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Research article

A priori estimate for resolving the boundary fractional problem

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Abstract: The energy inequality method (or a priori estimation) known in classical cases has been adopted for fractional evolution equations associated with initial conditions and boundary integral conditions. We prove the existence and uniqueness of the solution to the problem described in the following.

Keywords: existence and uniqueness; solvability; fractional equation; integral conditions; Caputo derivatives
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1. Introduction

Recently fractional calculus regained ground with its use in various fields, such as physical and chemical processes that occur in media with fractal geometry, as well as in the mathematical modeling of economic and social-biological phenomena [1, Chap. 5,6,11]. The reader can also find in the work of Alikhanov [1], Ladyzhenskaya [5], and Nakhushev [8] other rich domains in terms of applications of fractional calculus such as Abel's integral equation [2], fractional-order capacitor models [6,7], electroelectrical chemistry [3,4, 8–11]. Alikhanov [1] used the method of energy inequalities to obtain a priori estimates for solutions of boundary value problems for the diffusion-wave equation with the Caputo fractional derivative. Mesloub's [6] paper deals with a fractional two-times evolution equation associated with initial and purely boundary integral conditions. Our work was inspired by the latter but with a fractional one-time evolution equation with boundary integral conditions. In the present paper, we use the method of energy inequalities to obtain a priori estimates to prove the uniqueness and existence of the solution using some functional inequalities , the Caputo derivative and Caputo integrals. In Section 1, we formulate and set our nonlocal initial boundary value problem. In Section 2, we introduce some useful preliminaries and notations, and reformulate the problem (1.1)–(1.2) as a problem with homogeneous boundary conditions, (1.3) and (1.4) to simplify the calculations. In

Section 3, we apply the a priori estimation method to prove the uniqueness of the solution. We choose a certain functional differential operator multiplier (3.3). In Section 4, we discuss the existence of a solution. We always apply the same method.

1.1. Position of the problem

In the bounded domain, $Q^T = (0, T) \times (0, \alpha)$ such that $0 < \alpha < \infty, 0 < T < \infty$.
We consider the time fractional order problem with the Caputo-derivative.

$$ {}^{c}\partial_t^{\beta} u_t - (a(x,t)\frac{\partial u}{\partial x})_x + b(x,t)u_x + c(x,t)u = g(x,t). \tag{1.1}$$

The functions $a(x, t), b(x, t)$ and $c(x, t)$, satisfy the conditions

$$P_1 : \left\{ \begin{array}{c} 0 \le a(x,t) \le a_1, 0 \le a_x(x,t) \le a_2 \ , \\ 0 \le b(x,t) \le b_1, 0 \le b_x(x,t) \le b_2, \\ 0 \le c(x,t) \le c, \end{array} \right\} \forall\, (x,t) \in Q^T,$$

where

$$\begin{array}{rcl} a_i & \in & IR_*^+, \forall \overline{i = 1-2}, b_k \in IR_*^+, \forall \overline{k = 1-2}, \\ c & \in & IR_*^+. \end{array}$$

With problem (1.1), we associate the initial conditions

$$u(x, 0) = \varphi(x) \ge 0, \qquad 0 < x < \alpha, \tag{1.2}$$

and the boundary conditions

$$\int_0^{\alpha} u(x,t)dx = 0, \quad \int_0^{\alpha} xu(x,t)dx = 0, \quad u(\alpha,t)dx = 0. \tag{1.3}$$

$$\int_0^{\alpha} \varphi(x)dx = 0, \tag{1.4}$$

$$\int_0^{\alpha} x\varphi(x)dx = 0. \tag{1.5}$$

2. Prelemanaries

For a function $f(t)$, which has absolutely continuous derivatives up to order (n -1), the Riemann-Liouville fractional derivative of arbitrary order $n - 1 \le \beta \le n$ is defined as follows [6] :

$$D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds, \tag{2.1}$$

where Γ is the well-known gamma function. The following formula is true:

$$D^{\beta} f(t) = \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} f^k(0) + \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds. \tag{2.2}$$

