Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 607–621.
DOI: 10.3934/math.2023028
Received: 18 August 2022
Revised: 22 September 2022
Accepted: 25 September 2022
Published: 09 October 2022

Research article

Common fixed point of nonlinear contractive mappings

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Abstract: The purpose of this paper is to study the existence of a common fixed point for a pair of mappings without assumption of the contractive coefficient being fixed and less than 1. By replacing the fixed contractive coefficient with a nonlinear contractive function, we establish a unique common fixed point theorem for a pair of asymptotically regular self-mappings with either orbital continuity or q-continuity in a metric space. Moreover, by the asymptotical regularity of two approximate mappings, we prove that a pair of nonexpansive and continuous self-mappings, which are defined on a nonempty closed convex subset of a Banach space, have a common fixed point. Some examples are given to illustrate that our results are extensions of a recent result in the existing literature.

Keywords: common fixed point; nonlinear contraction; nonexpansive mapping; asymptotical regularity; complete metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Fixed point theory is an powerful tool of modern mathematics and may be viewed as a core context of nonlinear analysis. Since this theory has many applications in computer science, engineering, physics, economics, optimization theory, etc, a number of excellent books on it have been published. The reader is referred to the books [1, 6, 10] for more details on fixed point theory and applications. With the development of science, in recent years, the idea and techniques of fixed point theory have been used to study the problem of finite-/fixed-time synchronization for Clifford-valued recurrent neural networks with time-varying delays [4] and investigate control of a symmetric chaotic supply chain system [17].

Probably the most well known result in fixed point theory is Banach's contraction mapping principle. For the convenience of describing it, we need some notations. Let (X, d) be a metric space, $F : X \to X$ be a mapping and $\lambda \in [0, 1]$ with

$$d(Fx, Fy) \leq \lambda d(x, y), \quad \text{for all } x, y \in X. \tag{1.1}$$

If $\lambda < 1$ in (1.1), the mapping F is called a contractive mapping, and λ is called a contractive coefficient of F, whereas if $\lambda = 1$, F is called a nonexpansive mapping. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of all positive integer numbers. For simplification, for any $x \in X$ and $n \in \mathbb{N}$, we define $F^n x$ inductively by $F^1 x = Fx$ and $F^{n+1} x = F(F^n x)$. Moreover, we define $F^0 x = x$.

Banach's contraction mapping principle is stated as follows.

Theorem 1.1. Let (X, d) be a complete metric space and $F : X \to X$ be a contractive mapping. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$, the sequence $\{F^n x\}$ converges to u.

In order to extend Banach's contraction mapping principle, Geraghty [8] introduced a class of test functions.

Definition 1.2. [8] Let $\mathbb{R}^+ := \{t \in \mathbb{R} \mid t \geq 0\}$. $\mathcal{S}$ is defined as the set of functions $\psi : \mathbb{R}^+ \to [0, 1]$ with the properties:

(i) $0 \leq \psi(t) < 1$, for all $t > 0$,

(ii) $\lim_{n\to\infty} \psi(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Then using Definition 1.2, Geraghty [8] established the following result.

Theorem 1.3. Let (X, d) be a complete metric space, $F : X \to X$ be a mapping and $\psi \in \mathcal{S}$. If

$$d(Fx, Fy) \leq \psi(d(x, y))d(x, y), \quad \text{for all } x, y \in X,$$

then F has a unique fixed point $u \in X$.

One can deduce that the function F in Theorems 1.1 and 1.3 is a uniformly continuous function. In order to obtain a fixed point under weaker condition than Theorems 1.1 and 1.3, some useful concepts are given by some researchers. Browder and Petryshyn [5] introduced the concept of asymptotical regularity as follows.

Definition 1.4. Let (X, d) be a metric space, $F : X \to X$ be a mapping. F is said to be asymptotically regular on X, if

$$\lim_{n\to\infty} d(F^n x, F^{n+1} x) = 0, \quad \text{for all } x \in X.$$

Ćirić [7] introduced the definition of orbital continuity as follows.

Definition 1.5. [7] Let (X, d) be a metric space, $F : X \to X$ be a mapping. The set $O(F, z) := \{F^n z \mid n = 0, 1, 2, \ldots\}$ is called the orbit of F at the point $z \in X$. F is said to be orbitally continuous at $y \in X$ if for any sequence $\{x_k\} \subset O(F, x)$ for some $x \in X$, $\lim_{k\to\infty} x_k = y$ implies that $\lim_{k\to\infty} Fx_k = Fy$. We say that F is orbitally continuous on X if F is orbitally continuous at each point $y \in X$.

It is known that every continuous self-mapping is orbitally continuous, but the converse is not true [7].

Pant and Pant [15] introduced the definition of q-continuity for a $q \in \mathbb{N}$ as follows.

Definition 1.6. [15] Let (X, d) be a metric space, $F : X \to X$ be a mapping and $q \in \mathbb{N}$. F is said to be q-continuous at $y \in X$, if for each sequence $\{x_n\} \subset X$, $\lim_{n\to\infty} F^{q-1} x_n = y$ implies that $\lim_{n\to\infty} F^q x_n = Fy$. We say that F is q-continuous on X if F is q-continuous at each point $y \in X$.

Clearly, 1-continuity is the same as continuity. It is known that q-continuity implies that $q+1$-continuity, but the converse is not true [15].

