(3)} &= T((1-\alpha_n^{(2)})x_n^{(4)} + \alpha_n^{(2)}T\ x_n^{(4)}),\\
x_n^{(4)} &= T((1-\alpha_n^{(3)})x_n^{(5)} + \alpha_n^{(3)}T\ x_n^{(5)}),\\
&\ldots\\
&\ldots\\
&\ldots\\
x_n^{(k-1)} &= T((1-\alpha_n^{(k-2)})x_n^{(k)} + \alpha_n^{(k-2)}T\ x_n^{(k)}),\\
x_n^{(k)} &= (1-\alpha_n^{(k-1)})x_n^{(1)} + \alpha_n^{(k-1)}T\ x_n^{(1)}.
\end{aligned} \tag{3.1}$$

By setting $k = 1$, we obtain the Picard iteration (1.1). $k = 3$ gives the $k^\star$-iteration introduced by Ullah and Arshad. Now in digital metric spaces, we can represent (3.1) as

$$x_{n+1}^{(1)} = T\ x_n^{(2)},$$

$$x_n^{(2)} = T(D(x_n^{(3)}, (1 - \alpha_n^{(1)})) + D(T\ x_n^{(3)}), \alpha_n^{(1)})),$$
$$x_n^{(3)} = T(D(x_n^{(4)}, (1 - \alpha_n^{(2)})) + D(T\ x_n^{(4)}, \alpha_n^{(2)})),$$
$$...$$
$$...$$
$$...$$
$$x_n^{(k)} = D(x_n^{(1)}, (1 - \alpha_n^{(k-1)})) + D(T\ x_n^{(1)}, \alpha_n^{(k-1)}). \tag{3.2}$$

4. Main results

We prove the main results in two subsections.

4.1. Convergence and stability results of gerneralized $k^\star$-iteration

Theorem 4.1. *Let (E, μ, ρ) be a digital metric space having the linear digital structure D and $T : E \longrightarrow E$ be a contraction with $F(T) \neq \phi$. Then, for $x_0 \in E$, $\{x_n\}$ given by* (3.2)*, tends to the fixed point of T.*

Proof. Let $\{x_n\} \subseteq E$ and $p \in F(T)$ such that

$$\mu(x_{n+1}^{(1)}, p) = \mu(T\ x_n^{(2)}, T\ p) \leq \delta\mu(x_n^{(2)}, p).$$

Now, we have

$$\begin{aligned}
\mu(x_n^{(2)}, p) &= \mu(T[D(x_n^{(3)}, 1 - \alpha_n^{(1)}) + D(T\ x_n^{(3)}, \alpha_n^{(1)})], T\ p) \\
&\leq \delta\mu(D(x_n^{(3)}, 1 - \alpha_n^{(1)}) + D(T\ x_n^{(3)}, \alpha_n^{(1)}), p) \\
&= \delta[\mu(D(x_n^{(3)}, 1 - \alpha_n^{(1)}) + D(T\ x_n^{(3)}, \alpha_n^{(1)}), D(p, 1))] \\
&= \delta[\mu(D(x_n^{(3)}, (1 - \alpha_n^{(1)}) + D(T\ x_n^{(3)}), \alpha_n^{(1)})), D(p, (1 - \alpha_n^{(1)} + (\alpha_n^{(1)})] \\
&\leq \delta[\mu(D(x_n^{(3)}, (1 - \alpha_n^{(1)}) + D(T\ x_n^{(3)}), \alpha_n^{(1)})), D(p, (1 - \alpha_n^{(1)}) + D(p, \alpha_n^{(1)})].
\end{aligned}$$

Using the linear structure property, we get

$$\mu(x_n^{(2)}, p) \leq \delta[(1 - (1 - \delta)\alpha_n^{(1)}]\mu(x_n^{(3)}, p). \tag{4.1}$$

