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Research article

Fractional variable order differential equations with impulses: A study on the stability and existence properties

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Abstract: In this paper, for the first time, we study the existence and uniqueness of solutions of a Caputo variable-order initial value problem (IVP) in the impulsive settings. Our existence results are proved by using two fixed point theorems. The Ulam-Hyers stability of solutions is established for the variable order impulsive initial value problem. Finally, we provide an example to show the correctness of the results.

Keywords: impulsive condition; differential equations; fractional variable order; Ulam-Hyers stability; existence of solutions; fixed point theorem
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1. Introduction

Nowadays, the subject of fractional calculus has gained much attention and importance between the society of researchers. The existing differential equations in this theory are determined by generalizing integer order derivatives to arbitrary order ones. For the sake of the effective memory of fractional derivation operator, such classes of equations have been widely utilized to analyze and design many physical phenomena including fractional model of Langevin equation [1, 2], impulsive integro-differentials [3], thermostat model [4, 5], modeling of Syphilis [6], fractional structure of pantograph system [7], sequential fractional model [8, 9], fractional modeling based on diseases [10–14], p-Laplacian models [15–17], neural fractional network models [18], fractal-fractional modeling [19–22], etc.

With the complexity of the processes and dynamic behavior of systems, researchers were forced to enter a new field of fractional calculus, where a wide range of models can be studied in different piecewise continuous time periods. In fact, in variable order fractional calculus, fractional variable orders made it easy for researchers to study the behavior of systems in terms of different times. The fundamental idea of variable order fractional calculus is that we take the number τ from the constant-order fractional calculus as a function $\tau(.)$. Although this difference seems simple, the variable order operator can explain and model several physical and natural phenomena in comparison to constant order models. In recent years, some newly-published papers deal with this topic; see e.g., [23–31].

Along with these notions, in recent decades, we see some models in which there are impulsive boundary conditions [32–37]. Specifically, in [38], Benchohra and Seba studied an impulsive model of an IVP having the form

$$
\begin{gathered}
{}^{c}D^{\tau}x(t) = \Psi(t,x), \text{ for each }, \ t \in [0,M], \ t \neq M_{\vartheta}, \ \vartheta = 1,\dots,n, \\
\Delta x|_{t=M_{\vartheta}} = \Phi_{\vartheta}(x(M_{\vartheta}^{-})), \ \vartheta = 1,\dots,n, \\
x(0) = x_0,
\end{gathered}
$$

in which Ψ, Φ_{ϑ} are two given functions and ${}^{c}D^{\tau}$ illustrates the Caputo derivative of constant order τ.

Inspired by [38] and motivated by the above articles, we deal with the following impulsive initial value problems (IVP) of variable order

$$
{}^{c}D_{0^+}^{\tau(t)}x(t) = \Psi(t,x), \text{ for } t \in \varpi := [0,M], \ M > 1, \ t \neq M_{\vartheta}, \ \vartheta = 1,\dots,n, \tag{1.1}
$$

$$
\Delta x|_{t=M_{\vartheta}} = \Phi_{\vartheta}(x(M_{\vartheta}^{-})), \ \vartheta = 1,\dots,n, \tag{1.2}
$$

$$
x(0) = x_0, \tag{1.3}
$$

where $\tau : \varpi = [0,M] \to (0,1]$ is the variable order of the fractional derivative, $\Psi : \varpi \times \mathbb{R} \to \mathbb{R}$, $\Phi_{\vartheta} : \mathbb{R} \to \mathbb{R}$, $\vartheta = 1,\dots,n$ are defined continuous functions and ${}^{c}D_{0^+}^{\tau(t)}$ is the Caputo fractional derivative (CFD) of variable order $\tau(t)$ for function $x(t)$ defined by (see, for example, [39–41])

$$
{}^{c}D_{\rho_1^+}^{\tau(t)}x(t) = \int_{\rho_1}^{t} \frac{(t-\varrho)^{-\tau(t)}}{\Gamma(1-\tau(t))} x'(\varrho)d\varrho, \tag{1.4}
$$

for $t > \rho_1$, and the Riemann-Liouville integral (RLFI) of variable order $\tau(t)$ for x is defined by

$$
I_{\rho_1^+}^{\tau(t)}x(t) = \int_{\rho_1}^{t} \frac{(t-\varrho)^{\tau(t)-1}}{\Gamma(\tau(t))} x(\varrho)d\varrho, \ t > \rho_1, \tag{1.5}
$$

where $\Gamma(.)$ represents the Gamma function defined by

$$
\Gamma(\tau(t)) = \int_{0}^{\infty} \varrho^{\tau(t)-1} e^{-\varrho} d\varrho,
$$

and, $0 = M_0 < M_1 < \cdots < M_n < M_{n+1} = M$, $\Delta x|_{t=M_\vartheta} = x(M_\vartheta^+) - x(M_\vartheta^-)$, $x(M_\vartheta^+) = \lim_{h\to 0^+} x(M_\vartheta + h)$ and $x(M_\vartheta^-) = \lim_{h\to 0^-} x(M_\vartheta + h)$ stand for the right and left limits of $x(t)$ at $t = M_\vartheta$, $\vartheta = 1, \dots, n$. Note that Since variable order impulsive BVPs have complicated structure, so there exist limited studies in this regard, and accordingly, this research completes the basic gaps in this direction.