Recently, Bisht [2] weakened the conditions of Górnicki's [9] result from continuity to q-continuity or orbital continuity, and obtained the following result.

Theorem 1.7. Let (X, d) be a complete metric space, $F : X \to X$ be a mapping, $q \in \mathbb{N}$, $\theta \in [0, 1)$ and $\tau \geq 0$. Suppose that F is asymptotically regular, q-continuous or orbitally continuous on X, and satisfy

$$d(Fx, Fy) \leq \theta d(x, y) + \tau(d(x, Fx) + d(y, Fy)), \quad \text{for all } x, y \in X.$$

Then F has a unique fixed point $u \in X$ and $\lim_{n\to\infty} F^n x = u$.

One can observe that a fixed point of F in fact is a coincidence point of F with the identity mapping I on X. Therefore, replacing the identity mapping I by another self-mapping G of X to obtain a common fixed point of F and G is natural. In 2020, Bisht and Singh [3] extended Theorem 1.7 from a single self-mapping F to a pair of mappings F and G, and obtained the following theorem.

Theorem 1.8. Let (X, d) be a complete metric space, $F,\ G : X \to X$ be two mappings, $\theta \in [0, 1)$ and $\tau \geq 0$. Suppose that F is asymptotically regular with respect to G, and F and G are (F, G)-orbitally continuous and compatible. Further, F and G satisfy

$$d(Fx, Fy) \leq \theta d(Gx, Gy) + \tau(d(Fx, Gx) + d(Fy, Gy)), \quad \text{for all } x, y \in X.$$

Then F and G have a unique common fixed point $u \in X$.

In 2020, unlike conditions given in Theorem 1.8, Khan and Oyetunbi [14] also extended Theorem 1.7 to a pair of self-mapping F and G, and obtained the following result.

Theorem 1.9. Let (X, d) be a complete metric space, $F,\ G : X \to X$ be two mappings, $q \in \mathbb{N}$, $\theta \in [0, 1)$ and $\tau \geq 0$. Suppose that F and G are asymptotically regular, q-continuous or orbitally continuous on X, and satisfy

$$d(Fx, Gy) \leq \theta d(x, y) + \tau(d(x, Fx) + d(y, Gy)), \quad \text{for all } x, y \in X. \tag{1.2}$$

Then F and G have a unique common fixed point $u \in X$ and $\lim_{n\to\infty} F^n x = \lim_{n\to\infty} G^n x = u$.

Very recently, the study of fixed point theory has received a growing interest and made great progress. In 2020, Hassan et al. [12] introduced S^*-iteration scheme for approximation of fixed point of the nonexpansive mappings and proved that it is stable and faster than some iteration schemes existing in the literature. They also established some convergence theorems for Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces. In 2021, Hammad, Agarwal and Guirao [11] presented some tripled fixed point results for a pair of mappings under (φ, ρ, ι)-contraction in ordered partially metric spaces. As applications, they also discussed the existence and uniqueness of the solution to an initial value problem and a homotopy theory. In 2022, Rasham et al. [16] established some theorems on common fixed points of set-valued mappings under $\alpha_* - \psi$ Ćirić type contraction in a complete modular like metric space. They also proved that a pair of multi-graph dominated mappings with graph contractions have a common fixed point.

In Theorem 1.9, we observe that the contractive coefficient θ is fixed and less than 1. In Theorem 1.3, we also observe that the contractive coefficient θ in Theorem 1.1 can be replaced by a function

$\psi \in \mathcal{S}$. In this paper, motivated by the above observations, we extend Theorem 1.9 for a pair of self-mappings F and G by replacing θ in (1.2) with a function $\psi \in \mathcal{S}$ or 1 in order to get common fixed points for more mathematical models. Firstly, by replacing θ with a function $\psi \in \mathcal{S}$, we obtain that asymptotically regular, either orbitally continuous or q-continuous mappings F and G have a unique common fixed point in a complete metric space. Secondly, by replacing θ with 1 and modifying conditions of Theorem 1.9, we also obtain that F and G have a common fixed point on a nonempty, closed convex subset of a Banach space. Some examples are given to illustrate our extensions.

2. Main results

The following lemma plays an important role in the proof of Theorems 2.2 and 2.5.

Lemma 2.1. [18] Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be three real sequences with $u_n \geq 0$ and $v_n \in (0, 1)$. Suppose that

(i) $u_{n+1} \leq (1 - v_n)u_n + v_n w_n$;

(ii) $\sum\limits_{n=1}^{\infty} v_n = \infty$;

(iii) $\limsup\limits_{n\to\infty} w_n \leq 0$ or $\sum\limits_{n=1}^{\infty} v_n w_n$ is convergent.

Then $\lim\limits_{n\to\infty} u_n = 0$.

Now we establish our one main result.

Theorem 2.2. Let (X, d) be a complete metric space, and F and G be two asymptotically regular self-mappings on X. Suppose that there exist a function $\psi \in \mathcal{S}$ and a constant $\tau \in [0, +\infty)$ such that

$$d(Fx, Gy) \leq \psi(d(x, y))d(x, y) + \tau(d(x, Fx) + d(y, Gy)), \quad \text{for all } x, y \in X, \tag{2.1}$$

and that F and G are either orbitally continuous or q-continuous for some $q \in \mathbb{N}$ on X. Then F and G have a unique common fixed point on X.

proof. Take any $x \in X$. Define $x_n = F^n x$ and $y_n = G^n x$ for all $n \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned} d(F^{n+1}x, G^{n+1}x) &= d(F(F^n x), G(G^n x)) \\ &\leq \psi(d(F^n x, G^n x))d(F^n x, G^n x) + \tau(d(F^n x, F^{n+1}x) + d(G^n x, G^{n+1}x)). \end{aligned} \tag{2.2}$$

Since $0 \leq \psi(d(F^n x, G^n x)) \leq 1$, we can divide $\limsup\limits_{n\to\infty} \psi(d(F^n x, G^n x))$ into two cases.

