Mathematics

Research article

The dynamic and discrete systems of variable fractional order in the sense of the Lozi structure map

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Abstract: The variable fractional Lozi map (VFLM) and the variable fractional flow map are two separate systems that we propose in this inquiry. We study several key dynamics of these maps. We also investigate the sufficient and necessary requirements for the stability and asymptotic stability of the variable fractional dynamic systems. As a result, we provide VFLM with the necessary criteria to produce stable and asymptotically stable zero solutions. Furthermore, we propose a combination of these maps in control rules intended to stabilize the system. In this analysis, we take the 1D- and 2D-controller laws as givens.

1. Introduction

A significant issue in elasticity theory is how to represent the characteristics of materials that could alter as a result of various activities. For this reason, a number of researchers have put forth variable-order fractional operators (VOFO), or operators whose order varies over time or in response to particular state variables. The interest can be traced back to the early work of Samko and Ross [22, 24], and further advancements in the field of VOFOs were made at the beginning of the

previous decade [17, 27]. The VOFOs that explicitly depend on a temperature field are modeled as random noise were suggested [11]. As an alternative to the model [5], Beltempo et al. [4] have discussed the use of VOFO to handle the aging of materials, such as concrete and other solid materials and polymers. In order to simulate real-world structures on computers, VOFO has been used to model the aging of materials and provide relaxation functions that are mathematically consistent and can be coded finite-element specific algorithms [6, 7]. VOFOs are clearly a special case of ordinary and fractional differential equations, which are the generalization of these classes when the fractional order is a constant. In reality, a lot of physics, monetary, and biological processes seem to behave in fractional orders, which can change over time and/or space [16, 20, 25, 28].

Current investigation has focused on the stabilization and synchronization of two recently suggested fractional order (constant fractional power) chaotic maps, the generalized 3D fractional Logistic [23], Henon and Lozi maps [8, 13, 14], and the 2D fractional Logistic, Henon and Lozi map [2, 12, 21]. In this effort, we shall consider the VOFOs to generalize the Lozi system (similarly for other maps) to deliver the variable fractional Lozi map (VFLM) and the variable fractional flow map. The stability and stabilizing of the system are studied, and some variable fractional order examples are illustrated in the sequel. Finally, an analysis is presented to study the proposed system involving the equilibrium points and the set of fixed points.

2. Methods

We have the following concepts:

2.1. The VOFOs

The classic arbitrary integration operator is considered by the integral formula

$$\mathfrak{I}^{\wp}\varphi(\eta) = \frac{1}{\Gamma(\wp)} \int_0^\eta (\eta - \varsigma)^{\wp-1} \varphi(\varsigma)\, d\varsigma.$$

For a general function φ and $0 < \wp < 1$, the classical arbitrary differentiation is given by the formula

$$\mathfrak{R}^{\wp}\varphi(\eta) = \frac{1}{\Gamma(1 - \wp)} \frac{d}{d\eta} \int_0^\eta \frac{\varphi(\varsigma)}{(\eta - \varsigma)^{\wp}}\, d\varsigma.$$

The Caputo arbitrary differential operator of order $0 < \wp < 1$ is formulated by the equation

$$\mathsf{C}^{\wp}\varphi(\eta) = \frac{1}{\Gamma(1 - \wp)} \int_0^\eta \frac{\varphi'(\varsigma)}{(\eta - \varsigma)^{\wp}}\, d\varsigma.$$

The VOFOs of the above operators are presented in [26]. Let $\wp(\eta)$ be a continuous function, then the VOFO for integration is written by the equation

$$\mathfrak{I}^{\wp(\eta)}\varphi(\eta) = \frac{1}{\Gamma(\wp(\eta))} \int_0^\eta (\eta - \varsigma)^{\wp(\varsigma)-1} \varphi(\varsigma)\, d\varsigma.$$

The classical fractional derivative is formulated by the structure

$$\mathfrak{R}^{\wp(\eta)}\varphi(\eta) = \frac{1}{\Gamma(1 - \wp(\eta))} \frac{d}{d\eta} \int_0^\eta \frac{\varphi(\varsigma)}{(\eta - \varsigma)^{\wp(\varsigma)}}\, d\varsigma,$$

and for the Caputo operator is given by the formula

$$\mathbb{C}^{\wp(\eta)}\varphi(\eta) = \frac{1}{\Gamma(1 - \wp(\eta))} \int_0^{\eta} \frac{\varphi'(\varsigma)}{(\eta - \varsigma)^{\wp(\varsigma)}} \, d\varsigma.$$

We proceed to introduce the VOFO for discrete formula type Caputo calculus:

Definition 2.1. The VOFO in terms of Caputo calculus is given by [5]:

$$\mathbb{C}^{\wp(\eta)}_{\aleph}\varphi(\eta) = \Delta_{\aleph}^{-(v-\wp(\eta))}\Delta^v\varphi(\eta) \tag{2.1}$$

$$= \frac{1}{\Gamma(v - \wp(\eta))} \sum_{\varsigma=\aleph}^{\eta-(v-\wp(\eta))} (\eta - \varsigma - 1)^{v-\wp(\eta)-1}\Delta^v\varphi(\varsigma),$$

where $\varphi : \mathbb{N}_{\aleph} := \{\aleph, \aleph + 1, \aleph + 2, ...\} \to \mathbb{R}, \wp(\eta) \notin \mathbb{N}, v = 1 + \lceil \wp(\eta) \rceil$ and $\eta \in \mathbb{N}_{\aleph+v-\wp(\eta)}$. Furthermore, the term $\eta^{(\wp(\eta))}$ is given by the fraction

$$\eta^{(\wp(\eta))} = \frac{\Gamma(\eta + 1)}{\Gamma(\eta + 1 - \wp(\eta))}, \quad \wp(\eta) > 0.$$

The corresponding discrete integral equation can be formulated by the sum

$$\varphi(\eta) = \varphi_0(\eta) + \frac{1}{\Gamma(\wp(\eta))} \sum_{\varsigma=\aleph+v-\wp(\eta)}^{\eta-\wp(\eta)} (\eta - \varsigma - 1)^{\wp(\eta)-1}\varphi(\wp(\eta) + \varsigma - 1). \tag{2.2}$$

Note that when $\wp(\eta)$ is a constant function, we obtain the discrete form in [1].

2.2. The VFLM

By using the VOFOs in the above part, we have the VFLM. Rene Lozi introduced the Lozi chaotic map in [18] and it is formulated by the structure

$$\begin{cases} \phi(v + 1) = -\alpha|\phi(v)| + \psi(v) + 1 \\ \psi(v + 1) = \beta\phi(v), \end{cases} \tag{2.3}$$

where $v \in \mathbb{N}$, $\phi(v)$ and $\psi(v)$ are the functions and the certain parameters α and β which are in $\mathbb{R}$. It is discovered that (2.3) contains a chaotic attractor with $(\alpha, \beta) = (1.7, 0.5)$. In view of Definition 2.1, we have the following VFLM

$$\begin{cases} \mathbb{C}^{\wp(\eta)}_{\aleph}\phi(\eta) = -\alpha|\phi(\eta - 1 + \wp(\eta))| + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) \\ \mathbb{C}^{\wp(\eta)}_{\aleph}\psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{2.4}$$

Clearly, when $\wp(\eta)$ is a constant function, we obtain the system in [15]

$$\begin{cases} \mathbb{C}^{\wp}_{\aleph}\phi(\eta) = -\alpha|\phi(\eta - 1 + \wp)| + \psi(\eta - 1 + \wp) + 1 - \phi(\eta - 1 + \wp) \\ \mathbb{C}^{\wp}_{\aleph}\psi(\eta) = \beta\phi(\eta - 1 + \wp) - \psi(\eta - 1 + \wp). \end{cases} \tag{2.5}$$