Similarly

$$\mu(x_n^{(3)}, p) \leq \delta[(1 - (1 - \delta)\alpha_n^{(2)}]\mu(x_n^{(4)}, p), \tag{4.2}$$

$$\mu(x_n^{(k-1)}, p) \leq \delta[(1 - (1 - \delta)\alpha_n^{(k-2)}]\mu(x_n^{(k)}, p), \tag{4.3}$$

and for the final term $\mu(x_n^{(k)}, p)$, it becomes

$$\mu(x_n^{(k)}, p) \leq [(1 - (1 - \delta)\alpha_n^{(k-1)})]\mu(x_n^{(1)}, p). \tag{4.4}$$

Using the above equations, we have

$$\mu(x_{n+1}^{(1)}, p) \leq \delta^{k-1}\,[1-(1-\delta)\alpha_n^{(1)}]$$
$$\times [1-(1-\delta)\alpha_n^{(2)})]\,[1-(1-\delta)\alpha_n^{(3)}]...[1-(1-\delta)\alpha_n^{(k-1)}]\mu(x_n^{(1)}, p),$$

and so

$$\mu(x_{n+1}^{(1)}, p) \leq \ \delta^{k-1}\prod_{i=1}^{k}[1-(1-\delta)\alpha_n^{(i)}]\mu(x_n^{(1)}, p). \tag{4.5}$$

Now for $i = 1, 2, 3..., k$,

$$[1-(1-\delta)\alpha_n^{(k)}] \leq 1$$
$$\Rightarrow \delta^{k-1}\prod_{i=1}^{k-1}[1-(1-\delta)\alpha_n^{(i)}] \leq 1.$$

Hence, (4.5) yields $\lim_{n\to\infty}\mu(x_n^{(1)}, p) = 0$. Therefore, sequence $\{x_n\}$ converges to p. □

Next, we establish that the gerneralized $k^{\star}$-iteration algorithm converges more rapid than all aforesaid iterative algorithms and it is T-stable.

Theorem 4.2. *Let (E, μ, ρ) be a digital metric space having the linear digital structure D and $T : E \longrightarrow E$ be a contraction. Let T has a fixed point p. For $x_0 \in E$, $\{x_n\}$ defined iteratively by* (3.2) *be the gerneralized $k^{\star}$ iterative process, where $\alpha_n^{(k)} \in [0, 1]$ such that $\alpha_n^{(k)} < \alpha < 1$. Then, the $k^{\star}$-iteration is T-stable.*

Proof. Suppose that $\{x_n\} \subseteq E$ is defined by (3.2) and $\epsilon_n = \mu(x_{n+1}^{(1)}, Tx_n^{(2)})$, and $\lim\limits_{n\to\infty}\epsilon_n = 0$. Then, we show $\lim\limits_{n\to\infty} x_n = p$. We have

$$\begin{aligned}\mu(x_{n+1}^{(1)}, p) &\leq \mu(x_{n+1}^{(1)}, Tx_n^{(2)}) + \mu(Tx_n^{(2)}, p)\\ &= \mu(Tx_n^{(2)}, T\ p) + \epsilon_n\\ &\leq \delta\mu(x_n^{(2)}, p) + \epsilon_n.\end{aligned}$$

Using (4.1), we get

$$\mu(x_{n+1}^{(1)}, p) \leq \delta^2[(1-(1-\delta)\alpha_n^{(1)}]\mu(x_n^{(3)}, p) + \epsilon_n.$$

Similarly, using (4.2) and so on to (4.4), we get

$$\mu(x_{n+1}^{(1)}, p) \leq \delta^{k-1}\prod_{i=1}^{k-1}[1-(1-\delta)\alpha_n^{(i)}]\mu(x_n^{(1)}, p) + \epsilon_n.$$

Therefore, since $0 < \delta^{k-1}\prod_{i=1}^{k-1}[1-(1-\delta)\alpha_n^{(i)}] < 1$, by applying Lemma 2.1, we get $\lim_{n\longrightarrow\infty}\mu(x_n, p) = 0$, i.e., $\lim_{n\longrightarrow\infty} x_n = p$.

Conversely, let $\lim_{n\longrightarrow\infty} x_n = p$. Then we have to prove that $\lim_{n\longrightarrow\infty}\epsilon_n = 0$.