The structure of this research is as follows: First, we define some functional space in Section 2, and then collect some useful concepts and properties. An equivalent constant order impulsive model is derived from the given variable order impulsive IVP in Section 3. Also, in the same section, the existence and uniqueness theorems are proved. In the next section, UH stability is reviewed and finally, an illustrative variable order impulsive IVP is provided as an example in Section 5 to see the correctness of the findings. The research is completed by presenting conclusion section.

2. Auxiliary notions

In this section, we list some of the definitions and propositions that are used in the following sections.

Assume the collection of functions: $PC(\varpi, \mathbb{R}) = \{x : \varpi \to \mathbb{R}, x \in C((M_\vartheta, M_{\vartheta+1}], \mathbb{R})$, there exist $x(M_\vartheta^-), x(M_\vartheta^+)$ with $x(M_\vartheta^-) = x(M_\vartheta)\}$, for $\vartheta = 1, \dots, n\}$. Then $PC(\varpi, \mathbb{R})$ is a Banach space under the sup-norm

$$\|x\| = \sup\{|x(t)| : t \in \varpi\}.$$

Also, we denote by $PC^1(\varpi, \mathbb{R})$, the space $PC^1(\varpi, \mathbb{R}) = \{x \in PC(\varpi, \mathbb{R}), x \in C^1((M_\vartheta, M_{\vartheta+1}], \mathbb{R})$, there exist $x'(M_\vartheta^-), x'(M_\vartheta^+)$ with $x'(M_\vartheta^-) = x'(M_\vartheta), \vartheta = 1, \dots, n\}$.

It is clear that if we assume that $\tau(t)$ is a constant function, then RLFI and CFD are reduced to the usual Riemann-Liouville fractional integral $I^{\tau}_{\rho_1^+}$ and the Caputo fractional derivative ${}^cD^{\tau}_{\rho_1^+}$, respectively.

The following some important properties of ${}^cD^{\tau}_{\rho_1^+}$ and $I^{\tau}_{\rho_1^+}$ are useful for us in this research.

Proposition 2.1. *([42]) Let $\tau_1, \ \tau_2 > 0$, $\rho_1 > 0$, $\eta \in L^1(\rho_1, \rho_2)$, ${}^cD^{\tau_1}_{\rho_1^+}\eta \in L^1(\rho_1, \rho_2)$. Then, the unique solution of the following equation*

$${}^cD^{\tau_1}_{\rho_1^+}\eta(t) = 0,$$

is

$$\eta(t) = \omega_0 + \omega_1(t - \rho_1) + \omega_2(t - \rho_1)^2 + ... + \omega_{k-1}(t - \rho_1)^{k-1},$$

and

$$I^{\tau_1}_{\rho_1^+}\,{}^cD^{\tau_1}_{\rho_1^+}\eta(t) = \eta(t) + \omega_0 + \omega_1(t - \rho_1) + \omega_2(t - \rho_1)^2 + ... + \omega_{k-1}(t - \rho_1)^{k-1},$$

with $k = [\tau_1] + 1$, $\omega_\vartheta \in \mathbb{R}$, $\vartheta = 0, 1, ..., k - 1$.

Furthermore,

$${}^cD^{\tau_1}_{\rho_1^+} I^{\tau_1}_{\rho_1^+}\eta(t) = \eta(t),$$

and

$$I^{\tau_1}_{\rho_1^+} I^{\tau_2}_{\rho_1^+}\eta(t) = I^{\tau_2}_{\rho_1^+} I^{\tau_1}_{\rho_1^+}\eta(t) = I^{\tau_1+\tau_2}_{\rho_1^+}\eta(t).$$

Remark 2.1. *([43]) In the general case, note that*

$$I^{\tau_1(t)}_{\rho_1^+} I^{\tau_2(t)}_{\rho_1^+}\eta(t) \neq I^{\tau_1(t)+\tau_2(t)}_{\rho_1^+}\eta(t).$$

Example 2.1. *([43]) Let*

$$\tau_1(t) = t, \;\; t \in [0,4], \quad \tau_2(t) = \begin{cases} 2, \; t \in [0,1] \\ 3, \; t \in]1,4]. \end{cases} \quad \eta(t) = 2, \;\; t \in [0,4].$$

Simply, we see that

$$I_{0^+}^{\tau_1(t)} I_{0^+}^{\tau_2(t)} \eta(t)|_{t=3} \neq I_{0^+}^{\tau_1(t)+\tau_2(t)} \eta(t)|_{t=3}.$$

Proposition 2.2. *([44]) Let $\tau \in C(\varpi, (0,1])$ and $0 \le \gamma \le \min_{t\in\varpi} |\tau(t)|$; then for $\eta \in C_\gamma(\varpi, \mathbb{R})$, where*

$$C_\gamma(\varpi, \mathbb{R}) = \{\eta(t) \in C(\varpi, \mathbb{R}), \; t^\gamma \eta(t) \in C(\varpi, \mathbb{R})\},$$

the (RLFI) $I_{0^+}^{\tau(t)}\eta(t)$ exists for any $t \in \varpi$.

