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Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent

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Abstract: In this paper, we introduce grand weighted Herz spaces with variable exponent and prove the boundedness of fractional integrals on these spaces.

Keywords: fractional integrals; grand Herz spaces; weighted Herz spaces; grand weighted Herz spaces
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1. Introduction

During the previous decade, there was a vast boom of research in the so-called variable exponent spaces because they can be used to model electrorheological fluids, image restoration, and continuum medium mechanics see, for instance, [9–14, 17, 18, 21, 31, 32]. Consequently, over hundred scholars have contributed to the study of function spaces and related differential equations, so the theory of function spaces advanced quickly. For the time being, the theory of such variable exponent Lebesgue, Orlicz, Sobolev and Lorentz function spaces is widely developed, cf. [3, 4, 7, 8, 19, 20, 29, 33, 34]. The first generalization of variable exponent Herz spaces was established in [14]. The most general results were obtained in [1], where α was a variable. In [28], variable parameters were used to define continual Herz spaces, and the boundedness of sublinear operators (including the maximal function and Calderón-Zygmund singular operators)was proved in these spaces.

Boundedness of other operators, such as Riesz potential operators and the Marcinkiewicz integral, as proved in [22, 27]. The concept of Morrey spaces $L^{p,\lambda}$ was introduced by C. Morrey in 1938 (see [23]) in order to study regularity questions which appear in the calculus of variations.They describe local regularity more precisely than Lebesgue spaces and are widely used in not only harmonic analysis but also partial differential equations. In [22], Meskhi introduced the idea of grand Morrey spaces $L^{r),\theta,\lambda}$ and derived the boundedness of a class of integral operators (Hardy-Littlewood maximal functions, Calderón-Zygmund singular integrals and potentials) in these spaces. Muckenhoupt [24] has established the theory on weights called the Muckenhoupt A_p theory in the study of weighted function spaces and greatly developed real analysis. Recently, a generalization of the Muceknhoupt weights in terms of variable exponents has been studied in [2, 5]. Weighted norm inequalities for the maximal operator on variable Lebesgue spaces were proved in [5]. Boundedness of the fractional integrals on variable weighted Lebesgue spaces by using the extrapolation theorem can be checked in [6, 35].

Rafeiro et al. [25, 26] established the idea of grand variable Herz spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ and derived the boundedness of the sublinear operators and the Marcinkiewicz integral on $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$. Inspired by the concept, in this article we introduce the concept of grand weighted Herz spaces with variable exponent and prove the boundedness of the fractional integral operator in these spaces. There are four sections in this article. The first section is dedicated to the introduction, and the second section contains some basic definitions and lemmas. In the third section, we introduce the concept of grand weighted Herz spaces with variable exponent, and the boundedness of the fractional integral operator on grand weighted Herz spaces with variable exponent is proved in the last section.

2. Preliminaries

For this section, we refer to [8, 13, 15, 25, 27, 28, 30].

2.1. Lebesgue space with variable exponent

Assume that $G \subseteq \mathbb{R}^n$ is an open set, and $p(\cdot) : G \to [1, \infty)$ is a real-valued measurable function. Let the following condition holds:

$$1 \leq p_-(G) \leq p_+(G) < \infty, \tag{2.1}$$

where

i) $p_- := \operatorname{ess\,inf}_{g \in G} p(g)$,
ii) $p_+ := \operatorname{ess\,sup}_{g \in G} p(g)$.

Lebesgue space $L^{p(\cdot)}(G)$ is the space of measurable functions f on G such that,

$$I_{L^{p(\cdot)}}(f) = \int_G |f(g)|^{p(g)} dg < \infty,$$

and the norm is defined as

$$\|f\|_{L^{p(\cdot)}(G)} = \operatorname{ess\,inf} \left\{ \gamma > 0 : I_{L^{p(\cdot)}}\left(\frac{f}{\gamma}\right) \leq 1 \right\}.$$

This is the Banach function space and $p'(g) = \frac{p(g)}{p(g)-1}$ denotes the conjugate exponent of $p(g)$.

Next, we will define the space $L^{p(\cdot)}_{\text{loc}}(G)$ as,

$$L^{p(\cdot)}_{\text{loc}}(G) := \left\{\kappa : \kappa \in L^{p(\cdot)}(K) \textit{ for all compact subsets } K \subset G\right\}.$$

Now, to define the log-condition,

$$|\eta(z_1) - \eta(z_2)| \leq \frac{C}{-\ln|z_1 - z_2|}, \quad |z_1 - z_2| \leq \frac{1}{2}, \quad z_1, z_2 \in G, \tag{2.2}$$

where $C = C(\eta) > 0$ does not depend on z_1, z_2.

