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Research article

Existence and uniqueness results for coupled system of fractional differential equations with exponential kernel derivatives

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Abstract: In the framework of Caputo-Fabrizio derivatives, we study a new coupled system of fractional differential equations of higher orders supplemented with coupled nonlocal boundary conditions. The existence and uniqueness results of the solutions are proved. We consider the classical fixed-point theories due to Banach and Krasnoselskii for the main results. An example illustrating the main results is introduced.

Keywords: fractional differential equations; Caputo-Fabrizio; coupled system; existence; fixed-point theorems
Mathematics Subject Classification: 26A33, 34A08, 34A12, 47H10

1. Introduction

Fractional differential equations have been attractive to many researchers in the past few decades due to the non-localization properties of the fractional derivatives contrary to the integer-order derivatives [1–4]. Indeed, fractional derivatives supply memory description, hereditary properties and predict future dynamics of several materials. These properties have enabled many researchers to utilize fractional differential equations in modelling different complex phenomena [5–9]. Up to now, several fractional derivatives have been defined and extensively studied by many researchers, for instance see [10–15] and the references cited therein. Although Riemman-Liouvile and Caputo are the most common types of fractional derivatives, the singularity of their kernels prevents them from giving a full description of some advanced models [16, 17]. In seeking to develop the applicability of fractional calculus, eminent researchers introduced new types of non-singular kernel derivatives of fractional orders, such as Caputo-Fabrizio and Atangana-Baleanu [18, 19], which have been proved as a good tool to model several phenomena in engineering, physics, chemistry, biology, and other natural sciences; see for instance [20–22].

In [18], authors introduced the Caputo-Fabrizio derivative in which they have replaced the singularity of Caputo derivative's kernel with an exponential non-singular kernel. The new derivative is also known as a non-singular kernel or an exponential kernel derivative. Losada and Nieto [23] defined the corresponding fractional integral based on the new definition of Caputo-Fabrizio fractional derivative. In addition, they studied properties of this new type of derivatives as well as the solutions of some fractional differential equations. Abdeljawad in [24] introduced definitions of the Caputo-Fabrizio operators of higher arbitrary orders and proved the existence and uniqueness theorems for some initial value problem. The physical interpretation of Caputo-Fabrizio derivatives was presented recently in [25] and many properties of the operators of Caputo-Fabrizio type, have been established in [26, 27].

Nowadays, the theory of fractional operators in the frame of Caputo-Fabrizio attracted the interest of the researchers. Thus, several results related to these operators have been published. For instance, mathematical models for some diseases and epidemics in the frame of Caputo-Fabrizio derivatives were discussed in [28–33]. Some important numerical approaches regarding to this type of derivatives were studied in [34–38]. Moreover, many works studied boundary value problems involving such operators [39–46]. In [47, 48], authors investigated the existence and uniqueness results for some coupled systems involving Caputo-Fabrizio derivatives. However, this investigation for such coupled systems has been limited to only if the fractional orders are less than one. Motivated by these arguments, we intend to investigate a new coupled system involving fractional differential equations with Caputo-Fabrizio derivatives:

$$\begin{cases} {}^{\mathrm{CF}}D^{p}_{0+}x(t) = f(t, x(t), y(t)), & t \in \mathcal{I} := [0, T], \\ {}^{\mathrm{CF}}D^{q}_{0+}y(t) = g(t, x(t), y(t)), & 1 < p, q \le 2, \end{cases} \tag{1.1}$$

enhanced with the nonlocal boundary conditions defined by:

$$\begin{cases} x(0) = 0, & x(T) = \alpha y(\gamma), \\ y(0) = 0, & y(T) = \beta x(\eta), \qquad 0 < \gamma, \eta < T, \end{cases} \tag{1.2}$$

where ${}^{\mathrm{CF}}D^{\xi}$ is the Caputo-Fabrizio fractional derivative of order $\xi \in \{p, q\}$, $f, g : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\alpha, \beta \in \mathbb{R}^{+}$.

The present paper is outlined as follows: In Section 2, we recall some basic definitions and prove a basic lemma leads to transforming the proposed system into equivalent integral equations. In Section 3, the main results are established by using Krasnoselskii's and Banach's fixed point techniques. Finally, in Section 4, we discuss our results via an illustrating example.

2. Preliminaries

In this section, we present some basic definitions and lemmas which used throughout the paper.