The Riemann-Liouville fractional integral of order β, $0 \leq \beta \leq n$, is defined by

$$D_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\beta-n+1}} ds = D^\beta f(t). \tag{2.3}$$

The Riemann-Liouville fractional derivative is singular at the origin because of (2.3). Because of this fact, fractional differential equations in the sense of the Riemann-Liouville fractional derivative require initial conditions of a special form lacking clear physical interpretation. These shortcomings do not occur with the regularized Caputo derivative. The Caputo derivative of order β, n-1$\leq \beta \leq n, n \in IN$, is defined as the integral part of (2.3); that is,

$${}^c D^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds = D^\beta f(t) = D^\beta f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} f^k(0). \tag{2.4}$$

In the case of our problem, the Caputo and Riemann-Liiouville derivatives of order β, $0 \leq \beta \leq 1$, and their linked relationship formulas are defined as follows [4]:

$${}^c\partial_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\frac{\partial u(x,s)}{\partial s}}{(t-s)^\beta} ds = \partial_t^\beta u(x,t) - \frac{u(x,0)}{t^\beta \Gamma(1-\beta)}, \tag{2.5}$$

$$\partial_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u(x,s)}{(t-s)^\beta} ds. \tag{2.6}$$

(1) The Cauchy $\varepsilon-$inequality:

$$\alpha\beta \leq \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2. \tag{2.7}$$

(2) *Poincaré*-type inequalities [3]:

$$\left\| \Im_x u \right\|^2_{L^2(Q^T)} \leq \frac{\alpha}{2} \left\| u \right\|^2_{L^2(Q^T)}, \tag{2.8}$$

$$\left\| \Im^m u \right\|^2_{L^2(Q^T)} \leq \frac{a}{2} \left\| \Im_x^{m-1} u \right\|^2_{L^2(Q^T)} \forall m \in IN^*.$$

(3) Lemma. For any function $f(t)$ absolutely continuous on [0,T], owe have the inequality

$${}^c D_t^\beta f^2(t) \leqslant 2 f(t) {}^c D_t^\beta f(t). \tag{2.9}$$

Because the proofs are based on a priori estimates, we first write (1.1)–(1.3) in an equivalent operator form in order to establish the existence and uniqueness of the solution. The solution to (1.1)–(1.3) can be regarded as the solution of the operator equation

$$Lu = H, \tag{2.10}$$

where $L = (\mathfrak{L}, l_1)$, and operator L acts from B to F, with the domain of definition

$$D(L) = \left\{ \begin{array}{c} u \in L^2(Q^T), \partial_t^\beta u_t(x,t), u_x, u_t \in L^2(Q^T) \\ \int_0^\alpha u_t(x,t)dx = 0, \forall t \in (0,T) \\ u(x,0) = \varphi(x) \geq 0 \\ \int_0^\alpha \varphi(x)dx = 0 \end{array} \right\} \tag{2.11}$$

where B is the Banach space of functions u endowed by the finite norm

$$\|u\|_B^2 = \int_0^T \left(\partial_t^\beta \left\|\Im_x u_t\right\|_{L^2(0.\alpha)}^2 + \|u\|_{L^2(0.\alpha)}^2\right) dt. \tag{2.12}$$

F is a Hilbert space equipped with the scalar product:

$$(H, H^*)_F = \int_0^T \left((H, H^*)_{L^2(0.\alpha)}\right) dt. \tag{2.13}$$

The associated norm is

$$\|H\|_F^2 = \int_0^t \|H\|_{L^2(0.\alpha)} dt = \int_0^T (\|g\|_{L^2(0.\alpha)}^2 + \|\varphi\|_{L^2(0.\alpha)}^2) dt. \tag{2.14}$$