Proposition 2.3. *([44]) If $\tau \in C(\varpi, (0,1])$, then, $I_{0^+}^{\tau(t)}\eta(t) \in C(\varpi, \mathbb{R})$ for any $\eta \in C(\varpi, \mathbb{R})$.*

Proposition 2.4. *([45,46]) Let $\tau \in [0,1]$, we have*

$$\frac{\tau^2+1}{\tau+1} \le \Gamma(\tau+1) \le \frac{\tau^2+2}{\tau+2}.$$

Remark 2.2. *For $\tau \in [0,1]$, according to Proposition 2.4, we get*

$$\frac{1}{\Gamma(\tau+1)} \le \frac{1}{2(\sqrt{2}-1)}.$$

Definition 2.1. *([27,47]) The generalized interval $I \subset \mathbb{R}$ is either an interval or $\{\rho_1\}$ or $\emptyset$.*

The partition of the generalized interval I is a finite set $\mathcal{P}$ such that if every x in I is contained in only one of them between all the generalized intervals E in $\mathcal{P}$.

Let $g : I \to \mathbb{R}$ a function, g defines a piecewise constant function with respect to partition $\mathcal{P}$ of I if it admits constant values on E for every E in $\mathcal{P}$.

Theorem 2.1. *([42]) Let Λ be a convex set in the Banach space E and $\mathcal{F} : \Lambda \longrightarrow \Lambda$ be a continuous and compact function. Then, $\mathcal{F}$ has at least a fixed point in Λ.*

Theorem 2.2. *([48]) The variable order impulsive IVP (1.1)–(1.3) is **(UH)**-stable if there exists a real number $c_\Psi > 0$ such that for any $\epsilon > 0$, and for each $z \in PC^1(\varpi, \mathbb{R})$ satisfying*

$$|{}^cD_{0^+}^{\tau(t)} z(t) - \Psi(t, z(t))| \le \epsilon, \;\; t \in \varpi, \tag{2.1}$$

there exists a solution $x \in PC^1(\varpi, \mathbb{R})$ of IVP (1.1)–(1.3) such that

$$|z(t) - x(t)| \le c_\Psi \epsilon, \;\; t \in \varpi.$$

3. Existence and uniqueness of solutions

To complete the main results, some assumptions are needed:

(S1) Let $\mathcal{P} = \{[M_0, M_1], (M_1, M_2], (M_2, M_3], ...(M_n, M_{n+1}]\}$ be a partition of the interval J (with $M_0 = 0, M_{n+1} = T$) and let $\tau(t) : \varpi \to (0,1]$ be a piecewise constant function with respect to $\mathcal{P}$ and $\tau^* = \sup_{t\in\varpi} \tau(t)$; i.e.,

$$\tau(t)=\sum_{\vartheta=0}^{n}\tau_\vartheta I_\vartheta(t)=\begin{cases}\tau_0, & if\ t\in[M_0,M_1],\\ \tau_1, & if\ t\in(M_1,M_2],\\ \cdot \\ \cdot \\ \cdot \\ \tau_n, & if\ t\in(M_n,M_{n+1}],\end{cases}$$

where $0<\tau_\vartheta\le\tau^*\le 1$ are constants, and

$$I_\vartheta(t)=\begin{cases}1, & for\ t\in(M_\vartheta,M_{\vartheta+1}],\\ 0, & for\ elsewhere,\end{cases}\quad \vartheta=0,1,...,n.$$

Now, we will give the definition of the solution to the variable order impulsive IVP (1.1)–(1.3).

Definition 3.1. *$x\in PC(\varpi,\mathbb{R})$ is a solution of the variable order impulsive IVP* (1.1)–(1.3) *if x fulfills ${}^cD_{0^+}^{\tau(t)}x(t)=\Psi(t,x)$ for each $t\in\varpi/\{M_1,...,M_n\}$ and the conditions*

$$\Delta x|_{t=M_\vartheta}=\Phi_\vartheta(x(M_\vartheta^-)),\ \vartheta=1,\dots,n,$$

and

$$x(0)=x_0,$$

are satisfied.

First, we analyze the Eq (1.1) of the variable order impulsive IVP (1.1)–(1.3).

For any $t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=0,1,...,n$, the CFD of the variable order $\tau(t)$ for $x(t)\in C(\varpi,\mathbb{R})$, given by (1.4), is the sum of the CFDs of the constant-orders $\tau_0,\tau_1,...,\tau_\vartheta$, i.e.,

$${}^cD_{0^+}^{\tau(t)}x(t)=\int_0^{M_1}\frac{(t-\varrho)^{-\tau_0}}{\Gamma(1-\tau_0)}x'(\varrho)d\varrho+...+\int_{M_\vartheta}^{t}\frac{(t-\varrho)^{-\tau_\vartheta}}{\Gamma(1-\tau_\vartheta)}x'(\varrho)d\varrho. \tag{3.1}$$

Thus, according to (3.1), the Eq (1.1) of the variable order impulsive IVP (1.1)–(1.3) can be written for any $t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=0,1,...,n$ in the form

$$\int_0^{M_1}\frac{(t-\varrho)^{-\tau_0}}{\Gamma(1-\tau_0)}x'(\varrho)d\varrho+...+\int_{M_\vartheta}^{t}\frac{(t-\varrho)^{-\tau_\vartheta}}{\Gamma(1-\tau_\vartheta)}x'(\varrho)d\varrho=\Psi(t,x(t)). \tag{3.2}$$