Case I: $\limsup\limits_{n\to\infty} \psi(d(F^n x, G^n x)) = 1$.

In this case, there exists a subsequence $\{\psi(d(F^{n_k}x, G^{n_k}x))\}$ of $\{\psi(d(F^n x, G^n x))\}$ such that

$$\lim_{k\to\infty} \psi(d(F^{n_k}x, G^{n_k}x)) = 1. \tag{2.3}$$

Since ψ is a function satisfying Definition 1.2, with (2.3), we conclude

$$\lim_{k\to\infty} d(F^{n_k}x, G^{n_k}x) = 0. \tag{2.4}$$

In the following, we will prove that $\{F^{n_k}x\}$ is a Cauchy sequence. Suppose to the contrary that $\{F^{n_k}x\}$ is not a Cauchy sequence, then there exist a $\varepsilon_0 > 0$ and two integer number sequences $\{n_{\tilde{k}(i)}\}$, $\{n_{k(i)}\}$ of $\{n_k\}$ with $n_{\tilde{k}(i)} > n_{k(i)} > i$ such that

$$d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x) \geq \varepsilon_0, \quad i = 1, 2, \ldots. \tag{2.5}$$

Consequently, we have

$$\begin{aligned}\varepsilon_0 &\leq d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)\\ &\leq d(F^{n_{\tilde{k}(i)}}x, F^{n_{\tilde{k}(i)}-1}x) + d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{k(i)}-1}x) + d(F^{n_{k(i)}-1}x, F^{n_{k(i)}}x).\end{aligned} \tag{2.6}$$

Letting $i \to \infty$ in (2.6), by the asymptotical regularity of F, we obtain

$$\liminf_{i\to\infty} d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{k(i)}-1}x) \geq \varepsilon_0. \tag{2.7}$$

Now, by (2.1), we have

$$\begin{aligned}d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x) &\leq d(F^{n_{\tilde{k}(i)}}x, G^{n_{k(i)}}x) + d(G^{n_{k(i)}}x, F^{n_{k(i)}}x)\\ &\leq \psi(d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x))d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x)\\ &+ \tau\left(d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{\tilde{k}(i)}}x) + d(G^{n_{k(i)}-1}x, G^{n_{k(i)}}x)\right) + d(G^{n_{k(i)}}x, F^{n_{k(i)}}x)\\ &\leq \psi(d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x))\left(d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{\tilde{k}(i)}}x)\right.\\ &\left.+d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x) + d(F^{n_{k(i)}}x, G^{n_{k(i)}}x) + d(G^{n_{k(i)}}x, G^{n_{k(i)}-1}x)\right)\\ &+ \tau\left(d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{\tilde{k}(i)}}x) + d(G^{n_{k(i)}-1}x, G^{n_{k(i)}}x)\right) + d(G^{n_{k(i)}}x, F^{n_{k(i)}}x).\end{aligned} \tag{2.8}$$

Dividing both sides of inequality (2.8) by $d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)$, with (2.5), we conclude

$$\begin{aligned}1 \leq{}& \psi(d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x))\left(\frac{d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{\tilde{k}(i)}}x)}{d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)} + 1 + \frac{d(F^{n_{k(i)}}x, G^{n_{k(i)}}x)}{d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)} + \frac{d(G^{n_{k(i)}}x, G^{n_{k(i)}-1}x)}{d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)}\right)\\ &+ \tau\frac{d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{\tilde{k}(i)}}x) + d(G^{n_{k(i)}-1}x, G^{n_{k(i)}}x)}{d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)} + \frac{d(G^{n_{k(i)}}x, F^{n_{k(i)}}x)}{d(F^{n_{\tilde{k}(i)}}x, F^{n_{k(i)}}x)}.\end{aligned} \tag{2.9}$$

Letting $i \to \infty$ in (2.9), by (2.4), (2.5), asymptotical regularity of F and G, and the fact $0 \leq \psi(\cdot) \leq 1$, we deduce that

$$\lim_{i\to\infty} \psi(d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x)) = 1.$$

Since ψ satisfies Definition 1.2, we conclude

$$\lim_{i\to\infty} d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x) = 0. \tag{2.10}$$

Since

$$d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{k(i)}-1}x) \leq d(F^{n_{\tilde{k}(i)}-1}x, G^{n_{k(i)}-1}x) + d(G^{n_{k(i)}-1}x, F^{n_{k(i)}-1}x),$$

by combining (2.4) and (2.10), we conclude

$$\lim_{i\to\infty} d(F^{n_{\tilde{k}(i)}-1}x, F^{n_{k(i)}-1}x) = 0,$$

which contradicts (2.7). Therefore, $\{x_{n_k}\} = \{F^{n_k}x\}$ is a Cauchy sequence. Since X is complete, $\{x_{n_k}\}$ converges to a point u in X. Since

$$d(G^{n_k}x, u) \le d(G^{n_k}x, F^{n_k}x) + d(F^{n_k}x, u),$$

with (2.4), we deduce that $\{G^{n_k}x\}$ also converges to u.