Now the integral difference in the above section implies

$$
\begin{cases}
\phi(\eta) = \phi(\aleph) + \dfrac{1}{\Gamma(\wp(\eta))} \sum_{\varsigma=\aleph+1}^{\eta-\wp(\eta)} (\eta - \varsigma - 1)^{\wp(\eta)-1} \\
\quad (-\alpha|\phi(\tau - 1 + \wp(\eta))| + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta))) \\
\psi(\eta) = \psi(\aleph) + \dfrac{1}{\Gamma(\wp(\eta))} \sum_{\varsigma=\aleph+1}^{\eta-\wp(\eta)} (\eta - \varsigma - 1)^{\wp(\eta)-1} \\
\quad (\beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta))),
\end{cases}
\tag{2.6}
$$

where

$$
\frac{(\eta - \varsigma - 1)^{\wp(\eta)-1}}{\Gamma(\wp(\eta))} = \frac{\Gamma(\eta - \varsigma)}{\Gamma(\wp(\eta)) \, \Gamma(\eta - \varsigma - \wp(\eta) + 1)}
$$

indicates the discrete kernel function. Note that when $\wp(\eta)$ is a constant function, we obtain the system in [15], as follows:

$$
\begin{cases}
\phi(\eta) = \phi(\aleph) + \dfrac{1}{\Gamma(\wp)} \sum_{\varsigma=\aleph+1}^{\eta-\wp} (\eta - \varsigma - 1)^{\wp-1} \\
\quad (-\alpha|\phi(\tau - 1 + \wp)| + \psi(\eta - 1 + \wp) + 1 - \phi(\eta - 1 + \wp)) \\
\psi(\eta) = \psi(\aleph) + \dfrac{1}{\Gamma(\wp)} \sum_{\varsigma=\aleph+1}^{\eta-\wp} (\eta - \varsigma - 1)^{\wp-1} \\
\quad (\beta\phi(\eta - 1 + \wp) - \psi(\eta - 1 + \wp))
\end{cases}
.
\tag{2.7}
$$

For numerical structure, when $\aleph = 0$, we get (see Figures 1 and 2)

$$
\begin{cases}
\phi(v) = \phi(0) + \dfrac{1}{\Gamma(\wp(v))} \sum_{j=1}^{v} \left(\dfrac{\Gamma(n - j + \wp(v))}{\Gamma(n - j + 1)} \right) \\
\quad (-\alpha|\phi(j - 1)| + \psi(j - 1) + 1 - \phi(j - 1)), \\
\psi(v) = \psi(0) + \dfrac{1}{\Gamma(\wp(v))} \sum_{j=1}^{v} \left(\dfrac{\Gamma(n - j + \wp(v))}{\Gamma(n - j + 1)} \right) \\
\quad (\beta\phi(j - 1) - \psi(j - 1)),
\end{cases}
\quad v \in \mathbb{N}, \;\; \wp(v) \in (0, 1], \;\; j \le v.
\tag{2.8}
$$

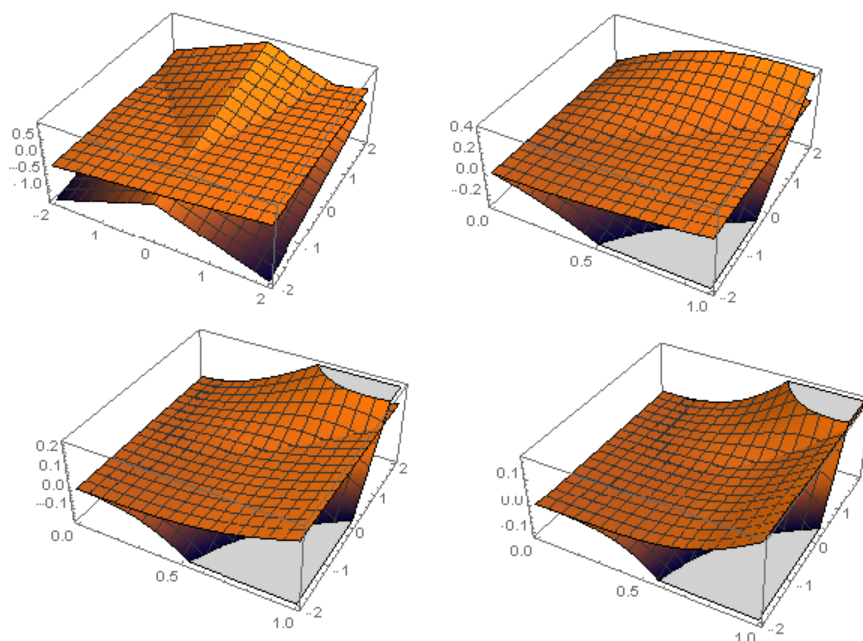Figure 3 shows the solution of the discrete systems, when $\wp(v) = 1/v, 0.99/v, 1.1/v, v \in \mathbb{N}$.

Figure 1. The 3D-plots of VFLM with $(\alpha, \beta) = (1.7, 0.5)$ for different values functional fractional order $\wp(\eta) \in (0, 1]$. From the left $\wp(\eta) = 0.9, \wp(\eta) = \eta, \wp(\eta) = \eta^2$ and $\wp(\eta) = \eta^3$.
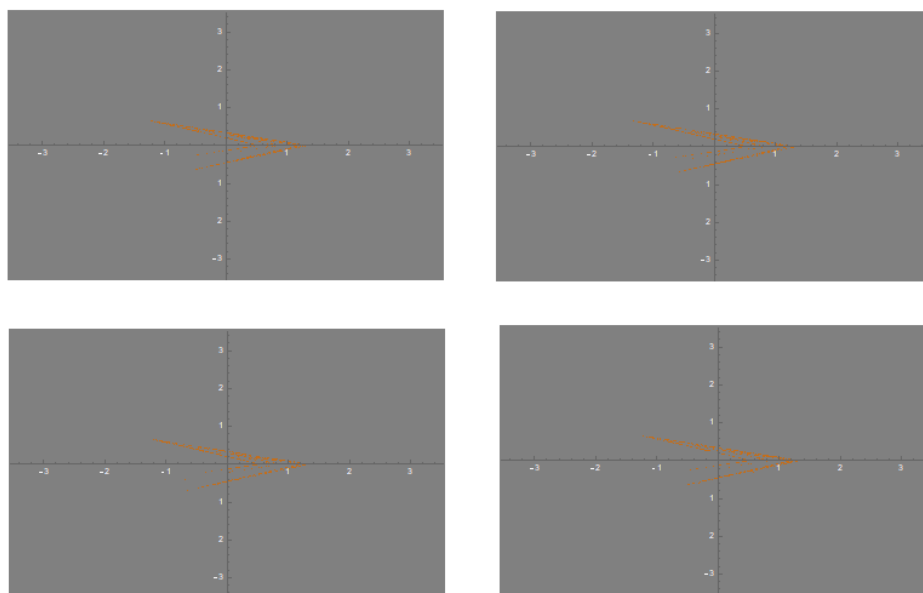