So, we assume that $x(t)\equiv 0$ on $t\in[0,M_\vartheta]/\{M_1,...,M_{\vartheta-1}\}$. Then, the Eq (1.1) of the variable order impulsive IVP (1.1)–(1.3) is reduced to constant order impulsive equation

$${}^cD_{0^+}^{\tau_\vartheta}x(t)=\Psi(t,x(t)),\ t\in[0,M_{\vartheta+1}]/\{M_1,...,M_\vartheta\}.$$

Proposition 3.1. *Let $\eta:\varpi\to\mathbb{R}$ be continuous. The solution of the following impulsive IVP*

$${}^cD_{0^+}^{\tau_\vartheta}x(t)=\eta(t),\ for\ each\ ,\ t\in[0,M_{\vartheta+1}]/\{M_1,...,M_\vartheta\},\quad \vartheta=0,1,\dots,n, \tag{3.3}$$

$$\Delta x|_{t=M_\vartheta}=\Phi_\vartheta(x(M_\vartheta^-)),\quad \vartheta=1,\dots,n, \tag{3.4}$$

$$x(0)=x_0, \tag{3.5}$$

is given by

$$x(t)=\begin{cases} x_0+\int_0^t \frac{(t-\varrho)^{\tau_0-1}}{\Gamma(\tau_0)}\eta(\varrho)d\varrho, & t\in[M_0,M_1],\\ x_0+\sum_{s=1}^{\vartheta}\int_{M_{s-1}}^{M_s}\frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\eta(\varrho)d\varrho+\int_{M_\vartheta}^t\frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\eta(\varrho)d\varrho & \\ +\sum_{s=1}^{\vartheta}\Phi_s(x(M_s^-)), & t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=1,...,n. \end{cases} \tag{3.6}$$

Proof. Let x be a solution of an equivalent impulsive IVP (3.3)–(3.5). If $t\in[M_0,M_1]$ and by Proposition 2.1, we get

$$x(t)=x_0+\int_0^t\frac{(t-\varrho)^{\tau_0-1}}{\Gamma(\tau_0)}\eta(\varrho)d\varrho.$$

If $t\in(M_1,M_2]$, then Proposition 2.1 implies

$$\begin{aligned} x(t) &= x(M_1^+)+\int_{M_1}^t\frac{(t-\varrho)^{\tau_1-1}}{\Gamma(\tau_1)}\eta(\varrho)d\varrho\\ &= \Delta x|_{t=M_1}+x(M_1^-)+\int_{M_1}^t\frac{(t-\varrho)^{\tau_1-1}}{\Gamma(\tau_1)}\eta(\varrho)d\varrho\\ &= I_1(x(M_1^-))+x_0+\int_0^{M_1}\frac{(M_1-\varrho)^{\tau_0-1}}{\Gamma(\tau_0)}\eta(\varrho)d\varrho+\int_{M_1}^t\frac{(t-\varrho)^{\tau_1-1}}{\Gamma(\tau_1)}\eta(\varrho)d\varrho. \end{aligned}$$

If $t\in(M_2,M_3]$, and by Proposition 2.1, we get

$$\begin{aligned} x(t) &= x(M_2^+)+\int_{M_2}^t\frac{(t-\varrho)^{\tau_2-1}}{\Gamma(\tau_2)}\eta(\varrho)d\varrho\\ &= \Delta x|_{t=M_2}+x(M_2^-)+\int_{M_2}^t\frac{(t-\varrho)^{\tau_2-1}}{\Gamma(\tau_2)}\eta(\varrho)d\varrho\\ &= I_2(x(M_2^-))+I_1(x(M_1^-))+x_0+\int_0^{M_1}\frac{(M_1-\varrho)^{\tau_0-1}}{\Gamma(\tau_0)}\eta(\varrho)d\varrho\\ &+ \int_{M_1}^{M_2}\frac{(M_2-\varrho)^{\tau_1-1}}{\Gamma(\tau_1)}\eta(\varrho)d\varrho+\int_{M_2}^t\frac{(t-\varrho)^{\tau_2-1}}{\Gamma(\tau_2)}\eta(\varrho)d\varrho. \end{aligned}$$

Then, if $t\in(M_\vartheta,M_{\vartheta+1}]$, Proposition 2.1 implies (3.6).

Conversely, assume that x solves the Eq (3.6). If $t\in[M_0,M_1]$, then $x(0)=x_0$. Employing the operator ${}^cD_{0^+}^{\tau_0}$, we get,

$${}^cD_{0^+}^{\tau_0}x(t)=\eta(t).$$

If $t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=0,1,...,n$ and using Proposition 2.1 and the fact that ${}^cD_{0^+}^{\tau_\vartheta}C=0$ where C is a constant, we find

$${}^cD_{0^+}^{\tau_\vartheta}x(t)=\eta(t).$$

Also, it is easy to show that

$$\Delta x|_{t=M_\vartheta}=\Phi_\vartheta(x(M_\vartheta^-)),\ \vartheta=1,...,n.$$

Now, we present our first result, assuming that the following assumptions are satisfied:

(S2) Let $t^{\gamma}\Psi : \varpi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function ($0 \leq \gamma \leq \min_{t\in\varpi} |\tau(t)|$), and there exists a constant $D_1 > 0$ such that:
$t^{\gamma}|\Psi(t, x_1) - \Psi(t, x_2)| \leq D_1|x_1 - x_2|$, for any $x_1,\ x_2 \in \mathbb{R}$ and $t \in \varpi$.