Definition 2.1 ([18]). *For any $f \in H^1(a, b)$, $a < b$, the left-sided Caputo-Fabrizio derivative is given by*

$$ {}^{\mathrm{CF}}D^{\rho}_{a^{+}}f(t) = \frac{M(\rho)}{1-\rho}\int_a^t f'(s)\exp\Big[-\frac{\rho(t-s)}{1-\rho}\Big]ds, \;\; \rho \in [0, 1], \tag{2.1}$$

where $M(\rho) > 0$ is the normalization constant satisfying $M(0) = M(1) = 1$ and $H^1(a,b)$ is the Sobolev space. Nevertheless, if $f \notin H^1(a,b)$, then the left-sided Caputo-Fabrizio derivative in this case is defined as

$$ {}^{\mathrm{CF}}D^{\rho}_{a^+} f(t) = \frac{M(\rho)}{1-\rho} \int_a^t \Big(f(t) - f(s)\Big) \exp\Big[-\frac{\rho(t-s)}{1-\rho}\Big] ds, \quad \rho \in [0,1]. \tag{2.2} $$

Remark 2.2 ([18]). *According to Definition 2.1, we have the following properties:*

(i) $\lim_{\rho\to1} {}^{\mathrm{CF}}D^{\rho}_{a^+} f(t) = f'(t)$,
(ii) $\lim_{\rho\to0} {}^{\mathrm{CF}}D^{\rho}_{a^+} f(t) = f(t) - f(a)$.

Definition 2.3 ([24]). *The associated fractional integral of Caputo-Fabrizio is given by*

$$ {}^{CF}I^{\rho}_{a^+} f(t) = \frac{1-\rho}{M(\rho)} f(t) + \frac{\rho}{M(\rho)} \int_a^t f(s) ds. \tag{2.3} $$

Caputo-Fabrizio operators can be extended to higher order as given by the following definition:

Definition 2.4 ([24]). *Let $n < \sigma \leq n+1$ and f be such that $f^{(n)} \in H^1(a,b)$. For $\rho = \sigma - n$, then*

$$ {}^{\mathrm{CF}}D^{\sigma}_{a^+} f(t) = {}^{\mathrm{CF}}D^{\rho}_{a^+} f^{(n)}(t), \tag{2.4} $$

$$ {}^{\mathrm{CF}}I^{\sigma}_{a^+} f(t) = I^{n}_{a^+}\Big({}^{\mathrm{CF}}I^{\rho}_{a^+} f(t)\Big), \tag{2.5} $$

where $\rho \in [0,1]$.

Lemma 2.5 ([24]). *For $f(t)$ defined on $[a,b]$ and $\sigma \in (n, n+1]$, for some $n \in \mathbb{N}$, we have*

$$ {}^{\mathrm{CF}}I^{\sigma}_{a^+}\Big({}^{\mathrm{CF}}D^{\sigma}_{a^+} f(t)\Big) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{2.6} $$

Next, we are proving an important lemma, which is vital in converting the given problem to a fixed point problem.

Lemma 2.6. *Let $\mathcal{I} := [0,T]$ and $\mathcal{X} = C(\mathcal{I}, \mathbb{R})$ be a Banach space with the norm $\|x\| = \max_{t\in\mathcal{I}} |f(t)|$ and $F, G \in \mathcal{X}$. Then the solution of the system:*

$$ \begin{cases} {}^{\mathrm{CF}}D^{p}_{0+} x(t) = F(t), t \in \mathcal{I}, \\ {}^{\mathrm{CF}}D^{q}_{0+} y(t) = G(t), 1 < p, q \leq 2, \\ x(0) = 0, \qquad x(T) = \alpha y(\gamma), \\ y(0) = 0, \qquad y(T) = \beta x(\eta), \qquad 0 < \gamma, \eta < T, \qquad \alpha, \beta \in \mathbb{R}^+, \end{cases} \tag{2.7} $$

is given by

$$\begin{aligned}
x(t) = & \frac{t}{T^2 - \alpha\beta\gamma\eta}\Big\{\frac{\alpha T}{M(q-1)}\int_0^{\gamma}\Big[(2-q)+(q-1)(\gamma-s)\Big]G(s)ds \\
& - \frac{T}{M(p-1)}\int_0^{T}\Big[(2-p)+(p-1)(T-s)\Big]F(s)ds \\
& + \frac{\alpha\beta\gamma}{M(p-1)}\int_0^{\eta}\Big[(2-p)+(p-1)(\eta-s)\Big]F(s)ds \\
& - \frac{\alpha\gamma}{M(q-1)}\int_0^{T}\Big[(2-q)+(q-1)(T-s)\Big]G(s)ds\Big\} \\
& + \frac{1}{M(p-1)}\int_0^{t}\Big[(2-p)+(p-1)(t-s)\Big]F(s)ds, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
y(t) = & \frac{t}{T^2 - \alpha\beta\gamma\eta}\Big\{\frac{\beta T}{M(p-1)}\int_0^{\eta}\Big[(2-p)+(p-1)(\eta-s)\Big]F(s)ds \\
& - \frac{T}{M(q-1)}\int_0^{T}\Big[(2-q)+(q-1)(T-s)\Big]G(s)ds \\
& + \frac{\alpha\beta\eta}{M(q-1)}\int_0^{\gamma}\Big[(2-q)+(q-1)(\gamma-s)\Big]G(s)ds \\
& - \frac{\beta\eta}{M(p-1)}\int_0^{T}\Big[(2-p)+(p-1)(T-s)\Big]F(s)ds\Big\} \\
& + \frac{1}{M(q-1)}\int_0^{t}\Big[(2-q)+(q-1)(t-s)\Big]G(s)ds, \qquad (2.9)
\end{aligned}$$

where $T^2 \neq \alpha\beta\gamma\eta$.