Now, assume that F is orbitally continuous. Since $\{x_{n_k}\}$ converges to u, the orbital continuity of F implies that $\{Fx_{n_k}\}$ converges to Fu. By asymptotical regularity of F, we have

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{n_k}) = \lim_{k\to\infty} d(F^{n_k+1}x, F^{n_k}x) = 0,$$

resulting in

$$\lim_{k\to\infty} Fx_{n_k} = \lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} x_{n_k} = u.$$

Uniqueness of limit implies that $Fu = u$.

Now, suppose that F is q-continuous. Since

$$\begin{aligned} d(x_{n_k+j}, x_{n_k}) &\le d(x_{n_k+j}, x_{n_k+j-1}) + \cdots + d(x_{n_k+1}, x_{n_k}) \\ &= d(F^{n_k+j}x, F^{n_k+j-1}x) + \cdots + d(F^{n_k+1}x, F^{n_k}x), \quad j = 1, 2, \ldots, q, \end{aligned}$$

using the asymptotical regularity of F, we conclude that

$$\lim_{k\to\infty} x_{n_k+j} = \lim_{k\to\infty} x_{n_k} = u, \quad j = 1, 2, \ldots, q, \tag{2.11}$$

especially,

$$\lim_{k\to\infty} F^{q-1}x_{n_k} = \lim_{k\to\infty} x_{n_k+q-1} = u. \tag{2.12}$$

Since F is q-continuous, with (2.12), we have

$$\lim_{k\to\infty} F^q x_{n_k} = Fu. \tag{2.13}$$

However, from (2.11), we have

$$\lim_{k\to\infty} F^q x_{n_k} = \lim_{k\to\infty} x_{n_k+q} = u. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain $Fu = u$.

Since $\{G^{n_k}x\}$ also converges to u, similarly, orbital continuity or q-continuity of G implies that $Gu = u$.

Now, u is a common fixed point of F and G. Assume that v is another common fixed point of F and G with $u \ne v$. Since $d(u, v) > 0$, it follows from Definition 1.2 that $0 \le \psi(d(u, v)) < 1$. By (2.1), we have

$$\begin{aligned} d(u, v) = d(Fu, Gv) &\le \psi(d(u, v))d(u, v) + \tau(d(u, Fu) + d(v, Gv)) \\ &= \psi(d(u, v))d(u, v) < d(u, v), \end{aligned}$$

which leads to a contradiction. Therefore, the common fixed point of F and G is unique.

Case II: $\limsup_{n\to\infty} \psi(d(F^n x, G^n x)) < 1$.

In this case, there exists a $\sigma \in (0, 1)$ such that $0 \le \psi(d(F^n x, G^n x)) < \sigma$. With (2.2), we have

$$d(F^{n+1}x, G^{n+1}x) \le \sigma d(F^n x, G^n x) + \tau(d(F^n x, F^{n+1}x) + d(G^n x, G^{n+1}x)). \tag{2.15}$$

Let $u_n := d(F^n x, G^n x)$, $v_n := 1-\sigma$, and

$$w_n := \frac{\tau(d(F^n x, F^{n+1} x) + d(G^n x, G^{n+1} x))}{1-\sigma}.$$

From (2.15), we have

$$u_{n+1} \leq (1-v_n)u_n + v_n w_n, \quad \text{for all } n \in \mathbb{N}.$$

Since F and G are asymptotically regular on X, we conclude that $\lim_{n\to\infty} w_n = 0$. Moreover,

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}(1-\sigma) = \infty.$$

By Lemma 2.1, we conclude

$$\lim_{n\to\infty} d(F^n x, G^n x) = 0.$$

Then for any subsequence $\{n_k\}$ of $\{n\}$, we have

$$\lim_{k\to\infty} d(F^{n_k} x, G^{n_k} x) = 0,$$

that is, (2.4) holds. From here, the proof is the same as previous Case I, so we omit it. The proof is completed.

In the following, we give two examples to illustrate that Theorem 2.2 is an extension of Theorem 1.9. Or, more specifically, Theorem 2.2 can be applied to the two examples but Theorem 1.9 can not.

Example 2.3. Let $X = [0,1]$ be equipped with the metric $d(x,y) = |x-y|$, and $F,\ G : X \to X$ be defined by

$$Fx = x - \frac{1}{4}x^2, \quad Gx = 0, \quad \text{for all } x \in X.$$

We first prove that F and G do not satisfy inequality (1.2). Suppose to the contrary that F and G satisfy inequality (1.2):

$$d(Fx, Gy) \leq \theta d(x,y) + \tau(d(x,Fx) + d(y,Gy)), \quad \text{for all } x,\ y \in X. \tag{2.16}$$

Taking $x = \frac{1}{n}$, $y = 0$ in (2.16) for all $n \in \mathbb{N}$, we have

$$\left|\frac{1}{n} - \frac{1}{4n^2}\right| \leq \theta\frac{1}{n} + \tau\frac{1}{4n^2},$$

and so

$$\left|1 - \frac{1}{4n}\right| \leq \theta + \tau\frac{1}{4n}. \tag{2.17}$$

By sending $n \to \infty$ in (2.17), we obtain $1 \leq \theta$, which contradicts $\theta \in [0,1)$. Therefore, F and G do not satisfy conditions of Theorem 1.9.