3. Uniqueness of the solution

Theorem 1. *In Q^T and for sufficiently small ε, there exists a positive constant K independent of u such that*

$$\int_0^T \left(\partial_t^\beta \left\|\Im_x u_t\right\|_{L^2(0.\alpha)}^2 + \|u\|_{L^2(0.\alpha)}^2\right) dt \le K \int_0^T \left(\|H\|_{L^2(0.\alpha)}^2\right) dt, \tag{3.1}$$

for all $u \in D(L)$, where

$$\|H\|_{L^2(Q^T)}^2 = \left(\|g\|_{L^2(Q^T)}^2 + \|\varphi\|_{L^2(0,\alpha)}^2\right)$$

and

$$K = \frac{\max(\varepsilon, a_1)}{3 \min(2 - (\frac{ab_2 + a_1 + \alpha c + b_1 + \alpha}{\varepsilon}), (a_1\varepsilon + c\varepsilon + b_1\varepsilon + b_2\varepsilon))}, \tag{3.2}$$

$$\Im_x u_t = \int_0^x u_t(\eta, t) d\eta. \tag{3.3}$$

Proof. Let $u \in D(L)$, and consider the equality

$$\begin{aligned}({}^c\partial_t^\beta u_t - ((a(x,t)\frac{\partial u}{\partial x})_x + b(x,t)u_x + c(x,t)u)\ .\ -\Im_x^2 u_t)_{L^2(Q^T)} &= (\mathcal{L}u\ .\ -\Im_x^2 u_t)_{L^2(Q^T)} \\ &= (\mathcal{L}u\ .\ -\Im_x^2 u_t)_{L^2(Q^T)} \\ &= (g\ .\ -\Im_x^2 u_t)_{L^2(Q^T)},\end{aligned} \tag{3.4}$$

where

$$\Im_x^2 u_t = \int_0^x \int_0^\eta u_t(\zeta, t) d\zeta d\eta.$$

Upon integration by parts and conditions (1.1)–(1.3), we evaluate each term of (3.4) to obtain

$$\left({}^c\partial_t^\beta u_t, -\Im_x^2 u_t\right)_{L^2(Q^T)} = -\int_0^T \int_0^\alpha {}^c\partial_t^\beta u_t\ \Im_x^2 u_t dx = -\int_0^T \Im_x\left({}^c\partial_t^\beta u_t\right) \Im_x^2 u_t \,\Big|_{x=0}^{x=\alpha} dt$$

$$+\int_0^T\int_0^\alpha \Im_x^c(\partial_t^\beta u_t)\Im_x u_t dx=\int_0^T\int_0^\alpha {}^c\partial_t^\beta\ (\Im_x u_t)\,\Im_x u_t dxdt. \tag{3.5}$$

Using the same method, we have

$$\left(-(a(x,t)\frac{\partial u}{\partial x})_x,-\Im_x^2 u_t\right)_{L^2(Q^T)}=\int_0^T\int_0^\alpha (a(x,t)\frac{\partial u}{\partial x})_x\Im_x^2 u_t dxdt$$

$$=\int_0^T\left(a(x,t)\frac{\partial u}{\partial x}\Im_x^2 u_t\right)|_{x=0}^{x=\alpha}\,dt-\int_0^T\int_0^\alpha a(x,t)\frac{\partial u}{\partial x}\Im_x u_t dxdt$$

$$=-\int_0^T\int_0^\alpha a(x,t)\frac{\partial u}{\partial x}\Im_x u_t dx=-\int_0^T a(x,t)u\Im_x u_t\,|_{x=0}^{x=\alpha}\,dt$$

$$+\int_0^T\int_0^\alpha a(x,t)uu_t dxdt+\int_0^T\int_0^\alpha a_x(x,t)u\Im_x u_t dxdt$$

$$=\frac{1}{2}\int_0^\alpha a(x,t)u^2(x,T)dx-\frac{1}{2}\int_0^\alpha a(x,t)\varphi^2(x)dx+\int_0^T\int_0^\alpha a_x(x,t)u\Im_x u_t dxdt \tag{3.6}$$