(S3) Let $\vartheta = 1, ..., n$, for $x \in \mathbb{R}$ and $t \in \varpi$, there exists $D_2 > 0$ such that,

$$|\Phi_{\vartheta}(x(t))| \leq D_2|x(t)|.$$

Theorem 3.1. *Let the conditions (S1)–(S3) be satisfied, and*

$$\Big[\frac{(n+1)D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + nD_2\Big] < 1. \tag{3.7}$$

Then, the variable order impulsive IVP (1.1)–(1.3) *possesses a solution on $PC(\varpi, \mathbb{R})$.*

Proof. Construct the operator

$$S : PC(\varpi, \mathbb{R}) \to PC(\varpi, \mathbb{R}),$$

as follows

$$\begin{aligned} Sx(t) &= x_0 + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s - \varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} \Psi(\varrho, x(\varrho)) d\varrho \\ &+ \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta - 1}}{\Gamma(\tau_\vartheta)} \Psi(\varrho, x(\varrho)) d\varrho + \sum_{0<M_s<t} \Phi_s(x(M_s^-)). \end{aligned} \tag{3.8}$$

The operator S defined in (3.8) is well defined from the continuity of function $t^{\gamma}\Psi$ and from the properties of fractional integrals.

Let the set

$$B_R = \{x \in PC(\varpi, \mathbb{R}),\ \|x\| \leq R\},$$

where

$$R \geq \frac{|x_0| + \Psi^{\star}\Big(\frac{(n+1)M^{\tau^*}}{2(\sqrt{2}-1)}\Big)}{1 - \Big[\frac{(n+1)D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + nD_2\Big]},$$

and

$$\Psi^{\star} = \sup_{t\in\varpi} |\Psi(t, 0)|.$$

Clearly B_R is nonempty, closed, convex and bounded. Now, we shall show that S satisfies the assumption of Theorem 2.1. The proof will be given in three steps:

Step 1: Claim: $S(B_R) \subseteq (B_R)$.

For $x \in B_R$, we get

$$\begin{aligned} |Sx(t)| &\leq |x_0| \\ &+ \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s - \varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} |\Psi(\varrho, x(\varrho)) - \Psi(\varrho, 0)| d\varrho \end{aligned}$$

$$
\begin{aligned}
& + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} |\Psi(\varrho,0)| d\varrho + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} |\Psi(\varrho,x(\varrho)) - \Psi(\varrho,0)| d\varrho \\
& + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} |\Psi(\varrho,0)| d\varrho + \sum_{0<M_s<t} |\Phi_s(x(M_s^-))| \\
\leq \ & |x_0| + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} \varrho^{-\gamma}(D_1|x(\varrho)|) d\varrho + \Psi^\star \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} d\varrho \\
& + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} \varrho^{-\gamma}(D_1|x(\varrho)|) + \Psi^\star \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} d\varrho + \sum_{0<M_s<t} D_2|x(M_s^-)| \\
\leq \ & |x_0| + \frac{(n+1)D_1 M^{1-\gamma} M^{\tau^*-1}}{(1-\gamma)\Gamma(\tau^*)} \|x\| + \Psi^\star \Big(\frac{(n+1)M^{\tau^*}}{2(\sqrt{2}-1)} \Big) + nD_2\|x\| \\
\leq \ & |x_0| + \Psi^\star \Big(\frac{(n+1)M^{\tau^*}}{2(\sqrt{2}-1)} \Big) + \Big[\frac{(n+1)D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + nD_2 \Big] \|x\| \\
\leq \ & R.
\end{aligned}
$$

Step 2: Claim: S is continuous.

Let's consider (x_n) as a sequence converging to x in $PC(\varpi,\mathbb{R})$. Then,

$$
\|(Sx_n) - (Sx)\| \to 0.
$$

For $t \in \varpi$, we have

$$
\begin{aligned}
& |(Sx_n)(t) - (Sx)(t)| \\
\leq \ & \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} |\Psi(\varrho,x_n(\varrho)) - \Psi(\varrho,x(\varrho))| d\varrho \\
& + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} |\Psi(\varrho,x_n(\varrho)) - \Psi(\varrho,x(\varrho))| d\varrho + \sum_{0<M_s<t} |\Phi_s(x_n(M_s^-)) - \Phi_s(x(M_s^-))| \\
\leq \ & \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})} \varrho^{-\gamma}(D_1|x_n(\varrho) - x(\varrho)|) d\varrho + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)} \varrho^{-\gamma}(D_1|x_n(\varrho) - x(\varrho)|) \\
& + \sum_{0<M_s<t} |\Phi_s(x_n(M_s^-)) - \Phi_s(x(M_s^-))| \\
\leq \ & \frac{(n+1)D_1 M^{1-\gamma} M^{\tau^*-1}}{(1-\gamma)\Gamma(\tau^*)} \|x_n - x\| + \sum_{0<M_s<t} |\Phi_s(x_n(M_s^-)) - \Phi_s(x(M_s^-))| \\
\leq \ & \Big[\frac{(n+1)D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} \Big] \|x_n - x\| + \sum_{0<M_s<t} |\Phi_s(x_n(M_s^-)) - \Phi_s(x(M_s^-))|.
\end{aligned}
$$

Since Φ_s is continuous, then

$$\|(Sx_n) - (Sx)\| \to 0 \ \ as \ \ n \to \infty.$$

Then, S is continuous.