For the decay condition: Let $\eta_\infty \in (1, \infty)$, such that

$$|\eta(z_1) - \eta_\infty| \leq \frac{C}{\ln(e + |z_1|)}, \tag{2.3}$$

$$|\eta(z_1) - \eta_0| \leq \frac{C}{\ln|z_1|}, |z_1| \leq \frac{1}{2}. \tag{2.4}$$

Equation (2.4) holds for $\eta_0 \in (1, \infty)$ in the case of homogenous Herz spaces. We adopted the following notations in this paper:

(i) The Hardy-Littlewood maximal operator M for $f \in L^1_{\text{loc}}(G)$ is defined as

$$Mf(g) := \sup_{t>0} t^{-n} \int_{D(g,r)} |f(g)|dg \quad (g \in G),$$

where $D(g, t) := \{y \in G : |g - y| < t\}$.

(ii) The set $\mathcal{P}(G)$ is the collection of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(iii) A weight is a locally integrable and positive function which is defined on $\mathbb{R}^n$, and it can be written as $\omega(G) := \int_G \omega(g)dg$ for a weight w and measurable set G.

(iv) The set of $p(\cdot)$ satisfying (2.3) and (2.4) is represented by $LH(\mathbb{R}^n)$.

C is a constant which is independent of the main parameters involved, and its value varies from line to line.

Lemma 2.1 (Generalized Hölder's inequality)**.** *Assume that G is a measurable subset of* $\mathbb{R}^n$*, and* $1 \leq p_-(G) \leq p_+(G) \leq \infty$*. Then,*

$$\|fg\|_{L^{r(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}$$

holds, where $f \in L^{p(\cdot)}(G)$*,* $g \in L^{q(\cdot)}(G)$*, and* $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ *for every* $z \in G$*.*

2.2. *Herz spaces with variable exponent*

We adopted the following notations in this subsection:

(a) $\chi_k = \chi_{R_k}$.

(b) $R_k = D_k \setminus D_{k-1}$.

(c) $D_k = D(0, 2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.

(d) $R_{t,\tau} := D(0, \tau)\backslash D(0, t)$.

Definition 2.2. Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogenous Herz space $\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.5}$$

where

$$\|f\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} f \chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}}.$$

Definition 2.3. Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.6}$$

where

$$\|f\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} f \chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}} + \|f\|_{L^{s(\cdot)}(D(0,1))}.$$

2.3. The variable exponent Muckenhoupt weights

Let $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and w is a weight. The weighted Lebesgue space $L^{r(\cdot)}$ is the set of all complex-valued measurable functions f such that $f w^{\frac{1}{r(\cdot)}} \in L^{r(\cdot)}(\mathbb{R}^n)$. $L^{r(\cdot)}(w)$ is a Banach space its norm is given by

$$\|f\|_{L^{r(\cdot)}(w)} := \|f w^{\frac{1}{r(\cdot)}}\|_{L^{r(\cdot)}},$$

where $r'(\cdot)$ is the conjugate exponent of $r(\cdot)$ given by $\frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)} = 1$. Next we will define Muckenhoupt classes by starting with classical Muckenhoupt weights.

Definition 2.4. Suppose $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is called an $A_{r(\cdot)}$ weight if

$$\sup_{D:ball} \frac{1}{|D|} \|w^{\frac{1}{r(\cdot)}} \chi_D\|_{L^{r(\cdot)}} \|w^{\frac{-1}{r(\cdot)}} \chi_D\|_{L^{r'(\cdot)}} < \infty. \tag{2.7}$$

The set $A_{r(\cdot)}$ consists of all $A_{r(\cdot)}$ weights.

Now, we shall give the definitions of the Muckenhoupt classes A_r with $r = 1, \infty$.

Definition 2.5. (i) A weight w is called a Muckenhoupt A_1 weight if $Mw(z) \leq w(z)$ holds for almost every $z \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights. For every $w \in A_1$,

$$[w]_{A_1} := \sup_{D:ball} \left(\frac{1}{|D|} \int_D w(z) dz . \|w^{-1}\|_{L^\infty(D)} \right).$$

Then a finite value of $[w]_{A_1}$ is called an A_1 constant.