Proof. When ${}^{\mathrm{CF}}I^{p}_{0+}, {}^{\mathrm{CF}}I^{q}_{0+}$ are applied to the fractional differential equations in (2.7) and Lemma 2.5 is used, the general solutions of the fractional differential equations in (2.7) for $t \in \mathcal{I}$ take the form:

$$\begin{aligned}
x(t) &= c_0 + c_1 t + {}^{\mathrm{CF}}I^{p}_{0+}F(t), \\
y(t) &= d_0 + d_1 t + {}^{\mathrm{CF}}I^{q}_{0+}G(t),
\end{aligned}$$

respectively, for some $c_i, d_i, \in \mathbb{R}$ $(i = 0, 1)$ to be determined. Since $1 < p, q \leq 2$, we can have $\rho_1,\ \rho_2$ such that $p = \rho_1 + 1,\ q = \rho_2 + 1$ and so we can apply the relation (2.5) to get

$$x(t) = c_0 + c_1 t + {}^{\mathrm{CF}}I^{\rho_1+1}_{0+}F(t) = c_0 + c_1 t + \frac{1}{M(p-1)}\int_0^t\Big[(2-p)+(p-1)(t-s)\Big]F(s)ds, \qquad (2.10)$$

$$y(t) = d_0 + d_1 t + {}^{\mathrm{CF}}I^{\rho_2+1}_{0+}G(t) = d_0 + d_1 t + \frac{1}{M(q-1)}\int_0^t\Big[(2-q)+(q-1)(t-s)\Big]G(s)ds. \qquad (2.11)$$

Applying conditions $x(0) = y(0) = 0$, in (2.10) and (2.11), we get $c_0 = d_0 = 0$ and so

$$x(t) = c_1 t + \frac{1}{M(p-1)}\int_0^t\Big[(2-p)+(p-1)(t-s)\Big]F(s)ds, \qquad (2.12)$$

$$y(t) = d_1 t + \frac{1}{M(q-1)} \int_0^t \Big[(2-q) + (q-1)(t-s)\Big] G(s)ds. \tag{2.13}$$

Using the boundary conditions $x(T) = \alpha y(\gamma), \ \ y(T) = \beta x(\eta)$, in (2.12) and (2.13) respectively, we get the following system:

$$\begin{cases} c_1 T - \alpha\gamma d_1 = J_1, \\ d_1 T - \beta\eta c_1 = J_2, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} J_1 = & \frac{\alpha}{M(q-1)} \int_0^\gamma \Big[(2-q) + (q-1)(\gamma-s)\Big] G(s)ds \\ & - \frac{1}{M(p-1)} \int_0^T \Big[(2-p) + (p-1)(T-s)\Big] F(s)ds, \end{aligned} \tag{2.15}$$

$$\begin{aligned} J_2 = & \frac{\beta}{M(p-1)} \int_0^\eta \Big[(2-p) + (p-1)(\eta-s)\Big] F(s)ds \\ & - \frac{1}{M(q-1)} \int_0^T \Big[(2-q) + (q-1)(T-s)\Big] G(s)ds. \end{aligned} \tag{2.16}$$

Solving the system (2.14) with aid of (2.15) and (2.16), we have

$$\begin{aligned} c_1 = & \frac{1}{T^2 - \alpha\beta\gamma\eta} \Big\{ \frac{\alpha T}{M(q-1)} \int_0^\gamma \Big[(2-q) + (q-1)(\gamma-s)\Big] G(s)ds \\ & - \frac{T}{M(p-1)} \int_0^T \Big[(2-p) + (p-1)(T-s)\Big] F(s)ds \\ & + \frac{\alpha\beta\gamma}{M(p-1)} \int_0^\eta \Big[(2-p) + (p-1)(\eta-s)\Big] F(s)ds \\ & - \frac{\alpha\gamma}{M(q-1)} \int_0^T \Big[(2-q) + (q-1)(T-s)\Big] G(s)ds \Big\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} d_1 = & \frac{1}{T^2 - \alpha\beta\gamma\eta} \Big\{ \frac{\beta T}{M(p-1)} \int_0^\eta \Big[(2-p) + (p-1)(\eta-s)\Big] F(s)ds \\ & - \frac{T}{M(q-1)} \int_0^T \Big[(2-q) + (q-1)(T-s)\Big] G(s)ds \\ & + \frac{\alpha\beta\eta}{M(q-1)} \int_0^\gamma \Big[(2-q) + (q-1)(\gamma-s)\Big] G(s)ds \\ & - \frac{\beta\eta}{M(p-1)} \int_0^T \Big[(2-p) + (p-1)(T-s)\Big] F(s)ds \Big\}. \end{aligned} \tag{2.18}$$