$$\left(b(x,t)u_x,-\Im_x^2 u_t\right)_{L^2(Q^T)}=-\int_0^T\int_0^\alpha b(x,t)u_x\Im_x^2 u_t dxdt.$$

$$=-\int_0^T b(x,t)u\Im_x^2 u_t\,|_{x=0}^{x=\alpha}\,dt+\int_0^T\int_0^\alpha b_x(x,t)u\Im_x^2 u_t dxdt+\int_0^T\int_0^\alpha b(x,t)u\Im_x u_t dxdt$$

$$=\int_0^T\int_0^\alpha b_x(x,t)u\Im_x^2 u_t dxdt+\int_0^T\int_0^\alpha b(x,t)u\Im_x u_t dxdt. \tag{3.7}$$

$$\left(c(x,t)u,-\Im_x^2 u_t\right)_{L^2(Q^T)}=\int_0^T\int_0^\alpha c(x,t)u\Im_x^2 u_t dxdt. \tag{3.8}$$

Substituting (3.5), (3.6), (3.7) and (3.8) into (3.5), we get

$$\int_0^T\int_0^\alpha {}^c\partial_t^\beta\ (\Im_x u_t)\,\Im_x u_t dxdt+\frac{1}{2}\int_0^\alpha a(x,t)u^2(x,T)dx$$

$$=\frac{1}{2}\int_0^\alpha a(x,t)\varphi^2(x)dx-\int_0^T\int_0^\alpha b_x(x,t)u\Im_x^2 u_t dxdt-\int_0^T\int_0^\alpha b(x,t)u\Im_x u_t dxdt$$

$$-\int_0^T\int_0^\alpha c(x,t)u\Im_x^2 u_t dxdt-\int_0^T\int_0^\alpha a_x(x,t)u\Im_x u_t dxdt+\left(g,-\Im_x^2 u_t\right)_{L^2(Q^T)}. \tag{3.9}$$

By using *poincaré*-type inequalities and a Cauchy-ε inequality estimate last five terms of the left side of (3.9), we get

$$-\int_0^T\int_0^\alpha b_x(x,t)u\Im_x^2 u_t dxdt\le\frac{b_2\varepsilon}{2}\|u\|^2_{L^2(Q^T)}+\frac{\alpha b_2}{2\varepsilon}\left\|\Im u_t\right\|^2_{L^2(Q^T)}, \tag{3.10}$$

$$-\int_0^T\int_0^\alpha b(x,t)u\Im_x u_t dxdt\le\frac{b_1\varepsilon}{2}\|u\|^2_{L^2(Q^T)}+\frac{b_1}{2\varepsilon}\left\|\Im_x u_t\right\|^2_{L^2(Q^T)}, \tag{3.11}$$

$$-\int_0^T\int_0^\alpha c(x,t)u\Im_x^2u_t dxdt \le \frac{c_1\varepsilon}{2}\|u\|^2_{L^2(Q^T)} + \frac{\alpha c}{2\varepsilon}\left\|\Im_x u_t\right\|^2_{L^2(Q^T)}, \tag{3.12}$$

$$-\int_0^T\int_0^\alpha a_x(x,t)u\Im_x u_t dxdt \le \frac{a_1\varepsilon}{2}\|u\|^2_{L^2(Q^T)} + \frac{a_1}{2\varepsilon}\left\|\Im_x u_t\right\|^2_{L^2(Q^T)}, \tag{3.13}$$

$$-\left(g, \Im_x^2 u_t\right)_{L^2(Q^T)} \le \frac{\varepsilon}{2}\|g\|^2_{L^2(Q^T)} + \frac{\alpha}{4\varepsilon}\left\|\Im_x u_t\right\|^2_{L^2(Q^T)}. \tag{3.14}$$