(ii) A weight is called a Muckenhoupt A_∞ weight if the weight belongs to the following set:

$$A_\infty := \bigcup_{1<r<\infty} A_r.$$

Definition 2.6. Suppose $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight is called an $A'_{r(\cdot)}$ weight if

$$\sup_{D:ball} |D|^{-P_D} \|w \chi_D\|_{L^1} \|w^{-1} \chi_D\|_{L^{r'(\cdot)/r(\cdot)}} < \infty, \tag{2.8}$$

where $P_D := (\frac{1}{|D|} \int_D \frac{1}{r(z)} dz)^{-1}$ is the harmonic average of $r(\cdot)$ over D. The set $A'_{r(\cdot)}$ consists of all $A'_{r(\cdot)}$ weights .

Definition 2.7. Let $0 < \alpha < n$ and $r_1(\cdot), r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{r_2(z)} \equiv \frac{1}{r_1(z)} - \frac{\alpha}{n}$. A weight w is called an $A(r_1(\cdot), r_2(\cdot))$ weight if

$$\|w\chi_D\|_{L^{r_2(\cdot)}}\|w^{-1}\chi_D\|_{L^{r_1'(\cdot)}} \leq |D|^{1-\frac{\alpha}{n}},$$

holds for all balls $D \subset \mathbb{R}^n$.

Lemma 2.8. *Assume that G is a Banach function space, and the Hardy-Littlewood maximal operator M is weakly bounded on G, that is,*

$$\|\chi_{(Mg>\lambda)}\|_G \leq \lambda^{-1}\|g\|_G, \tag{2.9}$$

is true for all $g \in G$ and all $\lambda > 0$. Then, we have

$$\sup_{D:ball} \frac{1}{|D|}\|\chi_D\|_G\|\chi_D\|_{G'} < \infty. \tag{2.10}$$

Lemma 2.9. *[16] Let X be a Banach function space, and M is bounded on the associate space X'. Then, there exists a constant $\delta \in (0,1)$ such that for all measurable sets $E \subset D$ and for all balls $D \subset \mathbb{R}^n$,*

$$\frac{\|\chi_E\|_X}{\|\chi_D\|_X} \leq \left(\frac{|E|}{|D|}\right)^{\delta}.$$

Let $r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{r_2(\cdot)} \in A_{r_2(\cdot)}$, $w^{-r_2'(\cdot)} \in A_{r_2'(\cdot)}$ and $\delta_1, \delta_2 \in (0,1)$.

$$\frac{\|\chi_E\|_{L^{r_2(\cdot)}(w^{r_2(\cdot)})}}{\|\chi_D\|_{L^{r_2(\cdot)}(w^{r_2(\cdot)})}} = \frac{\|\chi_E\|_{L^{r_2'(\cdot)}(w^{-r_2'(\cdot)})'}}{\|\chi_D\|_{L^{r_2'(\cdot)}(w^{-r_2'(\cdot)})'}} \leq \left(\frac{|E|}{|D|}\right)^{\delta_1}. \tag{2.11}$$

$$\frac{\|\chi_E\|_{L^{r_2'(\cdot)}(w^{r_2'(\cdot)})'}}{\|\chi_D\|_{L^{r_2'(\cdot)}(w^{r_2'(\cdot)})'}} \leq \left(\frac{|E|}{|D|}\right)^{\delta_2}. \tag{2.12}$$

For more details, see [16].

3. Grand weighted Herz spaces with variable exponent

In this section, first we will define grand Herz spaces and then introduce the concept of grand weighted Herz spaces with variable exponent.

Definition 3.1. [25] Let $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $r \in [1,\infty)$, $s : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand Herz spaces with variable exponent $\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)} = \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\epsilon)}\|f\chi_k\|^{r(1+\delta)}_{L^{s(\cdot)}}\right)^{\frac{1}{r(1+\delta)}} = \sup_{\delta>0}\delta^{\frac{\theta}{r(1+\delta)}}\|f\|_{\dot{K}^{\alpha,r),\theta}_{s(\cdot)}(\mathbb{R}^n)}.$$

Now, we will define a variable exponent weighted Lebesgue space.

Definition 3.2. [16] Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and w is a positive and locally integrable function on Ω. $L^{r(\cdot)}_{\mathrm{loc}}(\Omega, w)$ is the class of all functions f which satisfy the following condition: For all compact sets $E \subset \Omega$, there is a constant $\lambda > 0$ such that

$$\int_E \left|\frac{f(z)}{\lambda}\right|^{r(z)} w(z)dz < \infty.$$

Definition 3.3. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < r < \infty$, $\alpha \in \mathbb{R}$, $\theta > 0$. The homogeneous grand weighted Herz spaces with variable exponent $\dot{K}^{\alpha,r),\theta}_{q(\cdot)}(w)$ is the collection of $L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n/(\{0\}, w)$ is such that,

$$\dot{K}^{\alpha,r),\theta}_{q(\cdot)}(w) := \left\{L^{q(\cdot)}_{loc}(\mathbb{R}^n/\,(\{0\}, w)) : \|f\|_{\dot{K}^{\alpha,r),\theta}_{q(\cdot)}(w)} < \infty\right\}, \tag{3.1}$$

where

$$\|f\|_{\dot{K}^{\alpha,r),\theta}_{q(\cdot)}(w)} = \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)}\|f\chi_k\|^{r(1+\delta)}_{L^{q(\cdot)}(w)}\right)^{\frac{1}{r(1+\delta)}}.$$

Non-homogeneous grand weighted Herz spaces can be defined in a similar way.