We have

$$\epsilon_n = \mu(x_{n+1}^{(1)}, Tx_n^{(2)})$$

$$\leq \mu(x_{n+1}^{(1)}, p) + \mu(p, T x_n^{(2)})$$
$$\leq \mu(x_{n+1}^{(1)}, p) + \delta\mu(p, x_n^{(2)}).$$

Using (4.1), we get

$$\epsilon_n \leq \mu(x_{n+1}^{(1)}, p) + \delta^2[(1-(1-\delta)\alpha_n^{(1)}]\mu(x_n^{(3)}, p).$$

Using (4.2), (4.3) and (4.4), we finally have

$$\epsilon_n \leq \mu(x_{n+1}^{(1)}, p) + \delta^{k-1}\prod_{i=1}^{k-1}[1-(1-\delta)\alpha_n^{(i)}]\mu(x_n^{(1)}, p)) \longrightarrow 0,$$

as $n \longrightarrow \infty$. □

4.2. Convergence rate

Theorem 4.3. *Let (E, μ, ρ) be a digital metric space having the linear digital structure D and $T : E \longrightarrow E$ be a contraction. Let T has a fixed point p. For $x_0 \in E$, $\{x_n\}$ defined iteratively by* (3.2) *be the gerneralized $k^\star$ iterative process, where $\alpha_n^{(k)} \in [0,1]$ such that $\alpha_n^{(k)} < \alpha < 1$. Then, the gerneralized $k^\star$ iteration converges to p faster than the Ullah and Arshad iteration. Also, it converges more rapid than the explicit Mann and Ishikawa algorithms.*

Proof. For the Ullah and Arshad $k^\star$-iteration process, we have

$$\mu(x_{n+1}^{(1)}, p) = \mu(T\ x_n^{(2)}, T\ p) \leq \delta\mu(x_n^{(2)}, p),$$
$$\mu(x_n^{(2)}, p) \leq \delta[(1-(1-\delta)\alpha_n^{(1)}]\mu(x_n^{(3)}, p).$$

Similarly

$$\mu(x_n^{(3)}, p) \leq\ [(1-(1-\delta)\alpha_n^{(2)}]\mu(x_n^{(4)}, p).$$

Using the above equations, we have

$$\mu(x_{n+1}^{(1)}, p) \leq \delta^2\ [1-(1-\delta)\alpha_n^{(1)}]\ [1-(1-\delta)\alpha_n^{(2)})]\ [1-(1-\delta)\alpha_n^{(3)}]\mu(x_n^{(1)}, p),$$

$$\mu(x_{n+1}^{(1)}, p) \leq \delta^2\prod_{i=1}^{3}[1-(1-\delta)\alpha_n^{(i)}]\mu(x_n^{(1)}, p), \tag{4.6}$$

and for the gerneralized $k^\star$ iterative process, we have

$$\mu(x_{n+1}^{(1)}, p) \leq\ \delta^{k-1}\prod_{i=1}^{k}[1-(1-\delta)\alpha_n^{(i)}]\mu(x_n^{(1)}, p). \tag{4.7}$$

Now

$$\delta^2\prod_{i=1}^{3}[1-(1-\delta)\alpha_n^{(i)}] < \delta^{k-1}\prod_{i=1}^{k}[1-(1-\delta)\alpha_n^{(i)}]. \tag{4.8}$$

By considering the Berinde's definitions, (2.1) and (2.2), inequalities (4.6), (4.7) and (4.8) yield that gerneralized $k^\star$ iteration converges faster than the Ullah and Arshad $k^\star$ iteration. Now for the explicite Mann iteration, we get

$$\begin{aligned}\mu(x_{n+1}^{(1)}, p) &= \mu(D(x_n^{(1)}, 1-\alpha_n^{(1)}) + D(T\ x_n^{(1)}, \alpha_n^{(1)}), D(p, 1))\\ &= \mu(D(x_n^{(1)}, (1-\alpha_n^{(1)}) + D(T\ x_n^{(1)}), \alpha_n^{(1)})), D(p, (1-\alpha_n^{(1)} + (\alpha_n^{(1)}))\\ &\leq \delta[\mu(D(x_n^{(1)}, (1-\alpha_n^{(1)}) + D(T\ x_n^{(1)}), \alpha_n^{(1)})), D(p, (1-\alpha_n^{(1)}) + D(p, \alpha_n^{(1)})].\end{aligned}$$