Combining of equalities (3.10), (3.14) and (3.9) yields

$$\int_0^T\int_0^\alpha {}^c\partial_t^\beta\ (\Im_x u_t)\,\Im_x u_t dxdt + (1-(\frac{\alpha b_2 + a_1 + \alpha c + b_1 + \alpha}{2\varepsilon}))\left\|\Im_x u_t\right\|^2_{L^2(Q^T)}$$

$$-(\frac{a_1\varepsilon + c_1\varepsilon + b_1\varepsilon + b_2\varepsilon}{2})\|u\|^2_{L^2(Q^T)} \le \frac{\varepsilon}{2}\|g\|^2_{L^2(Q^T)} + \frac{a_1}{2}\|\varphi\|^2_{L^2(0,\alpha)}. \tag{3.15}$$

Consequently,

$$2\int_0^T\int_0^\alpha \left({}^c\partial_t^\beta\ \Im_x u_t\right)\Im_x u_t dxdt + \|u\|^2_{L^2(Q^T)} \le k\|H\|^2_{L^2(Q^T)}, \tag{3.16}$$

where

$$\|H\|^2_{L^2(Q^T)} = \left(\|g\|^2_{L^2(Q^T)} + \|\varphi\|^2_{L^2(0,\alpha)}\right)$$

$$K = \frac{\max(\varepsilon, a_1)}{\min(2-(\frac{\alpha b_2+a_1+\alpha c+b_1+\alpha}{\varepsilon}),\ (a_1\varepsilon + c\varepsilon + b_1\varepsilon + b_2\varepsilon))}.$$

Estimating the first term on the left side of (3.15), we get

$${}^c\partial_t^\beta\left\|\Im_x u_t\right\|^2_{L^2(Q^T)} = \int_0^T\int_0^\alpha {}^c\partial_t^\beta\ (\Im_x u_t)^2\,dxdt$$

$$\le 2\int_0^T\int_0^a {}^c\partial_t^\beta\ (\Im_x u_t)(\Im_x u_t)dxdt. \tag{3.17}$$

Thus, inequality (3.16) takes the form

$$\partial_t^\beta\left\|\Im_x u_t\right\|^2_{L^2(Q^T)} + \|u\|^2_{L^2(Q^T)} \le K\|H\|^2_{L^2(Q^T)}, \tag{3.18}$$

where

$$K = \frac{\max(\varepsilon, a_1)}{\min(2-(\frac{\alpha b_2+a_1+\alpha c+b_1+\alpha}{\varepsilon}),\ (a_1\varepsilon + c\varepsilon + b_1\varepsilon + b_2\varepsilon))}.$$

□

4. Solvability of the posed problem

To establish the existence of the solution to problems (1.1)–(1.6), we argueusing density argument. That is, we show that ImL, the image of the operator L is dense in the space $L^2(Q)$ for every element u in the Banach space B. For this, we consider the following theorem:

Theorem 2. *For all functions $u \in B$, and for some function $z \in L^2(Q^T)$,*

$$(Lu.Z)_{L^2(Q^T)} = 0. \tag{4.1}$$

Then z is zero a.e in the domain Q^T.

Proof. We see from (4.1) that

$$({}^c\partial_t^\beta u_t - (a(x,t)\frac{\partial u}{\partial x})_x + b(x,t)u_x + c(x,t)u, z)_{L^2(Q^T)} = 0. \tag{4.2}$$

Because (4.1) holds for any function u in B, it can be expressed in a particular form. Assume that a function $\sigma(x,t)$ satisfies the conditions (1.3)–(1.4) such that

$$\sigma, (x\Im_t\sigma(x,s))_x, {}^c\partial_t^\beta\sigma \in L^2(Q^T).$$

From the previous discussion, we introduce the function

$$u = \int_0^t \int_0^s \sigma(x,s)\,dsdt = \Im_t^2\sigma. \tag{4.3}$$

Equation (4.2) reduces to

$$({}^c\partial_t^\beta \Im_t\sigma - (a(x,t)\frac{\partial \Im_t^2\sigma}{\partial x})_x + b(x,t)\Im_t\sigma + c(x,t)\Im_t^2\sigma, z)_{L^2(Q^T)} = 0, \tag{4.4}$$

where

$$z(x,t) = -\int_0^x \int_0^\xi \int_0^t u_t(\eta,s)\,dsd\eta d\zeta = -\Im_x^2\Im_t\sigma. \tag{4.5}$$