Figure 2. The behavior of the discrete VFLM, when $(\alpha, \beta) = (1.7, 0.5)$. From the left $\wp(v) = \sin(\frac{1}{1+v}), \wp(v) = \sin(\frac{1}{0.1+v}), \wp(v) = \sin(\frac{1}{0.01+v})$ and $\wp(v) = \sin(\frac{1}{0.001+v}), v \in \mathbb{N}$. The iteration is running from 1 to 1000.
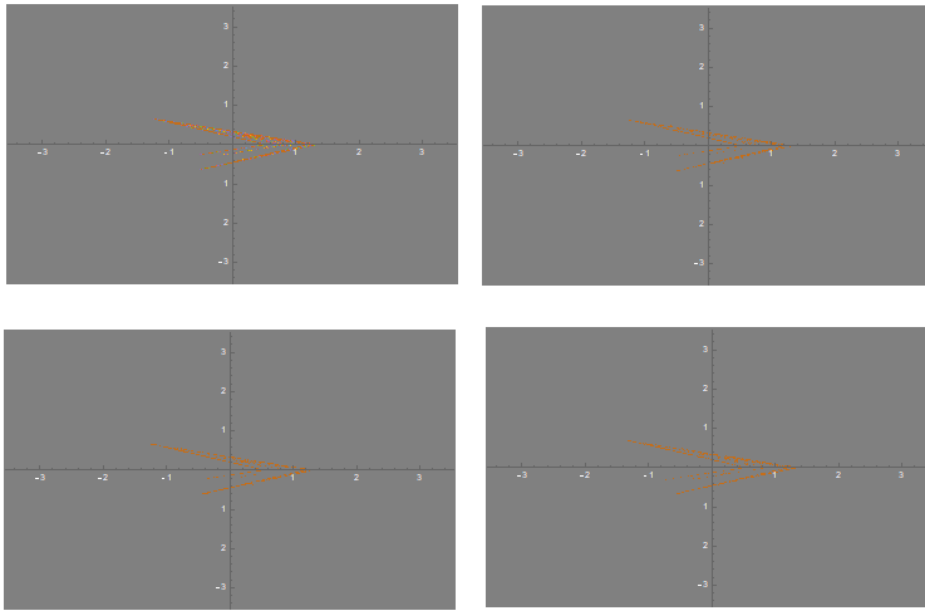
Figure 3. The solution of the discrete VFLM, when $(\alpha, \beta) = (1.7, 0.5)$. The left is the original system (2.3), $\wp(v) = 1/v$, $\wp(v) = 0.99/v$ and $\wp(v) = 1.1/v$, $v \in \mathbb{N}$. The iteration runs from 1 to 1000.

3. Stability

In this section, we look into the VFLM's overall stability.

3.1. Linear system

We begin with the definition below, which can be expanded into the n-dimensional space.

Definition 3.1. Suppose that $g(\eta) = g(\eta; \eta_0, g_0)$ is an outcome of the following equation

$$\mathsf{C}^{\wp(\eta)} g(\eta) = G(\eta, g), \quad \eta \in [\eta_0, \infty), \tag{3.1}$$

such that

- $g(\eta)$ is defined on the interval $[\eta_0, \infty)$;
- the point $(\eta, g(\eta)) \in \mathbb{S}$, where

$$\mathbb{S} := \{(\eta, g(\eta)) : \eta \in (s_1, \infty), \|g\| < \eta_0, \eta > \eta_0\}.$$

Then g is known as a stable outcome if there arises a positive real number $w > 0$ for all outcomes $g(\eta) = g(\eta; \eta_0, g_0) \in S$ owing the inequality

$$\|g_1 - g_0\| < w;$$

and for a given number $\varepsilon > 0$ there occurs $0 < \varpi \leq w$ with

$$\|g_1 - g_0\| < \varpi \Rightarrow$$

$$\|g(\eta; \eta_0, g_0) - g(\eta; \eta_0, g_1)\| < \varepsilon, \quad \eta \in [\eta_0, \infty).$$

Moreover, if

$$\lim_{\eta \to \infty} \|g(\eta; \eta_0, g_0) - g(\eta; \eta_0, g_1)\| = 0$$

then the solution g is asymptotically stable.

The following are the consequences:

Theorem 3.2. *Consider the linear system*

$$\mathbb{C}^{\wp(\eta)} \begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix} = \Xi_{2\times2} \begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix}. \tag{3.2}$$

Then all its outcomes are stable if and only if they are bounded.

Moreover, if the characteristic polynomial corresponds to Ξ is stable then the outcomes are asymptotically stable.

Proof. Via creating a matrix-valued function with two variables, Υ, as follows:

$$\Upsilon(\eta, \eta_0) = \mathbb{I}_d + \mathfrak{I}^\wp\Xi(\eta) + \mathfrak{I}^\wp\Xi(\eta)\mathfrak{I}^\wp\Xi(\delta_1) + ... + \mathfrak{I}^\wp\Xi(\eta)\mathfrak{I}^\wp\Xi(\delta_1)...\mathfrak{I}^\wp\Xi(\delta_{n-1}) + ...,$$

such that $\mathbb{I}_d$ presents the identity matrix. In view of $\mathfrak{I}^\wp$, we conclude that

$$\Upsilon(\eta_0, \eta_0) = \mathbb{I}_d.$$

Now, let the outcomes of system (3.2) be bounded. As a consequence, there occurs a fixed number $\kappa > 0$ satisfying the inequality $\|\Upsilon\| < \kappa$, where $\|.\|$ represents the max norm. This implies that

$$\|\phi(\eta) - \phi_0(\eta)\| < \frac{\varepsilon}{2\kappa}, \quad \|\psi(\eta) - \psi_0(\eta)\| < \frac{\varepsilon}{2\kappa}, \quad \varepsilon > 0.$$

Consequently, we obtain

$$\|\phi(\eta; \eta_0, \phi_0) - \phi(\eta; \eta_0, \chi_1)\| = \|\Upsilon(\eta, \eta_0)(\phi_0 - \phi_1)\| < \frac{\kappa\varepsilon}{2\kappa} = \frac{\varepsilon}{2}.$$

Similarly, we have

$$\|\psi(\eta; \eta_0, \psi_0) - \psi(\eta; \eta_0, \phi_1)\| = \|\Upsilon(x, x_0)(\psi_0 - \psi_1)\| < \frac{\kappa\varepsilon}{2\kappa} = \frac{\varepsilon}{2}.$$

Let $R = (\phi, \psi)^t$, then

$$\begin{aligned}
\|R(\eta) - R_0(\eta)\| &\leq \|\Upsilon(\eta, \eta_0)(R(\eta) - R_0(\eta))\| \\
&\leq \kappa\|R(\eta) - R_0(\eta)\| \\
&< \kappa(\frac{\varepsilon}{2\kappa} + \frac{\varepsilon}{2\kappa}) = \varepsilon.
\end{aligned}$$

Thus, all the solutions of system (3.1) are stable.

Contrariwise, the stability of the outcomes, including the zero solution yields that for a positive number $\varepsilon > 0$ there is a positive constant υ satisfying the inequality

$$\|R(\eta)\| < \upsilon \Rightarrow \|\Upsilon(\eta)R(\eta)\| < \varepsilon.$$

In particular,

$$\|\phi(\eta)\| = \|\phi(\eta; \eta_0, \phi_0)\| < \varepsilon/2$$

and

$$\|\psi(\eta)\| = \|\psi(\eta; \eta_0, \psi_0)\| < \varepsilon/2.$$

Which leads to all solutions are bounded.