Employing the values of $c_1, \ d_1$ in (2.12) and (2.13) respectively, gives the solutions (2.8) and (2.9). This completes the proof. □

3. Main results

As a result of Lemma 2.6, we define an operator $\mathcal{T}\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ as

$$\Omega(x,y)(t) = (\Omega_1(x,y)(t), \Omega_2(x,y)(t)), \tag{3.1}$$

where

$$\begin{aligned}
\Omega_1(x,y)(t) = {} & \frac{t}{T^2-\alpha\beta\gamma\eta}\Big\{\frac{\alpha T}{M(q-1)}\int_0^\gamma \Big[(2-q)+(q-1)(\gamma-s)\Big]g(s,x(s),y(s))ds \\
& - \frac{T}{M(p-1)}\int_0^T \Big[(2-p)+(p-1)(T-s)\Big]f(s,x(s),y(s))ds \\
& + \frac{\alpha\beta\gamma}{M(p-1)}\int_0^\eta \Big[(2-p)+(p-1)(\eta-s)\Big]f(s,x(s),y(s))ds \\
& - \frac{\alpha\gamma}{M(q-1)}\int_0^T \Big[(2-q)+(q-1)(T-s)\Big]g(s,x(s),y(s))ds\Big\} \\
& + \frac{1}{M(p-1)}\int_0^t \Big[(2-p)+(p-1)(t-s)\Big]f(s,x(s),y(s))ds,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\Omega_2(x,y)(t) = {} & \frac{t}{T^2-\alpha\beta\gamma\eta}\Big\{\frac{\beta T}{M(p-1)}\int_0^\eta \Big[(2-p)+(p-1)(\eta-s)\Big]f(s,x(s),y(s))ds \\
& - \frac{T}{M(q-1)}\int_0^T \Big[(2-q)+(q-1)(T-s)\Big]g(s,x(s),y(s))ds \\
& + \frac{\alpha\beta\eta}{M(q-1)}\int_0^\gamma \Big[(2-q)+(q-1)(\gamma-s)\Big]g(s,x(s),y(s))ds \\
& - \frac{\beta\eta}{M(p-1)}\int_0^T \Big[(2-p)+(p-1)(T-s)\Big]f(s,x(s),y(s))ds\Big\} \\
& + \frac{1}{M(q-1)}\int_0^t \Big[(2-q)+(q-1)(t-s)\Big]g(s,x(s),y(s))ds,
\end{aligned} \tag{3.3}$$

where $T^2 \neq \alpha\beta\gamma\eta$. Thus, the product space $(\mathcal{X}\times\mathcal{X}, \|(x,y)\|)$ is also a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|,\ x,y\in\mathcal{X}$. For brevity's sake, we will use the following notations with $T^2\neq\alpha\beta\gamma\eta$:

$$\bar{\rho}_1 = \frac{T}{M(p-1)}\Big[(2-p)+(p-1)\frac{T}{2}\Big], \tag{3.4}$$

$$\bar{\rho}_2 = \frac{T}{M(q-1)}\Big[(2-q)+(q-1)\frac{T}{2}\Big], \tag{3.5}$$

$$\kappa_1 = \frac{(2-p)}{M(p-1)}\Big[T^2+\alpha\beta\gamma\eta\Big] + \frac{(p-1)}{2M(p-1)}\Big[T^3+\alpha\beta\gamma\eta^2\Big], \tag{3.6}$$

$$\kappa_2 = \frac{\alpha\gamma T}{M(q-1)}\Big[2(2-q)+\frac{(q-1)}{2}(T+\gamma)\Big], \tag{3.7}$$

$$\bar{\kappa}_1 = \frac{\beta\eta T}{M(p-1)}\Big[2(2-p)+\frac{(p-1)}{2}(T+\eta)\Big], \tag{3.8}$$

$$\bar{\kappa}_2 = \frac{(2-q)}{M(q-1)}\Big[T^2+\alpha\beta\gamma\eta\Big] + \frac{(q-1)}{2M(q-1)}\Big[T^3+\alpha\beta\eta\gamma^2\Big], \tag{3.9}$$

$$\begin{aligned}\mathcal{K}_1 &= \frac{T}{M(p-1)}\bigg\{\Big(\frac{T^2}{|T^2-\alpha\beta\gamma\eta|}+1\Big)\Big[(2-p)+(p-1)\frac{T}{2}\Big] \\ &\quad + \frac{\alpha\beta\gamma\eta}{|T^2-\alpha\beta\gamma\eta|}\Big[(2-p)+(p-1)\frac{\eta}{2}\Big]\bigg\},\end{aligned} \tag{3.10}$$

$$\mathcal{K}_2 = \frac{\alpha\gamma T^2}{M(q-1)|T^2-\alpha\beta\gamma\eta|}\bigg\{2(2-q)+\frac{(q-1)}{2}(\gamma+T)\bigg\}, \tag{3.11}$$