Now, define $\psi : \mathbb{R}^+ \to [0,1]$ by

$$\psi(t) = \begin{cases} 1 - \frac{1}{2}t, & \text{for all } t \in [0,1], \\ \frac{1}{2}, & \text{for all } t \in (1,+\infty). \end{cases}$$

Then $\psi \in \mathcal{S}$. For any $x, y \in X$, since $|x-y| \le 1$, we have

$$\begin{aligned}
&\psi(d(x,y))d(x,y) + (d(x,Fx) + d(y,Gy)) - d(Fx,Gy)\\
&= \left(1 - \frac{1}{2}|x-y|\right)|x-y| + \frac{1}{4}x^2 + y - \left(x - \frac{1}{4}x^2\right)\\
&= |x-y| - \frac{1}{2}(x-y)^2 + \frac{1}{2}x^2 + y - x\\
&= \begin{cases} \frac{1}{2}x^2 - \frac{1}{2}(x-y)^2, & \text{if } x \ge y,\\ 2(y-x) - \frac{1}{2}(y-x)^2 + \frac{1}{2}x^2, & \text{if } x < y, \end{cases}\\
&\ge 0.
\end{aligned}$$

Therefore, F and G satisfy (2.1) with $\tau = 1$. Clearly, F and G are continuous on X and G is asymptotically regular on X. For any $x \in X$, since

$$0 \le F^{n+1}x = F^n x - \frac{1}{4}(F^n x)^2 \le F^n x,$$

we conclude that $\{F^n x\}$ is a decreasing sequence with a lower bound 0. Hence, $\{F^n x\}$ converges to a point in X. This implies that F is an asymptotically regular mapping. All conditions of Theorem 2.2 are satisfied, so F and G have a unique common fixed point. In fact, 0 is the unique common fixed point of F and G.

Example 2.4. Let $X = [-1, 1]$ be equipped with the metric $d(x,y) = |x-y|$, and $F,\ G : X \to X$ be defined by

$$Fx = \begin{cases} \frac{x}{1+x}, & 0 \le x \le 1,\\ \frac{1}{2}, & -1 \le x < 0, \end{cases} \tag{2.18}$$

$$Gx = \begin{cases} 0, & 0 \le x \le 1,\\ 1, & -1 \le x < 0. \end{cases}$$

We first prove that F and G do not satisfy inequality (1.2). Suppose to the contrary that F and G satisfy

$$d(Fx,Gy) \le \theta d(x,y) + \tau(d(x,Fx) + d(y,Gy)), \quad \text{for all } x,\ y \in X. \tag{2.19}$$

Taking $x = \frac{1}{n}$ and $y = 0$ in (2.19) for all $n \in \mathbb{N}$, we obtain

$$\frac{n}{n+1} \le \theta + \tau\frac{1}{n+1}. \tag{2.20}$$

By sending $n \to \infty$ in (2.20), we have $1 \le \theta$, which contradicts $\theta \in [0,1)$. Therefore, F and G do not satisfy conditions of Theorem 1.9.

Now, define $\psi : \mathbb{R}^+ \to [0,1]$ by

$$\psi(t) = \frac{1}{1+t}, \quad \text{for all } t \in [0, +\infty).$$

Then $\psi \in \mathcal{S}$. In the following, we will prove that for any $x, y \in X$,

$$\Psi := \psi(d(x,y))d(x,y) + 4(d(x,Fx) + d(y,Gy)) - d(Fx,Gy) \geq 0. \tag{2.21}$$

To prove inequality (2.21), we divide x and y into four cases.

Case 1: $x \geq 0$ and $y \geq 0$. In this case,

$$\begin{aligned}\Psi &= \frac{|x-y|}{1+|x-y|} + 4\frac{x^2}{1+x} + 4y - \frac{x}{1+x} \\ &\geq \begin{cases} \dfrac{|x-y|}{1+|x-y|} + 4\dfrac{x^2}{1+x} + 4x - x, & \text{if } y \geq x, \\ \dfrac{x-y}{1+x} + 4\dfrac{x^2}{1+x} + 4y - \dfrac{x}{1+x}, & \text{if } y < x, \end{cases} \\ &\geq 0.\end{aligned}$$

Case 2: $x \geq 0$ and $y < 0$. In this case,

$$\begin{aligned}\Psi &= \frac{|x-y|}{1+|x-y|} + 4\frac{x^2}{1+x} + 4|y-1| - \left|\frac{x}{1+x} - 1\right| \\ &= \frac{|x-y|}{1+|x-y|} + 4\frac{x^2}{1+x} + 4|y| + 4 + \frac{x}{1+x} - 1 \geq 0.\end{aligned}$$

Case 3: $x < 0$ and $y < 0$. In this case,

$$\Psi = \frac{|x-y|}{1+|x-y|} + 4\left|x - \frac{1}{2}\right| + 4(|y|+1) - \frac{1}{2} \geq 0.$$

Case 4: $x < 0$ and $y \geq 0$. In this case,

$$\begin{aligned}\Psi &= \frac{|x-y|}{1+|x-y|} + 4\left|x - \frac{1}{2}\right| + 4|y| - \frac{1}{2} \\ &= \frac{|x-y|}{1+|x-y|} + 4|x| + 2 + 4|y| - \frac{1}{2} \geq 0.\end{aligned}$$

From the above four cases, we deduce that (2.21) holds. Therefore, F and G satisfy (2.1) with $\tau = 4$.