Now, since the characteristic polynomial corresponding to Ξ is stable then the outcomes are asymptotically stable, because

$$\|\phi(\eta; \eta_0, \phi_0) - \phi(\eta; \eta_0, \phi_1)\| \leq \kappa \exp\left(\frac{\Xi(\eta - \eta_0)^{\wp(\eta)}}{\wp(\eta)}\right) \|\phi_1 - \phi_0\|$$

$$\leq \kappa \exp(-\varepsilon_1 \frac{\eta^{\wp(\eta)}}{\wp(\eta)}), \quad 0 < \varepsilon_1 < \varepsilon$$

$$= 0, \quad \eta \to \infty, \wp(\eta) \in (0, 1].$$

Similarly, for ψ, we have

$$\|\psi(\eta; \eta_0, \psi_0) - \psi(\eta; \eta_0, \psi_1)\| \leq c \exp\left(\frac{\Xi(\eta - \eta_0)^{\wp(\eta)}}{(\wp(\eta))}\right) \|\psi_1 - \psi_0\|$$

$$\leq \kappa \exp(-\varepsilon_1 \frac{\eta^{\wp(\eta)}}{\wp(\eta)}), \quad 0 < \varepsilon_1 < \varepsilon$$

$$= 0, \quad x \to \infty, \wp(\eta) \in (0, 1],$$

which implies the asymptotically stable outcomes. □

Corollary 3.3. *Consider the sup norm $\|\Xi\| < 1$ and all its eigenvalues are in the interval [0,1]. Then system* (3.2) *is asymptotically stable if and only if its outcomes are bounded.*

Proof. If $\|\Xi\| < 1$ and all its eigenvalues are in the interval [0, 1], then it is an invertible positive contraction [19]. Then $\Xi_{2\times2}^{-1} - \mathbb{I}_d$ is positive semi-definite, with $Det(\Xi_{2\times2}) > 0$ (see the proof of Proposition 3.5 [9]). This suggests that the Ξ characteristic polynomial is real stable. We have the solutions that are asymptotically stable in light of Theorem 3.2. □

Non-homogeneous case is given in the next outcome.

Theorem 3.4. *All of the continuous non-homogeneous system solutions that match Eq* (3.2)

$$\mathbb{C}^{\wp(\eta)}\begin{pmatrix}\phi(\eta)\\\psi(\eta)\end{pmatrix} = \Xi_{2\times2}\begin{pmatrix}\phi(\eta)\\\psi(\eta)\end{pmatrix} + \begin{pmatrix}\chi_1(\eta)\\\chi_2(\eta)\end{pmatrix} \tag{3.3}$$

are stable if and only if they are bounded and

$$\|X\| < \vartheta, \ \vartheta \in (0, \infty), X = (\chi_1, \chi_2)^t.$$

The solutions are additionally asymptotically stable if the characteristic polynomial Ξ is stable satisfying $\|\Xi\| < \kappa$ and

$$\kappa < \frac{e}{\vartheta}, \quad \vartheta > 0, \quad \|\Xi\| \leq \kappa.$$

Proof. Let $\|X(\eta)\| < \vartheta, \vartheta > 0$. The assumption of the theorem implies

$$\|\phi(\eta)\| \leq \kappa \exp\left(\kappa\vartheta \frac{\eta^{\wp(\eta)}}{\wp(\eta)}\right) \|\phi_0\|$$

$$\leq \kappa \exp\left((\kappa\vartheta - e)\frac{\eta^{\wp(\eta)}}{\wp(\eta)}\right) \|\phi_0\|$$

$$= 0, \quad \kappa\vartheta - e < 0, \wp(\eta) \in (0, 1], \eta \to \infty.$$

This proves the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An application of Theorem 3.4, is in the following outcome

Corollary 3.5. *Assume that the sup norm $\|\Xi\| < 1$ and all its eigenvalues are in the interval [0, 1] and $\|X\| < \vartheta, \vartheta > 0$. Consequently, system (3.3) admits asymptotically stable solutions whenever*

$$\kappa < \frac{e}{\vartheta}, \quad \vartheta > 0, \quad \|\Xi\| \leq \kappa.$$

3.2. Stability of VFLM

In this part, we discuss the stability of VFLM (continuous case) using the above results. We deliver the sufficient condition on the coefficients of the system in the following result:

Theorem 3.6. *Consider the continuous system VFLM*

$$\begin{cases} \mathbb{C}^{\wp(\eta)}\phi(\eta) = -\alpha|\phi(\eta)| + \psi(\eta) + 1 \\ \mathbb{C}^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta), \quad \wp(\eta) \in (0, 1]. \end{cases} \qquad (3.4)$$

If the following inequalities are satisfied

- $\dfrac{e-1}{2} < \alpha < e - 1$
- $0 < \beta < -\alpha + e - 1$
- $\alpha + \beta + 1 < \kappa < \dfrac{e}{\vartheta}$
- $1 < \vartheta < e/\kappa,$

then system (3.4) is asymptotically stable.

Proof. In matrix form, the system (3.3) becomes

$$\mathbb{C}^{\wp(\eta)}\begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix} = \begin{pmatrix} -\alpha & 1 \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad (3.5)$$

The characteristic polynomial takes the formula

$$-\beta + \alpha\eta + \eta^2 = 0,$$

with the two differences roots

$$r_1 = \frac{1}{2}\left(-\sqrt{\alpha^2 + 4\beta} - \alpha\right)$$

and

$$r_2 = \frac{1}{2}\left(\sqrt{\alpha^2 + 4\beta} - \alpha\right).$$

By letting $\sqrt{\alpha^2 + 4\beta} \geq 0$, we have $r_1 < 0$ providing $\alpha > 0$. Moreover, we have

$$\left\|\begin{pmatrix} -\alpha & 1 \\ \beta & 0 \end{pmatrix}\right\| = \alpha + \beta + 1 \leq \kappa, \quad \alpha + \beta + 1 < \kappa < \frac{e}{\vartheta}.$$

Hence, the characteristic polynomial is stable. All the conditions of Theorem 3.4 are achieved, then all the solutions are asymptotically stable. □

Figure 4 shows the dynamic of the characteristic polynomials for different values of $\alpha > 0$ and $\beta > 0$. Moreover, we have the following generalization result, with a proof similar to the one in [10].
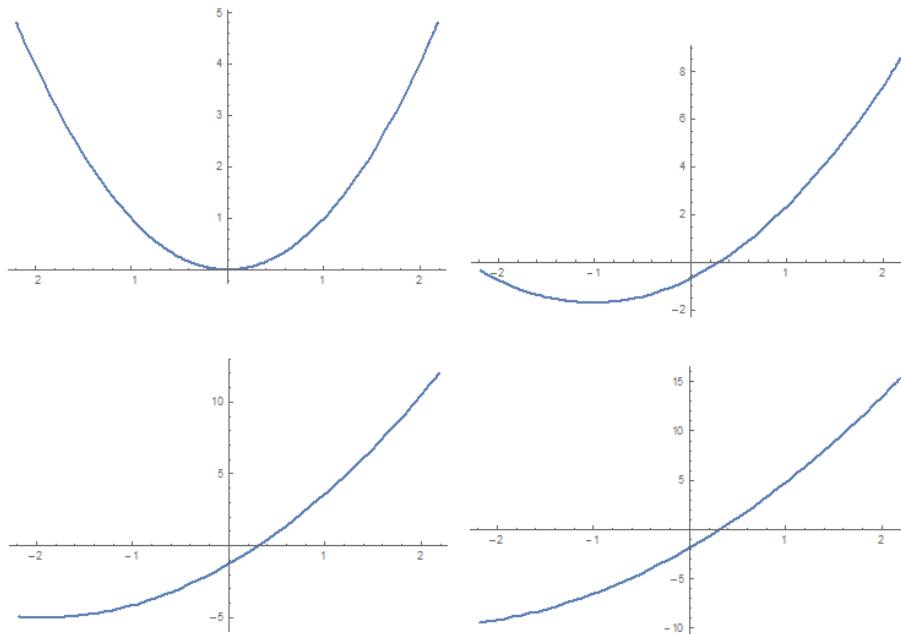

Figure 4. The characteristic polynomial with different values of α and β.