$$\bar{\mathcal{K}}_1 = \frac{\beta\eta T^2}{M(p-1)|T^2-\alpha\beta\gamma\eta|}\bigg\{2(2-p)+\frac{(p-1)}{2}(\eta+T)\bigg\}, \tag{3.12}$$

$$\begin{aligned}\bar{\mathcal{K}}_2 &= \frac{T}{M(q-1)}\bigg\{\Big(\frac{T^2}{|T^2-\alpha\beta\gamma\eta|}+1\Big)\Big[(2-q)+(q-1)\frac{T}{2}\Big] \\ &\quad + \frac{\alpha\beta\gamma\eta}{|T^2-\alpha\beta\gamma\eta|}\Big[(2-q)+(q-1)\frac{\gamma}{2}\Big]\bigg\}.\end{aligned} \tag{3.13}$$

In our first result, we investigate the existence of a solution for the coupled system (1.1) and (1.2) by applying Krasnoselskii fixed point theorem [49].

Theorem 3.1. *Assume that $f, g\colon \mathcal{I}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the assumptions:*
($\mathcal{H}_1$) there exists positive constants l_1, l_2 such that

$$\begin{aligned}&|f(t,x_1,x_2)-f(t,y_1,y_2)| \le l_1\Big[|x_1-x_2|+|y_1-y_2|\Big],\\ &|g(t,x_1,x_2)-g(t,y_1,y_2)| \le l_2\Big[|x_1-x_2|+|y_1-y_2|\Big], \forall x_i, y_i\in\mathbb{R}, i=1,2,\end{aligned}$$

and
($\mathcal{H}_2$) there exists positive constants $\mathcal{L}_1, \mathcal{L}_2$ such that

$$|f(t,x_1,x_2)| \le \mathcal{L}_1, \qquad |g(t,x_1,x_2)| \le \mathcal{L}_2, \tag{3.14}$$

for all $t\in\mathcal{I}$ and $x_i\in\mathbb{R}, i=1,2$. Then the considered problem (1.1), and (1.2) has at least one solution on $\mathcal{I}$ if

$$l_1\bar{\rho}_1 + l_2\bar{\rho}_2 < 1. \tag{3.15}$$

Proof. Let's define a closed ball $\mathcal{B}_\varepsilon = \{(x,y)\in\mathcal{X}\times\mathcal{X} : \|(x,y)\|\le\varepsilon\}$, which is bounded and convex subset of the Banach space $(\mathcal{X}\times\mathcal{X}, \|(x,y)\|)$ and choose $\varepsilon \ge \max\{\kappa_1\mathcal{L}_1+\kappa_2\mathcal{L}_2, \bar{\kappa}_1\mathcal{L}_1+\bar{\kappa}_2\mathcal{L}_2\}$ and split the operators Ω_1, Ω_2 as:

$$
\begin{aligned}
\Omega_{11}(x,y)(t) =& \frac{t}{T^2-\alpha\beta\gamma\eta}\Big\{\frac{\alpha T}{M(q-1)}\int_0^{\gamma}\Big[(2-q)+(q-1)(\gamma-s)\Big]g(s,x(s),y(s))ds\\
&-\frac{T}{M(p-1)}\int_0^{T}\Big[(2-p)+(p-1)(T-s)\Big]f(s,x(s),y(s))ds\\
&+\frac{\alpha\beta\gamma}{M(p-1)}\int_0^{\eta}\Big[(2-p)+(p-1)(\eta-s)\Big]f(s,x(s),y(s))ds\\
&-\frac{\alpha\gamma}{M(q-1)}\int_0^{T}\Big[(2-q)+(q-1)(T-s)\Big]g(s,x(s),y(s))ds\Big\},
\end{aligned}\tag{3.16}
$$

$$
\Omega_{12}(x,y)(t) =\frac{1}{M(p-1)}\int_0^{t}\Big[(2-p)+(p-1)(t-s)\Big]f(s,x(s),y(s))ds,\tag{3.17}
$$

and

$$
\begin{aligned}
\Omega_{21}(x,y)(t) =& \frac{t}{T^2-\alpha\beta\gamma\eta}\Big\{\frac{\beta T}{M(p-1)}\int_0^{\eta}\Big[(2-p)+(p-1)(\eta-s)\Big]f(s,x(s),y(s))ds\\
&-\frac{T}{M(q-1)}\int_0^{T}\Big[(2-q)+(q-1)(T-s)\Big]g(s,x(s),y(s))ds\\
&+\frac{\alpha\beta\eta}{M(q-1)}\int_0^{\gamma}\Big[(2-q)+(q-1)(\gamma-s)\Big]g(s,x(s),y(s))ds\\
&-\frac{\beta\eta}{M(p-1)}\int_0^{T}\Big[(2-p)+(p-1)(T-s)\Big]f(s,x(s),y(s))ds\Big\},
\end{aligned}\tag{3.18}
$$

$$
\Omega_{22}(x,y)(t) = \frac{1}{M(q-1)}\int_0^{t}\Big[(2-q)+(q-1)(t-s)\Big]g(s,x(s),y(s))ds.\tag{3.19}
$$

Notice that $\Omega_1(x,y)(t) = \Omega_{11}(x,y)(t) + \Omega_{12}(x,y)(t)$, and $\Omega_2(x,y)(t) = \Omega_{21}(x,y)(t) + \Omega_{22}(x,y)(t)$ on $\mathcal{B}_\varepsilon$.