Using the linear structure property, we get

$$\mu(x_{n+1}^{(1)}, p) \leq [(1-(1-\delta)\alpha_n^{(1)}]\mu(x_n^{(1)}, p). \tag{4.9}$$

Similarly, for the explicit Ishikawa iteration, we have

$$\mu(x_{n+1}^{(1)}, p) \leq [(1-(1-\delta)\alpha_n^{(1)}((1-(1-\delta)\alpha_n^{(2)})]\mu(x_n^{(1)}, p). \tag{4.10}$$

Now, the inequalities (4.7), (4.9) and (4.10) follow that the gerneralized $k^\star$ iteration converges more rapid than the explicit Mann and Ishikawa algorithms. □

In this position, we design an example to compare the convergence rate of our iterative algorithm with three other schemes such as Mann, Picard-S, and Noor. The convergence comparison is presented in some tables.

Example 4.4. *Consider $X = \{0, 1, 2, ...\}$ and the digital metric space (X, μ, ρ) equipped with the digital metric given by $d(x, y) = |x - y|$. For $T : (X, \mu, \rho) \to (X, \mu, \rho)$, define*

$$Tx = \frac{x}{2} + 3,$$

and $\alpha_n^{(1)} = \alpha_n^{(2)} = \alpha_n^{(3)} ... = \alpha_n^{(k)} = \frac{5}{6}$, $n = 1, 2, 3, \ldots$. From Table 1, we can observe that all the iterative algorithms converge to $p^\star = 6$. Evidently, our suggested iterative algorithm needs the least number of iterations as compared to other existing algorithms.

Table 1 is presented to show the number of iterations required by different schemes. From Tables 1–3, it is clear that our purposed method is more rapidly convergent compared to other scehmes.

Table 1. Number of iterations attaining the fixed point.

Scheme	number of iterations
Mann	48
Picard-S	15
Ishikawa	32
Noor	26
Agarwal	28
K.Ullah$K^\star(k = 3)$	11
$k = 4$	7
$k = 5$	6

Table 2. Convergence comparison with other schemes w.r.t number of iterations.

x_n	Picard-S	Noor	$k=3$	$k=4$	$k=5$
x_0	0	0	0	0	0
x_1	5.020833333333334	3.97569444444	5.4895833333	5.8511284722	5.9565791377
x_2	5.840205439814815	5.31703116962	5.9565791377	5.9963062114	5.9996857715
x_3	5.973922415525335	5.76957706707	5.9963062114	5.9999083500	5.9999977260
x_4	5.995744283089204	5.92225892946	5.9996857715	5.9999977260	5.9999999835
x_5	5.999305490643030	5.97377138650	5.9999732688	5.9999999436	5.9999999999
x_6	5.999886659931327	5.99115087866	5.9999977260	5.9999999986	6.0000000000
x_7	5.999981503530460	5.99701444575	5.9999998066	6.0000000000	.
x_8	5.999996981478930	5.99899272099	5.9999999835	.	.
x_9	5.999999507394131	5.99966015992	5.9999999986	.	.
x_{10}	5.999999919609460	5.99988534331	5.9999999999	.	.
x_{11}	5.999999986880711	5.99996131664	6.0000000000	.	.
x_{12}	5.999999997859005	5.99998694884	.	.	.
x_{13}	5.999999999650601	5.99999559674	.	.	.
x_{14}	5.999999999942980	5.99999851441	.	.	.
x_{15}	5.999999999990695	5.99999949879	.	.	.
x_{16}	5.999999999998481	5.99999983090	.	.	.
x_{17}	5.999999999999752	5.99999994295	.	.	.
x_{18}	5.999999999999959	5.99999998075	.	.	.
x_{19}	5.999999999999993	5.99999999351	.	.	.
x_{20}	5.999999999999999	5.99999999781	.	.	.
x_{21}	6.000000000000000	5.99999999926	.	.	.
x_{22}	.	5.99999999975	.	.	.
x_{23}	.	5.99999999992	.	.	.
x_{24}	.	5.99999999997	.	.	.
x_{25}	.	5.99999999999	.	.	.
x_{26}	.	6.00000000000	.	.	.