Theorem 3.7. *Consider the linear fractional-order discrete-time system, as follows:*

$$C_{\aleph}^{\wp(\eta)}\mathbb{X}(\eta) = \Pi_{n\times n}\mathbb{X}(\eta - 1 + \wp(\eta)), \quad \wp(\eta) \in (0, 1],$$

where $\mathbb{X} = (\chi_1(\eta), ..., \chi_n(\eta))^t, \eta \in \mathbb{N}_{\aleph+1-\wp(\eta)}.$

Then the zero equilibrium is asymptotically stable if and only if

$$\lambda \in S_{\|\wp\|} = \left\{\zeta : |\zeta| < \left(2\cos\left(\frac{|arg(\zeta)| - \pi}{2 - \|\wp\|}\right)\right)^{\|\wp\|}, |arg(\zeta)| > \frac{\pi\|\wp\|}{2}\right\},$$

for all the eigenvalues λ of $\Pi_{n \times n}$. Moreover, if all the complex eigenvalues are in the open unit disk

$$\lambda \in \mathbb{U} = \{\zeta : |\zeta| < 1\}$$

where

$$-\frac{\sqrt{2^{2\|\wp\|}}}{2} < \mathfrak{R}(\zeta) < \frac{\sqrt{2^{2\|\wp\|}}}{2}$$

and

$$\frac{-\sqrt{2^{2\|\wp\|} - \mathfrak{R}(\zeta)^2}}{2} < \mathfrak{I}(\zeta) < \frac{\sqrt{2^{2\|\wp\|} - \mathfrak{R}(\zeta)^2}}{2}, \quad 0 < \|\wp\| < 1;$$

or all the real eigenvalues in the unit interval [0,1], then the system is stable.

Proof. The first part of the theorem is similar to the proof of Theorem 1.4 [10]. For the second part, since all the eigenvalues are in the open unit disk, then the characteristic polynomial is stable which leads to the stability of the system. □

3.3. Nonlinear system

Consider the (2×2) VFLM (can be extended into $(n \times n)$ fractional system)

$$\mathbb{C}^{\wp(\eta)} \begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix} = \Xi_{2 \times 2} \begin{pmatrix} \phi(\eta) \\ \psi(\eta) \end{pmatrix} + \begin{pmatrix} h_1(\eta, \phi, \psi) \\ h_2(\eta, \phi, \psi) \end{pmatrix}, \tag{3.6}$$

where the continuous function $H(\eta, \phi, \psi) = (h_1(\eta, \phi, \psi), h_2(\eta, \phi, \psi))^t$ achieves the inequality

$$\|H\| \leq \mu \|R\| := \|(\phi, \psi)\|, \quad \mu > 0,$$

uniformly with respect to η.

Theorem 3.8. *Assume that Xi's characteristic polynomial is stable. If*

$$\frac{\mu \eta^{\|\wp\|}}{\Gamma(\|\wp\| + 1)} < 1, \quad \mu > 0, \eta \in [0, \infty)$$

then all the solutions of system (3.6) *are stable. Moreover, if $\kappa \mu < e$, then the zero solution of system* (3.6) *is asymptotically stable.*

Proof. In view the variable fractional integral formula, we have

$$\|R(\eta)\| \leq \|R_0(\eta)\| + \frac{1}{\Gamma(\wp(\eta))} \int_0^\eta \|(\eta - \varsigma)^{\wp(\eta)-1} H(\varsigma)\| d\varsigma$$

$$\leq \kappa + \frac{\mu \|R(\eta)\|}{\Gamma(\|\wp\| + 1)} \frac{\eta^{\|\wp\|}}{\|\wp\|}$$

which implies that

$$\|R\| \leq \frac{\kappa}{1 - \dfrac{\mu \eta^{\|\wp\|}}{\Gamma(\|\wp\| + 1)}}.$$

This yields that the solutions are bounded, then in view of the proof of Theorem 3.2, the solutions are stable. For the second part, we have

$$\|\phi(\eta)\| \leq \kappa \exp\left(\kappa\mu \frac{\eta^{\|\wp\|}}{\|\wp\|}\right) \|\phi_0\|$$

$$\leq \kappa \exp\left((\kappa\mu - e)\frac{\eta^{\|\wp\|}}{\|\wp\|}\right) \|\phi_0\|$$

$$= 0, \quad \kappa\mu - e < 0.$$

Similarly, for the variable ψ, we have

$$\|\psi(\eta)\| \leq c \exp\left(\kappa\mu \frac{\eta^{\|\wp\|}}{\|\wp\|}\right) \|\psi_0\|$$

$$\leq \kappa \exp\left((\kappa\mu - e)\frac{\eta^{\|\wp\|}}{\|\wp\|}\right) \|\psi_0\|$$

$$= 0, \quad \kappa\mu - e < 0.$$

Hence, the zero solution is asymptotically stable. □

4. Stabilizing

The exploration of chaotic systems, whether in discrete time or continuously, revolves around the development of control mechanisms to achieve stability. In this part, we will discuss some potential nonlinear control rules for stabilizing the aforementioned arbitrary order discrete-time systems. Stabilization involves applying a novel time-altering parameter, $\eth(\eta)$, to the particular system's states and devising an adaptive closed-form method for these parameters to quickly push the system's states to zero.

Theorem 4.1. *System* (2.4) *can be controlled by the 1D-control law*

$$\eth_\phi(\eta) = \alpha|\phi(\eta - 1 + \wp(\eta))| - \psi(\eta - 1 + \wp(\eta)) - 1.$$

Proof. In the controlled VFLM, which is represented by the symbol, the time-altering control parameter $\eth\phi(\eta)$ is employed.

$$\begin{cases} C^{\wp(\eta)}\phi(\eta) = -\alpha|\phi(\eta - 1 + \wp(\eta))| + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) + \eth_\phi(\eta) \\ C^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{4.1}$$

The simplified dynamics are obtained by substituting the proposed control law $\eth_\phi(x)$ into (4.1)

$$\begin{cases} C^{\wp(\eta)}\phi(\eta) = -\phi(\eta - 1 + \wp(\eta)) \\ C^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{4.2}$$

Hence, the set of eigenvalues is bounded where $\lambda_{1,2} = -1$ corresponding to the general eigenvectors $v = (\frac{1}{\beta}, 0), \beta \neq 0$. Then in view of Theorem 3.2, the zero solution is asymptotically stable. □

Theorem 4.2. *System* (2.4) *can be controlled by the 2D-control law*

$$U_\phi(\eta) = \alpha|\phi(\eta - 1 + \wp(\eta))| - \psi(\eta - 1 + \wp(\eta)) - 1 + (1 - \frac{\alpha}{4})\phi(\eta - 1 + \wp(\eta))$$

$$V_\psi(\eta) = -\beta\phi(\eta - 1 + \wp(\eta)) + \frac{3}{4}\psi(\eta - 1 + \wp(\eta)), \quad |\alpha| < 3.$$

Proof. The time-varying control parameter $(U_\phi(\eta), V_\psi(\eta))$ is employed in the 2D-controlled VFLM, which is formulated as follows:

$$\begin{cases} \mathbb{C}^{\wp(\eta)}\phi(\eta) = -\alpha|\phi(\eta - 1 + \wp(\eta))| + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) + U_\phi(\eta) \\ \mathbb{C}^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta - 1 + \sigma) - \psi(\eta - 1 + \wp(\eta)) + V_\psi(\eta). \end{cases} \tag{4.3}$$

The simplified dynamics are obtained by substituting the proposed 2D-control law $(U_\phi(\eta), V_\psi(\eta))$ into (4.3)

$$\begin{cases} \mathbb{C}^{\wp(\eta)}\phi(\eta) = \phi(\eta) = -\frac{\alpha}{4}\phi(\eta - 1 + \wp(\eta)) \\ \\ \mathbb{C}^{\wp(\eta)}\psi(\eta) = -\frac{1}{4}\psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{4.4}$$

Thus, the set of eigenvalues is bounded, with $\|\Xi\| < 1$, where

$$\lambda_1 = -1/4, \quad \lambda_2 = -\alpha/4, \quad |\alpha| < 3.$$

Therefore, the zero solution is asymptotically stable in light of Corollary 3.3. □

Theorem 4.2 is a one-dimensional parametric 2D-control law of the VFLM, which is α. The next theorem describes the two-dimensional parametric 2D-control law VFLM's stabilizing parameters, α and β.