First, to verify the first condition of Krasnoselskii's theorem, we will show that $\Omega\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$. Setting $x = (x_1, x_2), y = (y_1, y_2), \bar{x} = (\bar{x}_1, \bar{x}_2)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2) \in \mathcal{B}_\varepsilon$, and utilizing condition (3.14), we obtain

$$
\begin{aligned}
\|\Omega_{11}(x,y) + \Omega_{12}(\bar{x},\bar{y})\| &= \sup_{t\in\mathcal{I}} |\Omega_{11}(x,y)(t) + \Omega_{12}(\bar{x},\bar{y})(t)|\\
&\le \frac{T}{|T^2-\alpha\beta\gamma\eta|}\Big\{\frac{\alpha T\mathcal{L}_2}{M(q-1)}\int_0^{\gamma}\Big[(2-q)+(q-1)(\gamma-s)\Big]ds\\
&+\frac{T\mathcal{L}_1}{M(p-1)}\int_0^{T}\Big[(2-p)+(p-1)(T-s)\Big]ds\\
&+\frac{\alpha\beta\gamma\mathcal{L}_1}{M(p-1)}\int_0^{\eta}\Big[(2-p)+(p-1)(\eta-s)\Big]ds\\
&+\frac{\alpha\gamma\mathcal{L}_2}{M(q-1)}\int_0^{T}\Big[(2-q)+(q-1)(T-s)\Big]ds\Big\}
\end{aligned}
$$

$$+\frac{\mathcal{L}_1}{M(p-1)}\int_0^t\Big[(2-p)+(p-1)(t-s)\Big]ds.$$

According to (3.6), (3.7) and the definition of ε, we get

$$\|\Omega_{11}(x,y)+\Omega_{12}(\bar{x},\bar{y})\|\leq\kappa_1\mathcal{L}_1+\kappa_2\mathcal{L}_2\leq\varepsilon, \tag{3.20}$$

in a similar manner, we can find that

$$\|\Omega_{21}(x,y)+\Omega_{22}(\bar{x},\bar{y})\|\leq\bar{\kappa}_1\mathcal{L}_1+\bar{\kappa}_2\mathcal{L}_2\leq\varepsilon. \tag{3.21}$$

Clearly the inequalities (3.20) and (3.21) lead to the fact that $\Omega_1(x,y)+\Omega_2(\bar{x},\bar{y})\in\mathcal{B}_\varepsilon$.

Secondly, we have to prove that the operator $(\Omega_{12},\Omega_{22})$ is a contraction to satisfy the third condition of Krasnoselskii's theorem as the following:

For $(x_1,y_1),(x_2,y_2)\in\mathcal{B}_\varepsilon$, we have

$$\begin{aligned}\|\Omega_{12}(x_1,y_1)-\Omega_{12}(x_2,y_2)\| &= \sup_{t\in\mathcal{I}}|\Omega_{12}(x_1,y_1)(t)-\Omega_{12}(x_2,y_2)(t)|\\ &\leq\frac{1}{M(p-1)}\int_0^t\Big[(2-p)+(p-1)(t-s)\Big]\\ &\times\Big|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))\Big|ds,\end{aligned}$$

by applying the condition $(\mathcal{H}_1)$, we get

$$\begin{aligned}&\|\Omega_{12}(x_1,y_1)-\Omega_{12}(x_2,y_2)\|\\ &\leq\frac{l_1}{M(p-1)}\Big[\int_0^t\Big((2-p)+(p-1)(t-s)\Big)ds\Big]\Big[\|x_1-x_2\|+\|y_1-y_2\|\Big].\end{aligned}$$

As a result (3.4), we get

$$\|\Omega_{12}(x_1,y_1)-\Omega_{12}(x_2,y_2)\|\leq l_1\bar{\rho}_1\Big[\|x_1-x_2\|+\|y_1-y_2\|\Big]. \tag{3.22}$$

Similarly, we have

$$\|\Omega_{22}(x_1,y_1)-\Omega_{22}(x_2,y_2)\|\leq l_2\bar{\rho}_2\Big[\|x_1-x_2\|+\|y_1-y_2\|\Big]. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\|(\Omega_{12},\Omega_{22})(x_1,y_1)-(\mathcal{T}_{12},\Omega_{22})(x_2,y_2)\|\leq(l_1\bar{\rho}_1+l_2\bar{\rho}_2)\Big[\|x_1-x_2\|+\|y_1-y_2\|\Big], \tag{3.24}$$

which is a contraction by (3.15). Therefore, the third condition of Krasnoselskii's theorem is satisfied.