Table 3. Convergence comparison with other schemes w.r.t number of iterations.

x_n	$k = 3$	$k = 4$	$k = 5$
x_0	10	10	10
x_1	5.3670391374	5.0550582242	5.0086635272
x_2	5.0075092404	5.0001551356	5.0000023967
x_3	5.0001561926	5.0000005187	5.0000000007
x_4	5.0000035383	5.0000000019	5.0000000000
x_5	5.0000000852	5.0000000000	.
x_6	5.0000000021	.	.
x_7	5.0000000001	.	.
x_8	5.0000000000	.	.

Example 4.5. *Consider* $X = \{0, 1, 2, ...\}$ *and the digital metric space* (X, μ, ρ) *equipped with the digital metric given by* $d(x, y) = |x - y|$. *For* $T : (X, \mu, \rho) \to (X, \mu, \rho)$, *define*

$$Tx = \sqrt{x^2 - 8x + 40},$$

$$\alpha_n = \frac{2}{\sqrt{(7n+9)}},\ \beta_n = \frac{1}{\sqrt{(3n+7)}},\ \gamma_n = \frac{1}{\sqrt{(5n+7)}},\ \xi_n = \frac{1}{\sqrt{(3n+11)}},\ \textit{where } n = 1, 2, 3, \ldots.$$

The error is defined as $error = |xn - x_{n-1}|$ for three different iteration schemes presented in Table 4 (graphically in Figure 1).

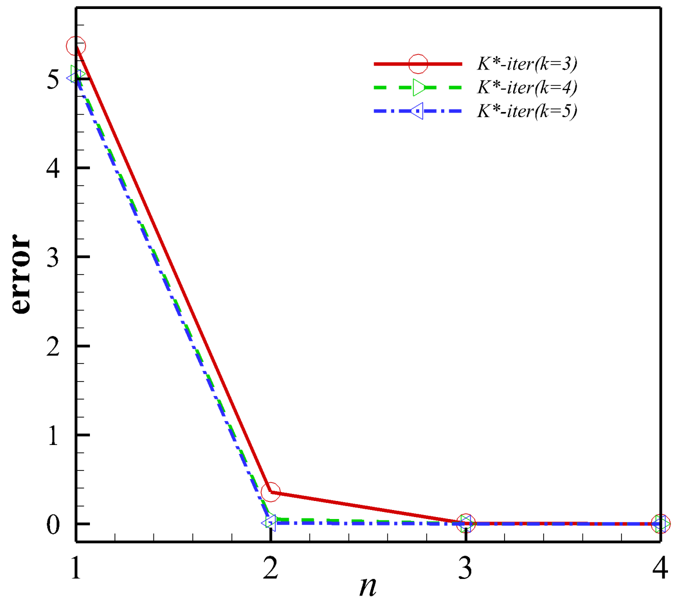

Figure 1. Comparison of error estimate for Iteration schemes.

Table 4. Errors for different steps.

x_n	$k = 3)$	$k = 4$	$k = 5$
x_1	5.3670391374	5.0550582242	5.0086635272
x_2	0.3595298970	0.0549030885	0.0086611305
x_3	0.0073530478	0.0001546169	0.0000023961
x_4	0.0001526543	0.0000005168	0.0000000007
x_5	0.0000034532	0.0000000019	0.0000000000
x_6	0.0000000830	0.0000000000	0.0000000000
x_7	0.0000000021	0.0000000000	0.0000000000

5. Conclusions

In this paper, we defined a generalized and novel k-step iterative algorithms in a digital metric space. Our results are listed as follows:

(1) The k-step iterative scheme is the general case of the Ullah and Arshad iteration and can be useful to choose the number of steps of the iterative schemes according to our need.
(2) Every increase in the step size increases the convergence speed.
(3) The speed of iterations depends on the parameters $\alpha_n^{(1)}, \alpha_n^{(2)}, \alpha_n^{(3)}, \ldots, \alpha_n^{(k)}$.

In the next works, we are going to investigate our iterative method in other generalized metric spaces equipped with special contractions such as α-ψ-contractions.

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Conflict of interest

The authors declare no conflicts of interest.

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