Theorem 4.3. *System* (2.4) *can be controlled by the 2D-control law*

$$P_\phi(\eta) = \alpha|\phi(\eta - 1 + \wp(\eta))| - \psi(\eta - 1 + \wp(\eta)) - 1 + (1 - \frac{\alpha}{4})\phi(\eta - 1 + \wp(\eta))$$

$$Q_\psi(\eta) = \frac{3}{4}\psi(\eta - 1 + \wp(\eta)), \quad |\alpha| < 3.$$

Proof. The time-altering control parameter $(P_\phi(\eta), Q_\psi(\eta))$ is utilized in the 2D-controlled VFLM, which is formulated as follows:

$$\begin{cases} \mathbb{C}^{\wp(\eta)}\phi(\eta) = -\alpha|\phi(\eta - 1 + \wp(\eta))| + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) + P_\phi(\eta) \\ \mathbb{C}^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)) + Q_\psi(x). \end{cases} \tag{4.5}$$

The proposed 2D-control formula is substituted to provide the simplified dynamics $(P_\phi(\eta), Q_\psi(\eta))$ into (4.3)

$$\begin{cases} \mathbb{C}^{\wp(\eta)}\phi(\eta) = -\frac{\alpha}{4}\phi(\eta - 1 + \wp(\eta)) \\ \\ \mathbb{C}^{\wp(\eta)}\psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \frac{1}{4}\psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{4.6}$$

Consequently, the collection of eigenvalues is constrained, with $\|\Xi\| < 1$, where

$$\lambda_1 = -\frac{1}{4}, \quad \lambda_2 = -\frac{\alpha}{4}, \quad |\alpha| < 3, \quad \beta \neq 0,$$

corresponding to the general eigenvectors $W = (-\dfrac{\alpha - 1}{4\beta}, 1)$. Therefore, the zero solution is asymptotically stable in light of Corollary 3.3. $\qquad \square$

5. Analysis of system VFLM (2.4)

System (2.4) can be viewed as two dynamical systems

$$\begin{cases} C_{\aleph}^{\wp(\eta)} \phi(\eta) = -\alpha\phi(\eta - 1 + \wp(\eta)) + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) \\ C_{\aleph}^{\wp(\eta)} \psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)) \end{cases} \tag{5.1}$$

and

$$\begin{cases} C_{\aleph}^{\wp(\eta)} \phi(\eta) = \alpha\phi(\eta - 1 + \wp(\eta)) + \psi(\eta - 1 + \wp(\eta)) + 1 - \phi(\eta - 1 + \wp(\eta)) \\ C_{\aleph}^{\wp(\eta)} \psi(\eta) = \beta\phi(\eta - 1 + \wp(\eta)) - \psi(\eta - 1 + \wp(\eta)). \end{cases} \tag{5.2}$$

The solutions of these systems occurred in two different domains.

5.1. System (5.1)

The Jacobian matrix corresponds to the system (5.1) is given in the following structure:

$$J_1 = \wp'(\eta) \begin{pmatrix} -\alpha - 1 & 1 \\ \beta & -1 \end{pmatrix}$$

with its eigenvalues

$$\lambda_1 = \wp'(\eta) \left(\frac{-\sqrt{\alpha^2 + 4\beta} - \alpha - 2}{2} \right) < 0, \quad \lambda_2 = \wp'(\eta) \left(\frac{\sqrt{\alpha^2 + 4\beta} - \alpha - 2}{2} \right) \geq 0,$$

satisfying the relation $\wp'(\eta) > 0$, $\sqrt{\alpha^2 + 4\beta} > \alpha + 2$. The solution of this inequality is $\alpha < -2, \beta \geq -\alpha^2/4$; or $\alpha \geq -2, \beta > \alpha + 1$, providing that $\wp(x) \in (0, 1]$ is not a constant function. In addition, the corresponding eigenvectors are

$$v_1 = \left(\frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta}, 1 \right), \quad v_2 = \left(\frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2\beta}, 1 \right).$$

Or, we have the converse case when $\wp'(\eta)$ is negative:

$$\lambda_1 = \wp'(x) \left(\frac{-\sqrt{\alpha^2 + 4\beta} - \alpha - 2}{2} \right) \geq 0, \quad \lambda_2 = \wp'(\eta) \left(\frac{\sqrt{\alpha^2 + 4\beta} - \alpha - 2}{2} \right) < 0,$$

where $\wp'(\eta) < 0$, $\sqrt{\alpha^2 + 4\beta} > \alpha + 2$. Hence, we have the following domains of solutions (see Figure 5):

$$[D_1]_{J_1} = \{(\phi, \psi) \in \mathbb{R}^2 : \lambda \geq 0\}$$
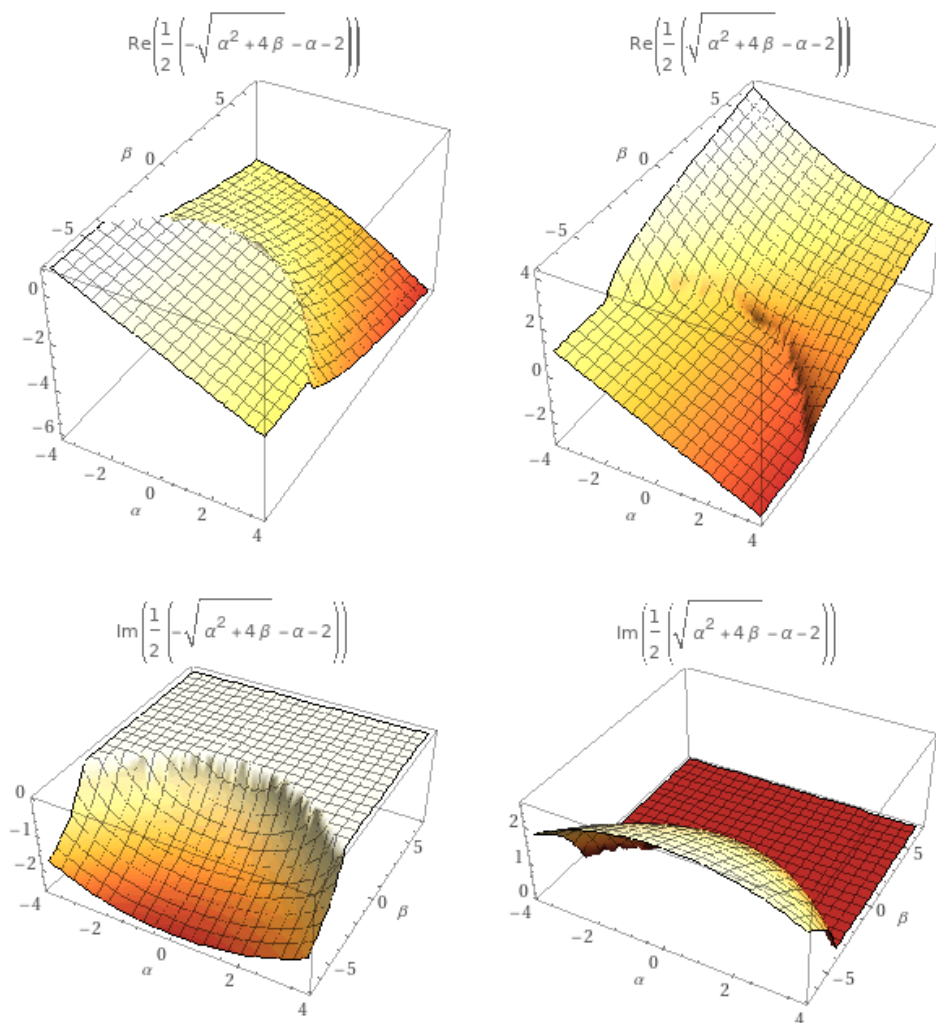$$[D_2]_{J_1} = \{(\phi, \psi) \in \mathbb{R}^2 : \lambda < 0\}.$$