Step 3: Claim: S is compact.

By Step 1, we have $\|S(x)\| \leq R$ for each $x \in B_R$, which gives the boundedness of $S(B_R)$. Now we will show that $S(B_R)$ is equicontinuous.

For $t_1, t_2 \in \varpi,\ t_1 < t_2$ and $x \in B_R$, estimate

$$|(Sx)(t_2) - (Sx)(t_1)|$$

$$\begin{aligned}
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big)|\Psi(\varrho, x(\varrho))| d\varrho \\
& + \frac{1}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} (t_2-\varrho)^{\tau_\vartheta - 1}|\Psi(\varrho, x(\varrho))| d\varrho + \sum_{0<M_s<t_2-t_1} |\Phi_s(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big)|\Psi(\varrho, x(\varrho)) - \Psi(\varrho, 0)| d\varrho \\
& + \frac{1}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big)|\Psi(\varrho, 0)| d\varrho \\
& + \frac{1}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} (t_2-\varrho)^{\tau_\vartheta - 1}|\Psi(\varrho, x(\varrho)) - \Psi(\varrho, 0)| d\varrho + \frac{1}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} (t_2-\varrho)^{\tau_\vartheta - 1}|\Psi(\varrho, 0)| d\varrho \\
& + \sum_{0<M_s<t_2-t_1} |D_2(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \varrho^{-\gamma}\Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big)(D_1|x(\varrho)|) d\varrho \\
& + \frac{\Psi^\star}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big) d\varrho \\
& + \frac{1}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} \varrho^{-\gamma}(t_2-\varrho)^{\tau_\vartheta - 1}(D_1|x(\varrho)|) d\varrho + \frac{\Psi^\star}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} (t_2-\varrho)^{\tau_\vartheta - 1} d\varrho + \sum_{0<M_s<t_2-t_1} |D_2(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \varrho^{-\gamma}\Big((t_2-t_1)^{\tau_\vartheta - 1}\Big)(D_1|x(\varrho)|) d\varrho + \frac{\Psi^\star}{\Gamma(\tau_\vartheta)} \int_0^{t_1} \Big((t_2-\varrho)^{\tau_\vartheta - 1} - (t_1-\varrho)^{\tau_\vartheta - 1}\Big) d\varrho \\
& + \frac{1}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} \varrho^{-\gamma}(t_2-\varrho)^{\tau_\vartheta - 1}(D_1|x(\varrho)|) d\varrho + \frac{\Psi^\star}{\Gamma(\tau_\vartheta)} \int_{t_1}^{t_2} (t_2-\varrho)^{\tau_\vartheta - 1} d\varrho + \sum_{0<M_s<t_2-t_1} |D_2(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}(D_1\|x\|)\Big(\frac{({t_1}^{1-\gamma})(t_2-t_1)^{\tau_\vartheta - 1}}{(1-\gamma)}\Big) + \frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\Big(\frac{t_2^{\tau_\vartheta}}{\tau_\vartheta} - \frac{(t_2-t_1)^{\tau_\vartheta}}{\tau_\vartheta} - \frac{t_1^{\tau_\vartheta}}{\tau_\vartheta}\Big)
\end{aligned}$$

$$+\frac{1}{\Gamma(\tau_\vartheta)}(D_1\|x\|)\frac{({t_2}^{1-\gamma}-{t_1}^{1-\gamma})(t_2-t_1)^{\tau_\vartheta-1}}{(1-\gamma)}+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\frac{(t_2-t_1)^{\tau_\vartheta}}{\tau_\vartheta}+\sum_{0<M_s<t_2-t_1}|D_2(x(M_s^-))|$$

$$\leq \frac{1}{\Gamma(\tau_\vartheta)}(D_1)\|x\|\frac{{t_1}^{1-\gamma}}{(1-\gamma)}\Big(t_2^{\tau_\vartheta-1}-t_1^{\tau_\vartheta-1}\Big)+\frac{\Psi^\star}{\Gamma(\tau_\vartheta+1)}\Big(t_2^{\tau_\vartheta}-t_1^{\tau_\vartheta}\Big)$$

$$+\frac{1}{\Gamma(\tau_\vartheta)}(D_1)\|x\|\frac{({t_2}^{1-\gamma}-{t_1}^{1-\gamma})}{(1-\gamma)}(t_2-t_1)^{\tau_\vartheta-1}+\sum_{0<M_s<t_2-t_1}|D_2(x(M_s^-))|.$$

Assuming $t_1 \to t_2$, the right-hand side converges to zero. Hence $|(Sx)(t_2)-(Sx)(t_1)| \to 0$. It implies that $S(B_R)$ is equicontinuous.