Following that, we can establish that the operator $(\Omega_{11},\Omega_{21})$ satisfies the second condition of Krasnoselskii's theorem.

Since the functions $f,\ g$ are continuous on $\mathcal{I}\times\mathbb{R}\times\mathbb{R}$, the operator $(\Omega_{11},\Omega_{21})$ is continuous. For each $(x,y)\in\mathcal{B}_\varepsilon$, applying (3.14) gives

$$
\begin{aligned}
\|\Omega_{11}(x,y)\| \leq & \frac{T}{|T^2-\alpha\beta\gamma\eta|}\Big\{\frac{\mathcal{L}_1}{M(q-1)}\Big[T\int_0^T\Big((2-p)+(p-1)(T-s)\Big)ds \\
& +\alpha\beta\gamma\int_0^\eta\Big((2-p)+(p-1)(\eta-s)\Big)ds\Big] \\
& +\frac{\mathcal{L}_2}{M(q-1)}\Big[\alpha T\int_0^\gamma\Big((2-q)+(q-1)(\gamma-s)\Big)ds \\
& +\alpha\gamma\int_0^T\Big((2-q)+(q-1)(T-s)\Big)ds\Big]\Big\}.
\end{aligned}
$$

From (3.6) and (3.7), we get

$$
\|\Omega_{11}(x,y)\| \leq \frac{T(\mathcal{L}_1\kappa_1+\mathcal{L}_2\kappa_2)}{|T^2-\alpha\beta\gamma\eta|} = R_1. \tag{3.25}
$$

Similarly, applying (3.14) and using the notations (3.8), (3.9) give

$$
\|\Omega_{21}(x,y)\| \leq \frac{T(\mathcal{L}_1\bar{\kappa}_1+\mathcal{L}_2\bar{\kappa}_2)}{|T^2-\alpha\beta\gamma\eta|} = R_2. \tag{3.26}
$$

The inequalities (3.25) and (3.26) lead to

$$
\|(\Omega_{11},\Omega_{21})(x,y)\| \leq R_1+R_2, \tag{3.27}
$$

and therefore the set $(\Omega_{11},\Omega_{21})\,\mathcal{B}_\varepsilon$ is uniformly bounded. The following step will demonstrate that the set $(\Omega_{11},\Omega_{21})\,\mathcal{B}_\varepsilon$ is equicontinuous. For $t_1,t_2\in\mathcal{I}$ with $t_1<t_2$ and for any $(x,y)\in\mathcal{B}_\varepsilon$ we get

$$
\begin{aligned}
& |\Omega_{11}(x,y)(t_2)-\Omega_{11}(x,y)(t_1)| \\
& \leq \frac{t_2-t_1}{|T^2-\alpha\beta\gamma\eta|}\Big\{\frac{\alpha T}{M(q-1)}\int_0^\gamma\Big[(2-q)+(q-1)(\gamma-s)\Big]|g(s,x(s),y(s))|ds \\
& +\frac{T}{M(p-1)}\int_0^T\Big[(2-p)+(p-1)(T-s)\Big]|f(s,x(s),y(s))|ds \\
& +\frac{\alpha\beta\gamma}{M(p-1)}\int_0^\eta\Big[(2-p)+(p-1)(\eta-s)\Big]|f(s,x(s),y(s))|ds \\
& +\frac{\alpha\gamma}{M(q-1)}\int_0^T\Big[(2-q)+(q-1)(T-s)\Big]|g(s,x(s),y(s))|ds\Big\}.
\end{aligned}
$$

Applying (3.14), using the notations (3.6) and (3.7), we have

$$
|\Omega_{11}(x,y)(t_2)-\Omega_{11}(x,y)(t_1)| \leq \frac{t_2-t_1}{|T^2-\alpha\beta\gamma\eta|}\Big(\kappa_1\mathcal{L}_1+\kappa_2\mathcal{L}_2\Big).
$$

Analogously, we can obtain from (3.8) and (3.9)

$$
|\Omega_{21}(x,y)(t_2)-\Omega_{21}(x,y)(t_1)| \leq \frac{t_2-t_1}{|T^2-\alpha\beta\gamma\eta|}\Big(\bar{\kappa}_1\mathcal{L}_1+\bar{\kappa}_2\mathcal{L}_2\Big).
$$

Therefore $|(\Omega_{11},\Omega_{21})(t_2)-(\Omega_{11},\Omega_{21})(t_1)|\to 0$ as $t_2\to t_1$ independent of $(x,y)\in\mathcal{B}_\varepsilon$. Thus the set $(\Omega_{11},\Omega_{21})\mathcal{B}_\varepsilon$ is equicontinuous. As an outcome, the Arzela-Ascoli theorem implies that the operator $(\Omega_{11},\Omega_{21})$ is compact on $\mathcal{B}_\varepsilon$. By the conclusion of Krasnoselskii's, the problem (1.1) and (1.2) has at least one solution on $\mathcal{I}$. □

In the next theorem we prove the existence of a unique solution of system (1.1) and (1.2) by applying the contraction mapping principle.