Clearly, F is 2-continuous at all points in $X \setminus \{0\}$ since F is continuous at these points. Now, assume that $\lim_{n\to\infty} Fx_n = 0$. By (2.18), we have $0 \leq x_n \leq 1$ and $0 \leq Fx_n \leq 1$ when n sufficiently large. Thus,

$$\lim_{n\to\infty} F^2 x_n = \lim_{n\to\infty} \frac{Fx_n}{1+Fx_n} = 0 = F0.$$

Therefore, F is 2-continuous at 0. Similarly, we can also prove that G is 2-continuous on X.

Since for any $x \in X$,

$$\lim_{n\to\infty} d(F^n x, F^{n+1}x) = \lim_{n\to\infty} \begin{cases} \left|\dfrac{x}{1+nx} - \dfrac{x}{1+(n+1)x}\right|, & \text{if } 0 \leq x \leq 1, \\ \left|\dfrac{1}{1+n} - \dfrac{1}{2+n}\right|, & \text{if } -1 \leq x < 0, \end{cases} = 0,$$

we deduce that F is asymptotically regular on X. Similarly, we can also prove that G is asymptotically regular on X.

We have verified that F and G satisfy all conditions of Theorem 2.2, and hence, they have a unique common fixed point on X. In fact, 0 is the unique common fixed point of F and G.

Now, we establish our another main result.

Theorem 2.5. Let X be a Banach space, D be a nonempty closed convex subset of X, $F,\ G : D \to D$ be two mappings, $r \in (0, 1)$, $\tau \in [0, +\infty)$, $z \in D$ and

$$\bar{F}x := rFx + (1 - r)z, \quad \bar{G}x := rGx + (1 - r)z, \quad \text{for all } x \in D.$$

Suppose that the following conditions hold:

(i) $\|Fx - Gy\| \le \|x - y\| + \tau(\|x - Fx\| + \|y - Gy\|), \quad \text{for all } x, y \in D;$

(ii) $\bar{F}$ and $\bar{G}$ are asymptotically regular on D;

(iii) $\lim_{n\to\infty} \left\|\bar{F}^n x - F(\bar{F}^n x)\right\| = 0, \quad \lim_{n\to\infty} \left\|\bar{G}^n x - G(\bar{G}^n x)\right\| = 0, \quad \text{for all } x \in D;$

(iv) F and G are continuous on D.

Then F and G have a common fixed point on D.

proof. Since the proof is similar with the proof Theorem 2.2, we just give a sketchy proof here. Take any $x \in D$. For all $n \in \mathbb{N}$, by condition (i), we have

$$\begin{aligned} \left\|\bar{F}^{n+1}x - \bar{G}^{n+1}x\right\| &= r\left\|F(\bar{F}^n x) - G(\bar{G}^n x)\right\| \\ &\le r\left\|\bar{F}^n x - \bar{G}^n x\right\| + \tau r\left(\left\|\bar{F}^n x - F(\bar{F}^n x)\right\| + \left\|\bar{G}^n x - G(\bar{G}^n x)\right\|\right). \end{aligned} \tag{2.22}$$

Let $u_n := \left\|\bar{F}^n x - \bar{G}^n x\right\|$, $v_n := 1 - r$, and

$$w_n := \frac{\tau r\left(\left\|\bar{F}^n x - F(\bar{F}^n x)\right\| + \left\|\bar{G}^n x - G(\bar{G}^n x)\right\|\right)}{1 - r}.$$

With Lemma 2.1, condition (iii) and (2.22), we conclude

$$\lim_{n\to\infty} \left\|\bar{F}^n x - \bar{G}^n x\right\| = 0. \tag{2.23}$$

Now, we prove that $\{\bar{F}^n x\}$ is a Cauchy sequence. Suppose to the contrary that $\{\bar{F}^n x\}$ is not a Cauchy sequence, then there exist a $\varepsilon_0 > 0$ and two integer number sequences $\{\tilde{n}(i)\}$, $\{n(i)\}$ with $\tilde{n}(i) > n(i) > i$ such that

$$\left\|\bar{F}^{\tilde{n}(i)}x - \bar{F}^{n(i)}x\right\| \ge \varepsilon_0, \quad i = 1, 2, \ldots. \tag{2.24}$$

Now, by condition (i), we have

$$\begin{aligned} &\left\|\bar{F}^{\tilde{n}(i)}x - \bar{F}^{n(i)}x\right\| \le \left\|\bar{F}^{\tilde{n}(i)}x - \bar{G}^{n(i)}x\right\| + \left\|\bar{G}^{n(i)}x - \bar{F}^{n(i)}x\right\| \\ &= r\left\|F(\bar{F}^{\tilde{n}(i)-1}x) - G(\bar{G}^{n(i)-1}x)\right\| + \left\|\bar{G}^{n(i)}x - \bar{F}^{n(i)}x\right\| \\ &\le r\left\|\bar{F}^{\tilde{n}(i)-1}x - \bar{G}^{n(i)-1}x\right\| \\ &+ r\tau\left(\left\|\bar{F}^{\tilde{n}(i)-1}x - F(\bar{F}^{\tilde{n}(i)-1}x)\right\| + \left\|\bar{G}^{n(i)-1}x - G(\bar{G}^{n(i)-1}x)\right\|\right) + \left\|\bar{G}^{n(i)}x - \bar{F}^{n(i)}x\right\| \\ &\le r\left(\left\|\bar{F}^{\tilde{n}(i)-1}x - \bar{F}^{\tilde{n}(i)}x\right\| + \left\|\bar{F}^{\tilde{n}(i)}x - \bar{F}^{n(i)}x\right\| + \left\|\bar{F}^{n(i)}x - \bar{G}^{n(i)}x\right\| + \left\|\bar{G}^{n(i)}x - \bar{G}^{n(i)-1}x\right\|\right) \\ &+ r\tau\left(\left\|\bar{F}^{\tilde{n}(i)-1}x - F(\bar{F}^{\tilde{n}(i)-1}x)\right\| + \left\|\bar{G}^{n(i)-1}x - G(\bar{G}^{n(i)-1}x)\right\|\right) + \left\|\bar{G}^{n(i)}x - \bar{F}^{n(i)}x\right\|. \end{aligned} \tag{2.25}$$