Thus, by Theorem (2.1) the variable order impulsive IVP (1.1)–(1.3) possesses a solution in B_R. Since $B_R \subset PC(\varpi,\mathbb{R})$, the claim of Theorem (3.1) is verified.

Introduce the following assumption:

(S4) For $s = 1, ..., n$, there exists $D_3 > 0$ such that for any $x, y \in \mathbb{R}$ and $t \in \varpi$, $|\Phi_s(x(t)) - \Phi_s(y(t))| \leq D_3|x(t) - y(t)|$.

Theorem 3.2. *Let (S1), (S2), (S4), and*

$$\Big[\frac{(n+1)D_1M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)}+nD_3\Big]<1. \tag{3.9}$$

Then, the variable order impulsive IVP (1.1)–(1.3) *possesses a solution uniquely on $PC(\varpi,\mathbb{R})$.*

Proof. For $t \in \varpi$ and $x \in PC(\varpi,\mathbb{R})$, we have

$$|(Sx)(t)-(Sy)(t)|$$

$$\leq \sum_{0<M_s<t}\int_{M_{s-1}}^{M_s}\frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}|\Psi(\varrho,x(\varrho))-\Psi(\varrho,y(\varrho))|d\varrho$$

$$+\int_{M_\vartheta}^{t}\frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}|\Psi(\varrho,x(\varrho))-\Psi(\varrho,y(\varrho))|d\varrho+\sum_{0<M_s<t}|\Phi_s(x(M_s^-))-\Phi_s(y(M_s^-))|$$

$$\leq \sum_{0<M_s<t}\int_{M_{s-1}}^{M_s}\frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\varrho^{-\gamma}(D_1|x(\varrho)-y(\varrho)|)d\varrho$$

$$+\int_{M_\vartheta}^{t}\frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\varrho^{-\gamma}(D_1|x(\varrho)-y(\varrho)|)+\sum_{0<M_s<t}D_3|x(t)-y(t)|$$

$$\leq \Big[\frac{(n+1)D_1M^{1-\gamma}M^{\tau^*-1}}{(1-\gamma)\Gamma(\tau^*)}+nD_3\Big]\|x-y\|$$

$$\leq \Big[\frac{(n+1)D_1M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)}+nD_3\Big]\|x-y\|.$$

Accordingly, by (3.9), the operator S has a contraction structure. Thus, S involves a fixed point uniquely which is the unique solution of the variable order impulsive IVP (1.1)–(1.3).

4. Ulam-Hyers stability

we will discuss here the Ulam-Hyers stability for solutions of the supposed variable order impulsive IVP (1.1)–(1.3).

Theorem 4.1. *Consider the hypotheses of Theorem 3.2. Then, the variable order impulsive IVP* (1.1)–(1.3) *is* ***(UH)*** *stable.*

Proof. Assume $z(t)$ satisfies the inequality (2.1); then the integral inequality

$$\Big|z(t) - z_0 + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\Psi(\varrho, z(\varrho))d\varrho - \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\Psi(\varrho, z(\varrho))d\varrho - \sum_{0<M_s<t} \Phi_s(z(M_s^-))\Big| \le \epsilon \frac{M^{\tau^*}}{2(\sqrt{2}-1)}$$

holds.

Let x be a the unique solution of the variable order impulsive IVP (1.1)–(1.3). According to Proposition 3.1, x is given by

$$x = x_0 + \sum_{s=1}^{\vartheta} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\Psi(\varrho)d\varrho + \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\Psi(\varrho)d\varrho + \sum_{s=1}^{\vartheta} \Phi_s(x(M_s^-)), \quad t \in (M_\vartheta, M_{\vartheta+1}],\ \vartheta = 1, ..., n. \tag{4.1}$$

Let $t \in (M_\vartheta, M_{\vartheta+1}],\ \vartheta = 1, ..., n$. Then,

$$\begin{aligned}
&|z(t) - x(t)| \\
= &\Big|z(t) - x_0 - \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\Psi(\varrho, x(\varrho))d\varrho \\
&- \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\Psi(\varrho, x(\varrho))d\varrho - \sum_{0<M_s<t} \Phi_s(x(M_s^-))\Big| \\
\le &\Big|z(t) - z_0 + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\Psi(\varrho, z(\varrho))d\varrho - \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\Psi(\varrho, z(\varrho))d\varrho \\
&- \sum_{0<M_s<t} \Phi_s(z(M_s^-))\Big| + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\Big|\Psi(\varrho, z(\varrho)) - \Psi(\varrho, x(\varrho))\Big|d\varrho \\
&+ \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta-1}}{\Gamma(\tau_\vartheta)}\Big|\Psi(\varrho, z(\varrho)) - \Psi(\varrho, x(\varrho))\Big|d\varrho + \sum_{0<M_s<t} \Big|\Phi_s(z(M_s^-)) - \Phi_s(x(M_s^-))\Big| \\
\le &\ \epsilon \frac{M^{\tau^*}}{2(\sqrt{2}-1)} + \sum_{0<M_s<t} \int_{M_{s-1}}^{M_s} \frac{(M_s-\varrho)^{\tau_{s-1}-1}}{\Gamma(\tau_{s-1})}\varrho^{-\gamma}(D_1|z(\varrho) - x(\varrho)|)d\varrho
\end{aligned}$$