Figure 5. The plots of the eigenvalues of J_1 showing the relation between the values of α and β when when $\wp(\eta) = 1/\eta$ and $\wp(\eta) = \sin(1/\eta)$ for $\wp'(\eta) = 1/\eta^2 < 0$ and $\wp'(\eta) = -\dfrac{\cos(1/\eta)}{\eta^2} < 0$ respectively. In addition, $\wp(\eta) = \cos(1/\eta)$ with $\wp'(\eta) = \dfrac{\sin(1/\eta)}{\eta^2} > 0$.

The equilibrium point of system (5.1) is

$$(\phi_0, \psi_0) = \left(\frac{1}{\alpha - \beta + 1}, \frac{-\beta}{-\alpha + \beta - 1} \right).$$

And the fixed point is

$$\left(\phi_{fix}, \psi_{fix}\right) = \left(\frac{2}{2\alpha - \beta + 4}, \frac{-\beta}{-2\alpha + \beta - 4}\right).$$

5.2. System (5.2)

The Jacobian matrix corresponds to the system (5.2) is, as follows:

$$J_2 = \wp'(\eta) \begin{pmatrix} \alpha - 1 & 1 \\ \beta & -1 \end{pmatrix}$$

with its eigenvalues

$$\lambda_1 = \wp'(\eta) \left(\frac{-\sqrt{\alpha^2 + 4\beta} + \alpha - 2}{2}\right), \quad \lambda_2 = \wp'(\eta) \left(\frac{\sqrt{\alpha^2 + 4\beta} + \alpha - 2}{2}\right),$$

where $\alpha \geq \sqrt{\alpha^2 + 4\beta} + 2$ satisfying the inequality

$$0 \leq \left(\frac{-\sqrt{\alpha^2 + 4\beta} + \alpha - 2}{2}\right) < \left(\frac{\sqrt{\alpha^2 + 4\beta} + \alpha - 2}{2}\right).$$

Thus, the solution is that $\alpha > 2, \beta \geq -\alpha^2/4$. If $\wp(\eta) \in (0, 1]$ is not a constant function with a positive derivative $\wp'(\eta) > 0$, then we have $\lambda_1 \geq$ and $\lambda_2 > 0$. Hence, we receive a unique domain of definite solutions

$$R_1 = \{(\phi, \psi) \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 > 0\}.$$

Moreover, if $\wp'(\eta) < 0$, we have only one domain for the solutions (see Figure 6)

$$R_2 = \{(\phi, \psi) \in \mathbb{R}^2 : \lambda_1 \leq 0, \lambda_2 < 0\}.$$

And the corresponding eigenvectors are

$$w_1 = \left(-\frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta}, 1\right), \quad w_2 = \left(-\frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2\beta}, 1\right).$$

The equilibrium point of system (5.2) is

$$(\phi_0, \psi_0) = \left(\frac{1}{-\alpha - \beta + 1}, \frac{-\beta}{\alpha + \beta - 1}\right).$$

And the fixed point is

$$\left(\phi_{fix}, \psi_{fix}\right) = \left(\frac{-2}{2\alpha + \beta - 4}, \frac{-\beta}{2\alpha + \beta - 4}\right).$$
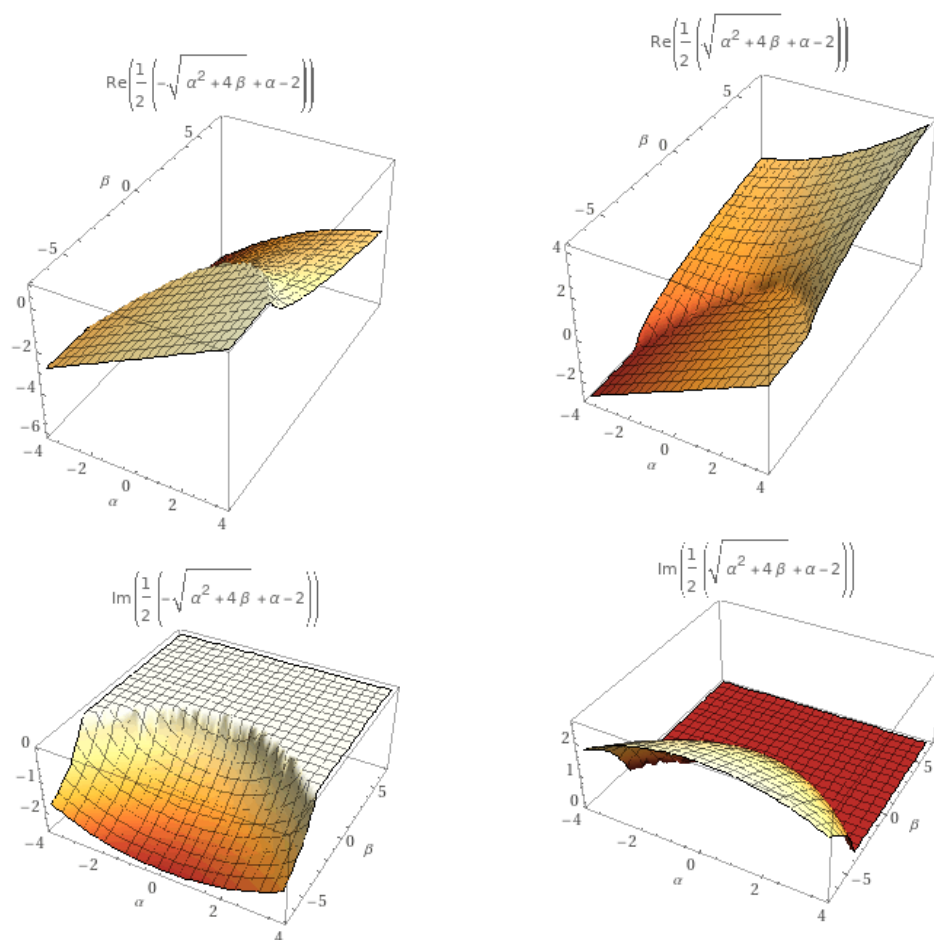
Figure 6. The plots of the eigenvalues of J_2 showing the relation between the values of α and β when $\wp(\eta) = 1/\eta$ and $\wp(\eta) = \sin(1/\eta)$ for $\wp'(\eta) = 1/\eta^2 < 0$ and $\wp'(\eta) = -\dfrac{\cos(1/\eta)}{\eta^2} < 0$ respectively. In addition, $\wp(\eta) = \cos(1/\eta)$ with $\wp'(\eta) = \dfrac{\sin(1/\eta)}{\eta^2} > 0$.