Dividing both sides of inequality (2.25) by $\left\|\bar{F}^{\tilde{n}(i)}x - \bar{F}^{n(i)}x\right\|$, and with (2.24), we have

$$\begin{aligned}1 \leq\ & r\left(\frac{\left\|\bar{F}^{\tilde{n}(i)-1}x - \bar{F}^{\tilde{n}(i)}x\right\|}{\varepsilon_0} + 1 + \frac{\left\|\bar{F}^{n(i)}x - \bar{G}^{n(i)}x\right\|}{\varepsilon_0} + \frac{\left\|\bar{G}^{n(i)}x - \bar{G}^{n(i)-1}x\right\|}{\varepsilon_0}\right) \\ & + r\tau\frac{\left\|\bar{F}^{\tilde{n}(i)-1}x - F(\bar{F}^{\tilde{n}(i)-1}x)\right\| + \left\|\bar{G}^{n(i)-1}x - G(\bar{G}^{n(i)-1}x)\right\|}{\varepsilon_0} + \frac{\left\|\bar{G}^{n(i)}x - \bar{F}^{n(i)}x\right\|}{\varepsilon_0}.\end{aligned} \tag{2.26}$$

Letting $i \to \infty$ in (2.26), by using (2.23), conditions (ii) and (iii), we deduce that $1 \leq r$. This is a contradiction since $r < 1$. Consequently, $\{\bar{F}^n x\}$ is a Cauchy sequence. Since D is a closed set, $\{\bar{F}^n x\}$ converges to a point u in D. This and (2.23) imply that $\{\bar{G}^n x\}$ also converges to u. From condition (iv), F and G are continuous on D, and with condition (iii) we get

$$Fu = u = Gu.$$

Hence u is a common fixed point of F and G.

Remark 2.6. In the following, we will give two examples to illustrate that Theorem 2.5 is an extension of Theorem 1.9, and the common fixed point of F and G which satisfy the conditions of Theorem 2.5 probably is not unique.

Example 2.7. Let $X = \mathbb{R}$ be equipped with the norm $\|x\| = |x|$, $D = [-1, 1]$, and $F,\ G : D \to D$ be defined by

$$Fx = x, \quad \text{for all } x \in D,$$

and

$$Gx = \begin{cases} 0, & x \in [0, 1], \\ x, & x \in [-1, 0). \end{cases}$$

Obviously, the metric induced by the norm is

$$d(x, y) = |x - y|, \quad \forall\ x, y \in X.$$

We first prove that F and G do not satisfy inequality (1.2) on D. Suppose to the contrary that F and G satisfy inequality (1.2):

$$d(Fx, Gy) \leq \theta d(x, y) + \tau(d(x, Fx) + d(y, Gy)), \quad \text{for all } x,\ y \in D. \tag{2.27}$$

Taking $x = 1$ and $y = 0$ in (2.27), we obtain $1 \leq \theta$, which contradicts $\theta \in [0, 1)$. Therefore, F and G do not satisfy conditions of Theorem 1.9.

For any $x, y \in D$, we have

$$\begin{aligned}&|x - y| + (|x - Fx| + |y - Gy|) - |Fx - Gy| \\ &= \begin{cases} |x - y| + |y| - |x|, & \text{if } y \geq 0, \\ |x - y| - |x - y|, & \text{if } y < 0, \end{cases} \\ &\geq 0,\end{aligned}$$

where the last inequality holds by the triangle inequality of $|\cdot|$. Therefore, F and G satisfy condition (i) of Theorem 2.5 with $\tau = 1$. Let $\bar{F}x := \frac{1}{2}Fx + \frac{1}{2}0 = \frac{1}{2}Fx$, $\bar{G}x = \frac{1}{2}Gx$. Then we can calculate that for all $x \in D$,

$$\lim_{n\to\infty} \left\|\bar{F}^n x - \bar{F}^{n+1} x\right\| = \lim_{n\to\infty} \left|\frac{1}{2^n}x - \frac{1}{2^{n+1}}x\right| = 0,$$

$$\lim_{n\to\infty} \left\|\bar{G}^n x - \bar{G}^{n+1} x\right\| = \lim_{n\to\infty} \begin{cases} 0, & \text{if } x \geq 0, \\ \left|\frac{1}{2^n}x - \frac{1}{2^{n+1}}x\right|, & \text{if } x < 0, \end{cases} = 0.$$

$$\lim_{n\to\infty} \left\|\bar{F}^n x - F(\bar{F}^n x)\right\| = \lim_{n\to\infty} \left|\frac{1}{2^n}x - \frac{1}{2^n}x\right| = 0,$$

$$\lim_{n\to\infty} \left\|\bar{G}^n x - G(\bar{G}^n x)\right\| = \lim_{n\to\infty} \begin{cases} 0, & \text{if } x \geq 0, \\ \left|\frac{1}{2^n}x - \frac{1}{2^n}x\right|, & \text{if } x < 0, \end{cases} = 0.$$

Therefore, conditions (ii) and (iii) of Theorem 2.5 are satisfied. Obviously, F and G are continuous. All conditions of Theorem 2.5 are satisfied. Hence, F and G have a common fixed point. In fact, all points in $[-1, 0]$ are common fixed points of F and G.