In this case, equation (4.4) can be written in the form

$$\left(\partial_t^\beta\Im_t\sigma, -\Im_x^2\Im_t u_t\right)_{L^2(Q^T)} - \left((a(x,t)\frac{\partial\Im_t^2\sigma}{\partial x})_x, -\Im_x^2\Im_t\sigma\right)_{L^2(Q^T)}$$

$$+\left(b(x,t)\Im_t\sigma, -\Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} + \left(c(x,t)\Im_t^2\sigma, -\Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} = 0. \tag{4.6}$$

Integrating by parts, and keeping in mind that the function satisfies (1.2)–(1.4), we deduce the following expressions for each term in (4.5)

$$-\left(\partial_t^\beta\Im_t\sigma, \Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} = -\int_0^T\int_0^\alpha \left(\partial_t^\beta\,\Im_t\sigma\right)\Im_x^2\Im_t\sigma dxdt$$

$$= -\int_0^T \Im_x\left({}^c\partial_t^\beta\,\Im_t\sigma\right)\Im_x^2\Im_t\sigma\,|_{x=0}^{x=\alpha}\,dt + \int_0^T\int_0^\alpha (\Im_x^c\partial_t^\beta\Im_t\sigma)\Im_x\Im_t\sigma dxdt$$

$$= \left(\partial_t^\beta(\Im_x\Im_t\sigma), \Im_x\Im_t\sigma\right)_{L^2(Q^T)}. \tag{4.7}$$

$$\left((a(x,t)\frac{\partial\Im_t^2\sigma}{\partial x})_x, \Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} = \int_0^T\int_0^\alpha (a(x,t)\frac{\partial\Im_t^2\sigma}{\partial x})_x(\Im_x^2\Im_t\sigma)dxdt$$

$$= \int_0^T a(x,t)\frac{\partial \Im_t^2\sigma}{\partial x}(\Im_x^2\Im_t\sigma)\mid_{x=0}^{x=\alpha} dt - \int_0^T\int_0^\alpha a(x,t)\frac{\partial \Im_t^2\sigma}{\partial x}\Im_x\Im_t\sigma dxdt$$

$$= -\int_0^T\int_0^\alpha a(x,t)\frac{\partial \Im_t^2\sigma}{\partial x}\Im_x\Im_t\sigma dxdt$$

$$= -\int_0^T a(x,t)\Im_t^2\sigma(\Im_x\Im_t\sigma)\mid_{x=0}^{x=\alpha} dt + \int_0^T\int_0^\alpha a(x,t)(\Im_t^2\sigma)\Im_t\sigma dxdt$$

$$+ \int_0^T\int_0^\alpha a_x(x,t)(\Im_t^2\sigma)\Im_x\Im_t\sigma dxdt$$

$$= \frac{1}{2}\int_0^\alpha a(x,t)(\Im_t^2\sigma)^2\mid_{t=0}^{t=T} dx + \int_0^T\int_0^\alpha a_x(x,t)(\Im_t^2\sigma)\Im_x\Im_t\sigma dxdt$$

$$= \frac{1}{2}\int_0^\alpha a(x,t)(\Im_t^2\sigma(x,T))^2 + \int_0^T\int_0^\alpha a_x(x,t)(\Im_t^2\sigma)\Im_x\Im_t\sigma dxdt. \tag{4.8}$$

$$\left(b(x,t)\Im_t\sigma, -\Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} = -\int_0^T\int_0^\alpha b(x,t)\Im_t\sigma(\Im_x^2\Im_t\sigma)dxdt$$