6. Conclusions

We suggested two distinct systems in this study: The variable fractional Lozi map (VFLM) and the variable fractional flow map. We looked into a few of these maps' crucial dynamics. Additionally, we looked into the prerequisites that the variable fractional dynamic systems must meet to be stable and asymptotically stable. To obtain a stable and asymptotically stable zero solutions, we therefore imposed the VFLM with the essential requirements. To stabilize the system, we also suggested combining these maps with control rules. The 1D and 2D controller rules were taken as givens in this analysis. For future works, one can extend this analysis into other types of variable fractional calculus such as the ABC-operator [3].

Conflict of interest

The authors declare no conflicts of interest.

References

1. T. Abdeljawad, On Riemann and Caputo fractional differences, *Comput. Math. Appl.*, **62** (2011), 1602–1611. https://doi.org/10.1016/j.camwa.2011.03.036

2. N. M. G. Al-Saidi, S. S. Al-Bundi, N. J. Al-Jawari, A hybrid of fractal image coding and fractal dimension for an efficient retrieval method, *Comput. Math. Appl.*, **37** (2018), 996–1011. https://doi.org/10.1007/s40314-016-0378-9

3. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A

4. Z. P. Bazant, S. Baweja, Creep and shrinkage prediction model for analysis and design of concrete structures: Model B3, *ACI Spec. Publ.*, **194** (2000), 1–84.

5. A. Beltempo, M. Zingales, O. S. Bursi, L. Deseri, A fractional order model for agingmaterials: An application to concrete, *Int. J. Solids Struct.*, **138** (2018), 13–23. https://doi.org/10.1016/j.ijsolstr.2017.12.024

6. A. Beltempo, O. S. Bursi, C. Cappello, D. Zonta, M. Zingales, A viscoelastic model for the long-term deflection of segmental prestressed box girders, *Comput.-Aided Civ. Infrastruct. Eng.*, **33** (2018), 64–78. https://doi.org/10.1111/mice.12311

7. A. Beltempo, A. Bonelli, O. S. Bursi, M. Zingales, A numerical integration approach for fractional-order viscoelastic analysis of hereditary-aging structures, *Int. J. Numer. Methods Eng.*, **121** (2020), 1120–1146. https://doi.org/10.1002/nme.6259

8. S. Bendoukha, Stabilization and synchronization of discrete-time fractional chaotic systems with non-identical dimensions, *Acta Math. Appl. Sin. Engl. Ser.*, **37** (2021), 523–538. https://doi.org/10.1007/s10255-021-1029-5

9. J. Borcea, P. Brändén, T. M. Liggett, Negative dependence and the geometry of polynomials, *J. Amer. Math. Soc.*, **22** (2009), 521–567. https://doi.org/10.1090/S0894-0347-08-00618-8

10. J. Čermák, I. Győri, L. Nechvátal, On explicit stability conditions for a linear fractional difference system, *FCAA*, **18** (2015), 651–672. https://doi.org/10.1515/fca-2015-0040

11. N. Colinas-Armijo, M. Di Paola, A. Di Matteo, Fractional viscoelastic behaviour under stochastic temperature process, *Probab. Eng. Mech.*, **54** (2018), 37–43. https://doi.org/10.1016/j.probengmech.2017.06.005

12. A. K. Farhan, N. M. G. Al-Saidi, A. T. Maolood, F. Nazarimehr, I. Hussain, Entropy analysis and image encryption application based on a new chaotic system crossing a cylinder, *Entropy*, **21** (2019), 958. https://doi.org/10.3390/e21100958

13. X. Han, J. Mou, H. Jahanshahi, Y. Cao, F. Bu, A new set of hyperchaotic maps based on modulation and coupling, *Eur. Phys. J. Plus*, **137** (2022), 523. https://doi.org/10.1140/epjp/s13360-022-02734-3

14. R. W. Ibrahim, D. Baleanu, Global stability of local fractional Henon-Lozi map using fixed point theory, *AIMS Math.*, **7** (2022), 11399–11416. https://doi.org/10.3934/math.2022636

15. A. Khennaoui, A. Ouannas, S. Bendoukha, G. Grassi, R. P. Lozi, V. Pham, On fractional-order discrete-time systems: Chaos, stabilization and synchronization, *Chaos Solitons Fract.*, **119** (2019), 150–162. https://doi.org/10.1016/j.chaos.2018.12.019

16. J. F. Li, H. Jahanshahi, S. Kacar, Y. M. Chu, J. F. Gomez-Aguilar, N. D. Alotaibi, et al., On the variable-order fractional memristor oscillator: Data security applications and synchronization using a type-2 fuzzy disturbance observer-based robust control, *Chaos Solitons Fract.*, **145** (2021), 110681. https://doi.org/10.1016/j.chaos.2021.110681

17. C. F. Lorenzo, T. T. Hartley Variable order and distributed order fractional operators, *Nonlinear Dyn.*, **29** (2002), 57–98. https://doi.org/10.1023/A:1016586905654

18. R. Lozi, Un attracteur étrange (?) du type attracteur de Hénon, *J. Phys. Colloquia*, **39** (1978), C5-9. https://doi.org/10.1051/jphyscol:1978505

19. R. Lyons, A note on tail triviality for determinantal point processes, *Electron. Commun. Probab.*, **23** (2018), 1–3. https://doi.org/10.1214/18-ECP175

20. S. Nemati, P. M. Lima, D. F. M. Torres, Numerical solution of variable-order fractional differential equations using Bernoulli polynomials, *Fractal Fract.*, **5** (2021), 219. https://doi.org/10.3390/fractalfract5040219

21. A. Ouannas, A. Khennaoui, I. M. Batiha, V. Pham, Stabilization of different dimensional fractional chaotic maps, *Fractional-Order Des.*, 2022, 123–155. https://doi.org/10.1016/B978-0-32-390090-4.00010-X

22. B. Ross, S. G. Samko, Fractional integration operator of variable order in the Holder space $H\lambda(x)$, *Int. J. Math. Math. Sci.*, **18** (1995), 713098. https://doi.org/10.1155/S0161171295001001

23. S. H. Salih, N. M. G. Al-Saidi, 3D-Chaotic discrete system of vector borne diseases using environment factor with deep analysis, *AIMS Math.*, **7** (2022), 3972–3987. https://doi.org/10.3934/math.2022219

24. S. G. Samko, B. Ross, Integration and differentiation to a variable fractional order, *Integr. Transf. Spec. F.*, **1** (1993), 277–300. https://doi.org/10.1080/10652469308819027

25. M. Uddin, I. U. Din, A numerical method for solving variable-order solute transport models, *Compu. Appl. Math.*, **39** (2020), 320. https://doi.org/10.1007/s40314-020-01355-9

26. S. Umarov, S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes, *ZAA*, **28** (2009), 431–450. https://doi.org/10.4171/ZAA/1392

27. D. Valério, J. Sá da Costa, Variable-order fractional derivatives and their numerical approximations, *Signal Process.*, **91** (2011), 470–483. https://doi.org/10.1016/j.sigpro.2010.04.006

28. L. Wei, W. Li, Local discontinuous Galerkin approximations to variable-order time-fractional diffusion model based on the Caputo-Fabrizio fractional derivative, *Math. Comput. Simul.*, **188** (2021), 280–290. https://doi.org/10.1016/j.matcom.2021.04.001