Example 2.8. Let $X = \mathbb{R}$ be equipped with the usual norm $|\cdot|$, $D = \left[0, \frac{1}{2}\right]$, and $F,\ G : D \to D$ be defined by

$$Fx = x, \quad G(x) = x^2, \quad \text{for all } x \in D.$$

Obviously, the metric induced by the norm is

$$d(x, y) = |x - y|, \quad \forall\ x, y \in X.$$

We first prove that F and G do not satisfy inequality (1.2) on D. Suppose to the contrary that F and G satisfy

$$d(Fx, Gy) \leq \theta d(x, y) + \tau(d(x, Fx) + d(y, Gy)), \quad \text{for all } x,\ y \in D. \tag{2.28}$$

Taking $x = 1$ and $y = 0$ in (2.28), we obtain $1 \leq \theta$, which contradicts $\theta \in [0, 1)$. Therefore, F and G do not satisfy the conditions of Theorem 1.9.

For any $x, y \in D$, we have

$$|x - y| + (|x - Fx| + |y - Gy|) - |Fx - Gy| = |x - y| + \left|y - y^2\right| - \left|x - y^2\right| \geq 0.$$

Thus F and G satisfy condition (i) of Therem 2.5 with $\tau = 1$. Let

$$\bar{F}x := \frac{3}{4}Fx + \frac{1}{4}0 = \frac{3}{4}x, \quad \bar{G}x := \frac{3}{4}Gx + \frac{1}{4}0 = \frac{3}{4}x^2, \quad \text{for all } \ x \in D.$$

Then we can calculate that for all $x \in D$,

$$\lim_{n\to\infty} \left\|\bar{F}^n x - \bar{F}^{n+1} x\right\| = \lim_{n\to\infty} \left|\frac{3^n}{4^n}x - \frac{3^{n+1}}{4^{n+1}}x\right| = 0,$$

$$\lim_{n\to\infty} \left\|\bar{G}^n x - \bar{G}^{n+1} x\right\| = \lim_{n\to\infty} \left|\frac{3^{2^n - 1}}{4^{2^n - 1}}x^{2^n} - \frac{3^{2^{n+1} - 1}}{4^{2^{n+1} - 1}}x^{2^{n+1}}\right| = 0.$$

$$\lim_{n\to\infty}\left\|\bar{F}^n x - F(\bar{F}^n x)\right\| = \lim_{n\to\infty}\left|\frac{3^n}{4^n}x - \frac{3^n}{4^n}x\right| = 0,$$

$$\lim_{n\to\infty}\left\|\bar{G}^n x - G(\bar{G}^n x)\right\| = \lim_{n\to\infty}\left|\frac{3^{2^n-1}}{4^{2^n-1}}x^{2^n} - \left(\frac{3^{2^n-1}}{4^{2^n-1}}x^{2^n}\right)^2\right| = 0.$$

Therefore, F and G satisfy conditions (ii) and (iii) of Theorem 2.5. Clearly, F and G satisfy condition (iv) since they are continuous on D. Now, all conditions of Theorem 2.5 are satisfied, and so F and G have a common fixed point. In fact, $u = 0$ is the unique common fixed point of F and G on X.

3. Conclusions

In this paper, following the argument of [14], we studied the common fixed point of two self-mappings. In this direction, most of results in the literature required that the contractive coefficient is constant and in interval $[0, 1)$. However, in some mathematical models, a fixed contractive coefficient in $[0, 1)$ does not exist. In order to overcome this difficulty, we replaced the contractive coefficient with a contractive function with its range in $[0, 1]$. By using the concepts of asymptotical regularity, orbital continuity and q-continuity, and some techniques of mathematical analysis, a unique common fixed point theorem for two self-mappings has been established in a metric space. Furthermore, for a pair of nonexpansive mappings, by constructing two approximate mappings of them, the concepts of asymptotical regularity and continuity, a common fixed point has been derived. We also presented some examples to illustrate that our results extended the main result in [14]. The obtained results will bring potential applications in the existence of solution of some mathematical models.

In future studies, we will extend our results to more general enriched Banach contractions mappings in a Banach space, and obtain an iterative sequence, which converges to a common fixed point of them. Moreover, following the idea of our previous work [13], we will consider the common fixed point of finite self-mappings without contraction conditions in a Banach space, and its applications in variational inequalities.

Acknowledgments

The author sincerely thanks the anonymous reviewers for their insightful comments which greatly improved the quality of the paper. This work was supported by the National Natural Science Foundation of China (12061085) and the Applied Basic Research Project of Yunnan Province (202001BB050036).

Conflict of interest

The authors declare that they have no competing interests.

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