4. Boundedness of the fractional integrals

Definition 4.1. Fractional integrals are given by the following.

Let $0 < \zeta < n$, and then the fractional integral operator I^{ζ} is defined by

$$I^{\zeta} f(z_1) := \int_{\mathbb{R}^n} \frac{f(z_2)}{|z_1 - z_2|^{n-\zeta}} dz_2. \tag{4.1}$$

Theorem 4.2. *[16] Let $r_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $0 < \zeta < n/r_1+$ and $\sigma := (n/\zeta)'$. Define $r_2(\cdot)$ by $1/r_2(\cdot) = 1/r_1(\cdot) - \zeta/n$. Then, for all weights w such that $(r_2(\cdot)/\sigma, w^{\sigma})$ is an M-pair, I^{ζ} is bounded from $L^{r_1(\cdot)}(w^{r_1(\cdot)})$ to $L^{r_2(\cdot)}(w^{r_2(\cdot)})$.*

Theorem 4.3. *Let $1 < r < \infty$, $q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{q_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0, 1)$ be the constants appearing in* (2.11) *and* (2.12)*, respectively. α and ζ are such that*

(i) $-n\delta_1 < \alpha < n\delta_2 - \zeta$.
(ii) $0 < \zeta < n(\delta_1 + \delta_2)$.

Define $q_1(\cdot)$ by $1/q_2(\cdot) = 1/q_1(\cdot) - \zeta/n$. Then, the fractional integral operator I^{ζ} is a bounded operator from $\dot{K}^{\alpha,r),\theta}_{q_2(\cdot)}(w^{q_2(\cdot)})$ to $\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})$.

Proof. Let $f \in \dot{K}^{\alpha,r),\theta}_{q_2(\cdot)}(w^{q_2(\cdot)})$, and $f_j := f\chi_j$ for any $j \in \mathbb{Z}$. Then, $f = \sum_{j=-\infty}^{\infty} f_j$, and we have

$$\begin{aligned}\|I^{\zeta} f\|_{\dot{K}^{\alpha,r),\theta}_{q_2(\cdot)}(w^{q_2(\cdot)})} &= \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)}\|\chi_k I^{\zeta} f\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\ &= \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)}\sum_{j=-\infty}^{\infty}\|\chi_k(I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\end{aligned}$$

$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\leq k-2}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$+\sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j=k-1}^{k+1}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$+\sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\geq k+2}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$=: E_1+E_2+E_3.$$

Operator I^{ζ} is bounded on weighted Lebesgue space, so for E_2,

$$E_2 \leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j=k-1}^{k+1}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j=k-1}^{k+1}\|(f\chi_j)\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\|(f\chi_k)\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \|f\|_{\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

For E_1, by using the size condition and Hölder's inequality,

$$|I^{\zeta}(f_j)(z_1)|\chi_k(z_1) \leq \chi_k(z_1)\int_{\mathbb{R}^n}|z_1-z_2|^{\zeta-n}|f_j(z_2)|dz_2$$
$$\leq 2^{k(\zeta-n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}.\chi_k(z_1).$$

By using Lemma (2.8), we get

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{(q_1(\cdot)}(w^{q_1(\cdot)}))'}2^{-kn}\|\chi_{D_k}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.$$

By using (2.12), we have

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}$$
$$= 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\frac{\|\chi_{D_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}{\|\chi_{D_k}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}$$
$$\leq 2^{k\zeta}2^{n\delta_2(j-k)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.$$

By the boundednesss of $I^{\zeta}: L^{q_1(\cdot)}(w^{q_1(\cdot)}) \to L^{q_2(\cdot)}(w^{q_2(\cdot)})$, and the inequality $2^{j\zeta}\chi_{D_j} \leq (I^{\zeta}f_{D_j})(x)$, we have

$$\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{-j\zeta}\|I^{\zeta}\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{-j\zeta}\|\chi_{D_j}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

By using Lemma (2.8) again, we obtain

$$\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq 2^{-j\zeta}\|\chi_{D_j}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \leq 2^{j(n-\zeta)}\|\chi_{D_j}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \leq 2^{j(n-\zeta)}\|\chi_j\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}.$$