$$
+ \int_{M_\vartheta}^{t} \frac{(t-\varrho)^{\tau_\vartheta - 1}}{\Gamma(\tau_\vartheta)} \varrho^{-\gamma} (D_1 |z(\varrho) - x(\varrho)|) d\varrho + \sum_{0<M_s<t} D_3 |z(t) - x(t)|
$$

$$
\leq \epsilon \frac{M^{\tau^*}}{2(\sqrt{2}-1)} + \Big[\frac{(n+1) D_1 M^{1-\gamma} M^{\tau^*-1}}{(1-\gamma)\Gamma(\tau^*)} + n D_3 \Big] \|z - x\|
$$

$$
\leq \epsilon \frac{M^{\tau^*}}{2(\sqrt{2}-1)} + \Big[\frac{(n+1) D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + n D_3 \Big] \|z - x\|.
$$

Then,

$$
\|z - x\| \Big[1 - \Big(\frac{(n+1) D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + n D_3 \Big) \Big] \leq \epsilon \frac{M^{\tau^*}}{2(\sqrt{2}-1)}.
$$

Thus, we obtain

$$
|z(t) - x(t)| \leq \|z - x\| \leq \frac{M^{\tau^*}}{2(\sqrt{2}-1) \Big[1 - \Big(\frac{(n+1) D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + n D_3 \Big) \Big]} \epsilon := c_\Psi \epsilon.
$$

Consequently by Theorem 2.2, the variable order impulsive IVP (1.1)–(1.3) is **(UH)** stable.

5. Example

Consider the variable order impulsive IVP

$$
{}^c D_{M_\vartheta^+}^{\tau(t)} x(t) = \frac{e^{-3t}}{t^{\frac{1}{3}} (e^{e^{\frac{3t^2}{1+t}}} + 5)(1 + |x(t)|)}, \quad t \in \varpi := \varpi_0 \cup \varpi_1, \tag{5.1}
$$

$$
\Delta x|_{t=\frac{1}{2}} = \frac{|(x(\frac{1}{2}^-))|}{10 + |(x(\frac{1}{2}^-))|}, \tag{5.2}
$$

$$
x(0) = x_0, \tag{5.3}
$$

where

$$
M_0 = 0, \quad M_1 = \frac{1}{2}, \quad M_2 = M = 1, \quad n = 1, \quad \varpi := [0,1], \quad \varpi_0 = [0, \frac{1}{2}], \quad \varpi_1 = [\frac{1}{2}, 1],
$$

and

$$
\tau(t) = \begin{cases} \frac{1}{2}, & t \in \varpi_0, \\ \frac{3}{4}, & t \in \varpi_1. \end{cases} \tag{5.4}
$$

Let

$$
\Psi(t, x) = \frac{e^{-3t}}{t^{\frac{1}{3}} (e^{e^{\frac{3t^2}{1+t}}} + 5)(1 + |x(t)|)}, \ (t, x) \in \varpi \times \mathbb{R}.
$$

For each $t \in \varpi$ and $x, y \in \mathbb{R}$,

$$
t^{\frac{1}{3}} |\Psi(t, x) - \Psi(t, y)| \leq \frac{1}{e+5} |x - y|.
$$

Thus, assumption (S2) is satisfied with $D_1 = \frac{1}{e+5}$ and $\gamma = \frac{1}{3}$.

Let

$$\Phi_1(x) = \frac{|x|}{7+|x|}, \quad x \in \mathbb{R}.$$

For $x, y \in [0, \infty)$, we have

$$|\Phi_1(x) - \Phi_1(y) \leq \frac{1}{7}|x-y|.$$

Then, the assumption (S4) holds with $D_3 = \frac{1}{7}$.

We shall check that assumption (3.9) is fulfilled with $M = 1$, $n = 1$, $\gamma = \frac{1}{3}$, $D_1 = \frac{1}{e+5}$, $D_3 = \frac{1}{7}$ and $\tau^* = \frac{3}{4}$. Indeed,

$$\Big[\frac{(n+1)D_1 M^{\tau^*-\gamma}}{(1-\gamma)\Gamma(\tau^*)} + nD_3\Big] = \frac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{4})} \simeq 0.3171 < 1.$$

Hence, assumption (3.9) is satisfied.

By Theorem 3.2, the variable order impulsive IVP (5.1)–(5.3) has a unique solution on $PC(\varpi, \mathbb{R})$. According to Theorem 4.1, the variable order impulsive IVP (5.1)–(5.3) is **(UH)** stable.

6. Conclusions

A variable order impulsive IVP was studied in this paper by terms of analytical properties. In more precise, an equivalent constant order impulsive model is derived from the given variable order impulsive IVP by using the properties of piecewise constant functions. In this direction, the existence and uniqueness theorems were discussed via notions in functional analysis. In the following, UH stability was checked. Lastly, an illustrative variable order impulsive IVP was provided as an example in the sequel to see the correctness of the findings. Since variable order impulsive BVPs have complicated structure, so there exist limited studies in this regard, and accordingly, we will extend our studies on different impulsive BVPs (implicit, resonance, thermostat model, etc.) by changing conditions (terminal, integral conditions, etc.) or taking $1 < \tau(t) \leq 2$ in the future.

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Conflict of interest

The authors declare no conflict of interest.

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