$$-\int_0^T b(x,t)\Im_x\Im_t\sigma(\Im_x^2\Im_t\sigma)\mid_{x=0}^{x=\alpha} dt + \int_0^T\int_0^\alpha b(x,t)(\Im_x\Im_t\sigma)^2 dxdt$$

$$+ \int_0^T\int_0^\alpha b(x,t)\Im_x\Im_t\sigma(\Im_x^2\Im_t\sigma)dxdt$$

$$= \int_0^T\int_0^\alpha b(x,t)(\Im_x\Im_t\sigma)^2 dxdt + \int_0^T\int_0^\alpha b(x,t)\Im_x\Im_t\sigma(\Im_x^2\Im_t\sigma)dxdt. \tag{4.9}$$

$$\left(c(x,t)\Im_t^2\sigma, -\Im_x^2\Im_t\sigma\right)_{L^2(Q^T)} = -\int_0^T\int_0^\alpha c(x,t)\Im_t^2\sigma(\Im_x^2\Im_t\sigma)dxdt. \tag{4.10}$$

Substituting formulas (4.7) and (4.10) into (4.6), we get

$$\left(\partial_t^\beta(\Im_x\Im_t\sigma), \Im_x\Im_t\sigma\right)_{L^2(Q^T)}$$

$$= -\frac{1}{2}\int_0^\alpha a(x,t)(\Im_t^2\sigma(x,T))^2 - \int_0^T\int_0^\alpha a_x(x,t)(\Im_t^2\sigma)\Im_x\Im_t\sigma dxdt$$

$$-\int_0^T\int_0^\alpha b(x,t)(\Im_x\Im_t\sigma)^2 dxdt - \int_0^T\int_0^\alpha b(x,t)\Im_x\Im_t\sigma(\Im_x^2\Im_t\sigma)dxdt$$

$$+ \int_0^T\int_0^\alpha c(x,t)\Im_t^2\sigma(\Im_x^2\Im_t\sigma)dxdt. \tag{4.11}$$

By using Poincaré-type inequalities and the Cauchy-ε inequality estimate for the last four terms of left side of (4.12), we get

$$-\int_0^T\int_0^\alpha a_x(x,t)(\Im_t^2\sigma)\Im_x\Im_t\sigma dxdt \le \frac{a_2\varepsilon}{2}\left\|\Im_t^2\sigma\right\|_{L^2(Q^T)}^2 + \frac{a_2}{2\varepsilon}\left\|\Im_x\Im_t\sigma\right\|_{L^2(Q^T)}^2. \tag{4.12}$$

$$\int_0^T \int_0^\alpha b(x,t)(\Im_x \Im_t \sigma)^2 dxdt \leq \; b_1 \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)} \tag{4.13}$$

$$\int_0^T \int_0^\alpha b(x,t)\Im_x \Im_t \sigma (\Im_x^2 \Im_t \sigma) dxdt \leq \frac{b_1 \varepsilon}{2} \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)} + \frac{b_1 \alpha}{4\varepsilon} \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)}. \tag{4.14}$$

$$\int_0^T \int_0^\alpha c(x,t)\Im_t^2 \sigma (\Im_x^2 \Im_t \sigma) dxdt \leq \frac{c_1 \varepsilon}{2} \left\| \Im_t^2 \sigma \right\|^2_{L^2(Q^T)} + \frac{c_1 \alpha}{2\varepsilon} \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)}. \tag{4.15}$$

Combining equalities (4.12)–(4.15) and (4.12) yields

$$\left(\partial_t^\beta (\Im_x \Im_t \sigma), \Im_x \Im_t \sigma\right)_{L^2(Q^T)} - (b_1 + (\frac{2a_2 + b_1 \alpha + 2c_1 \alpha}{4\varepsilon})) \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)}$$

$$\leq (\frac{a_2 \varepsilon + c_1 \varepsilon}{2}) \left\| \Im_t^2 \sigma \right\|^2_{L^2(Q^T)} - \frac{1}{2} \left\| \sqrt{a(x,t)} (\Im_t^2 \sigma(x,T)) \right\|^2_{L^2(0.\alpha)}. \tag{4.16}$$