By using the above inequalities, we get

$$\begin{aligned}
&\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
\leq&2^{k\zeta}2^{n\delta_2(j-k)}2^{j(n-\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{D_j}\|^{-1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_{D_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\\
=&2^{(\zeta-n\delta_2)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\left(2^{-jn}\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_{D_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\right)^{-1}\\
\leq&2^{(\zeta-n\delta_2)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

It is known that $\zeta - n\delta_2 + \alpha < 0$, so we will consider two cases: $1 < r(1+\delta) < \infty$ and $0 < r(1+\delta) \leq 1$. By considering, the first case, $1 < r(1+\delta) < \infty$, and by applying Hölder's inequality, we get

$$\begin{aligned}
E_1 &\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha r(1+\delta)}\left(\sum_{j=-\infty}^{k-2}\|\chi_k I^{\zeta}(f_j)\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\left(2^{\alpha j}\sum_{j=-\infty}^{k-2}2^{(\zeta-n\delta_2+\alpha)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\Bigg[\delta^{\theta}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)\\
&\times\left(\sum_{j=-\infty}^{k-2}2^{(\zeta-n\delta_2+\alpha)(k-j)(r(1+\delta))'/2}\right)^{\frac{r(1+\delta)}{(r(1+\delta))'}}\Bigg]^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{l=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq\|f\|_{\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

For $0 < r(1+\delta) \leq 1$, we get

$$\begin{aligned}
E_1 &\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\left(2^{\alpha j}\sum_{j=-\infty}^{k-2}2^{(\zeta-n\delta_2+\alpha)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}
\end{aligned}$$

$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{l=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \|f\|_{\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

Now, we will estimate E_3, by using the size condition, and Hölder's inequality, for $j,k\in\mathbb{Z}$ with $j\geq k+2$, We have

$$|I^{\zeta}(f_j)(z_1)|\chi_k(z_1)\leq \chi_k(z_1)\int_{D_j}|z_1-z_2|^{\zeta-n}|f_j(z_2)|dz_2$$
$$\leq 2^{j(\zeta-n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\chi_k(z_1).$$

By taking the $L^{p_2(\cdot)}(w^{p_2(\cdot)})$-norm, we have

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{j(-n+\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{j(-n+\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\frac{\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}$$
$$\leq 2^{j(-n+\zeta)}2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.$$

By the definition of $A(p_1(\cdot),p_2(\cdot))$, we obtain

$$\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\leq \|\chi_{D_j}\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq \|w^{-1}\chi_{D_j}\|_{L^{q_1'(\cdot)}}\|w\chi_{D_j}\|_{L^{q_2(\cdot)}}$$
$$\leq 2^{jn(1-\zeta/n)}.$$

Hence, we have

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{j(-n+\zeta)}2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq 2^{j(-n+\zeta)}2^{n\delta_1(k-j)}2^{jn(1-\zeta/n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$$
$$\leq 2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

Therefore, we get

$$E_3=\sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\geq k+2}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\geq k+2}2^{n\delta_1(k-j)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}$$

$$\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2}2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}.$$

For $\alpha + n\delta_1 > 0$, we will consider the two cases: $1 < r(1 + \delta) < \infty$ and $0 < r(1 + \delta) \leq 1$. Now, by considering the first case, $1 < r(1 + \delta) < \infty$, and by using Hölder's inequality, we have

$$\begin{aligned}
E_3 &\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2}2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\Bigg[\delta^{\theta}\sum_{k=-\infty}^{\infty}\left(\sum_{j\geq k+2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)\\
&\times\left(\sum_{j\geq k+2}2^{(n\delta_1+\alpha)(k-j)(r(1+\delta))'/2}\right)^{\frac{r(1+\delta)}{(r(1+\delta))'}}\Bigg]^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\sum_{j\geq k+2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \|f\|_{\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

For $0 < r(1 + \delta) \leq 1$, we get

$$\begin{aligned}
E_3 &\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2}2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{k=-\infty}^{\infty}\sum_{j\geq k+2}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \sup_{\delta>0}\left(\delta^{\theta}\sum_{j=-\infty}^{\infty}2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\
&\leq \|f\|_{\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(w^{q_1(\cdot)})},
\end{aligned}$$

which completes the proof. □

5. Conclusions

In this work, we have introduced a new type of space called grand weighted Herz spaces with variable exponents, and we have proved the boundedness of the fractional integrals on those spaces. This spaces will open the door for many future research work in this field.

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Conflict of interest

The authors declare no conflicts of interest.

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