We have, by lemma 3,

$$\partial_t^\beta (\Im_x \Im_t \sigma)^2 \leq 2 \left(\partial_t^\beta (\Im_x \Im_t \sigma), \Im_x \Im_t \sigma\right)_{L^2(Q^T)}. \tag{4.17}$$

Consequently, in light of (4.18) and (4.17), inequality (4.17) becomes

$$\partial_t^\beta (\Im_x \Im_t \sigma)^2 - (2b_1 + (\frac{2a_2 + b_1 \alpha + 2c_1 \alpha}{2\varepsilon})) \left\| \Im_x \Im_t \sigma \right\|^2_{L^2(Q^T)}$$

$$\leq (a_2 \varepsilon + c_1 \varepsilon) \left\| \Im_t^2 \sigma \right\|^2_{L^2(Q^T)} - \left\| \sqrt{a(x,t)} (\Im_t^2 \sigma\,(x,T) \right\|^2_{L^2(0.\alpha)}. \tag{4.18}$$

Take ε, such that

$$(2b_1 + (\frac{2a_2 + b_1 \alpha + 2c_1 \alpha}{2\varepsilon})) \geq 0.$$

Consequently,

$$\partial_t^\beta (\Im_x \Im_t \sigma)^2$$

$$\leq (a_2 \varepsilon + c_1 \varepsilon) \left\| \Im_t^2 \sigma \right\|^2_{L^2(Q^T)} - \sqrt{a(x,t)} \left\| (\Im_t^2 \sigma\,(x,T) \right\|^2_{L^2(0.\alpha)} \leq 0. \tag{4.19}$$

We conclude that z is zero a.e in the domain Q^T. □

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. A. Alikhanov, A priori estimates for solutions of boundary value problems for fractional-order equations, *Diff. Equat.*, **46** (2010), 660–666. https://doi.org/10.1134/S0012266110050058
2. N. H. Abel, Resolution d'un probleme de mecanique, In: *Oeuvres completes de Niels Henrik Abel*, Cambridge University Press, 2012, 97–101. https://doi.org/10.1017/CBO9781139245807.010
3. M. Caputo, *Elasticita e Dissipazione*, Bologna: Zanichelli, 1969.

4. A. M. Keightley, J. C. Myland, K. B. Oldham, P. G. Symons, Reversible cyclic voltammetry in the presence of product, *J. Electroanal. Chem.*, **322** (1992), 25–54. https://doi.org/10.1016/0022-0728(92)80065-C

5. O. A. Ladyzhenskaya, *Kraevye zadachi matematicheskoi fiziki*, (Russian), Moscow: Nauka, 1973.

6. S. Messloub, Existence and uniqueness results for a fractional two-times evolution problem with constraints of purely integral type, *Math. Method. Appl. Sci.*, **39** (2016), 1558–1567. https://doi.org/10.1002/mma.3589

7. S. Mesloub, A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation, *J. Math. Anal. Appl.*, **316** (2006), 189–209. https://doi.org/10.1016/j.jmaa.2005.04.072

8. A. M. Nakhushev, Drobnoe ischislenie i ego primenenie, (Russian), Moscow, 2003.

9. A. V. Pskhu, Uravneniya v chastnykh proizvodnykh drobnogo poryadka, (Russian), Moscow: Nauka, 2005.

10. S. G. Samko, A. A. Kilbas, O. I. Marichev, Integraly i proizvodnye drobnogo poryadka i nekotorye ikh prilozheniya, (Russian), Minsk: Naukai Tekhnika, 1987.

11. F. I. Taukenova, M. Kh. Shkhanukov-Lafishev, Difference methods for solving boundary value problems for fractional differential quations, *Comput. Math. and Math. Phys.*, **46** (2006), 1785–1795. https://doi.org/10.1134/S0965542506100149