Theorem 3.2. *Assume that* $f, g : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions satisfying the assumption* $(\mathcal{H}_1)$ *in Theorem 3.1. If*

$$(\mathcal{K}_1 + \bar{\mathcal{K}}_1)l_1 + (\mathcal{K}_2 + \bar{\mathcal{K}}_2)l_2 < 1, \tag{3.28}$$

then the boundary value problem (1.1) and (1.2) has a unique solution on $\mathcal{I}$, *where* $\mathcal{K}_i, \bar{\mathcal{K}}_i,\ i = 1, 2$ *are given by (3.10)–(3.13) respectively.*

Proof. Define $\sup_{t\in\mathcal{I}} f(t, 0, 0) = N_1 < \infty$, $\sup_{t\in\mathcal{I}} g(t, 0, 0) = N_2 < \infty$ and $r > 0$ such that

$$r \geq \frac{(\mathcal{K}_1 + \bar{\mathcal{K}}_1)N_1 + (\mathcal{K}_2 + \bar{\mathcal{K}}_2)N_2}{1 - (\mathcal{K}_1 + \bar{\mathcal{K}}_1)l_1 - (\mathcal{K}_2 + \bar{\mathcal{K}}_2)l_2}.$$

Firstly, we show that $\Omega\mathcal{B}_r \subset \mathcal{B}_r$ when operator Ω is given by (3.1) and $\mathcal{B}_r = \{(x, y) \in \mathcal{X} \times \mathcal{X}: \|(x, y)\| \leq r\}$. By assumption $(\mathcal{H}_1)$, for $(x, y) \in \mathcal{B}_r, t \in \mathcal{I}$, we have

$$\begin{aligned} |f(t, x(t), y(t))| &\leq l_1 r + N_1, \\ |g(t, x(t), y(t))| &\leq l_2 r + N_2. \end{aligned}$$

Then we obtain

$$\begin{aligned} |\Omega_1(x, y)(t)| \leq{}& \frac{T}{|T^2 - \alpha\beta\gamma\eta|}\Big\{\frac{\alpha T}{M(q-1)} \int_0^{\gamma} \Big[(2-q) + (q-1)(\gamma - s)\Big](l_2 r + N_2)ds \\ &+ \frac{T}{M(p-1)} \int_0^{T} \Big[(2-p) + (p-1)(T - s)\Big](l_1 r + N_1)ds \\ &+ \frac{\alpha\beta\gamma}{M(p-1)} \int_0^{\eta} \Big[(2-p) + (p-1)(\eta - s)\Big](l_1 r + N_1)ds \\ &+ \frac{\alpha\gamma}{M(q-1)} \int_0^{T} \Big[(2-q) + (q-1)(T - s)\Big](l_2 r + N_2)ds\Big\} \\ &+ \frac{1}{M(p-1)} \int_0^{t} \Big[(2-p) + (p-1)(t - s)\Big](l_1 r + N_1)ds. \end{aligned}$$

Using notations (3.10)–(3.13) guides to

$$|\Omega_1(x, y)(t)| \leq (\mathcal{K}_1 l_1 + \mathcal{K}_2 l_2)r + \mathcal{K}_1 N_1 + \mathcal{K}_2 N_2. \tag{3.29}$$

Similarly,

$$|\Omega_2(x, y)(t)| \leq (\bar{\mathcal{K}}_1 l_1 + \bar{\mathcal{K}}_2 l_2)r + \bar{\mathcal{K}}_1 N_1 + \bar{\mathcal{K}}_2 N_2. \tag{3.30}$$

As a result, (3.29) and (3.30) follow $\|\Omega(x, y)\| \leq r$, and thus $\Omega\mathcal{B}_r \subset \mathcal{B}_r$. Now, for $(x_1, y_2), (x_2, y_2) \in \mathcal{X}\times\mathcal{X}$ and any $t \in \mathcal{I}$, we get

$$
\begin{aligned}
&|\Omega_1(x_2,y_2)(t)-\Omega_1(x_1,y_1)(t)|\\
\leq&\frac{l_1T}{M(p-1)}\bigg\{\Big(\frac{T^2}{|T^2-\alpha\beta\gamma\eta|}+1\Big)\Big[(2-p)+(p-1)\frac{T}{2}\Big]\\
&+\frac{\alpha\beta\gamma\eta}{|T^2-\alpha\beta\gamma\eta|}\Big[(2-p)+(p-1)\frac{\eta}{2}\Big]\bigg\}\Big[\|x_2-x_1\|+\|y_2-y_1\|\Big]\\
&+\frac{\alpha\gamma T^2 l_2}{M(q-1)|T^2-\alpha\beta\gamma\eta|}\bigg\{2(2-q)+\frac{(q-1)}{2}(\gamma+T)\bigg\}\Big[\|x_2-x_1\|+\|y_2-y_1\|\Big],
\end{aligned}
$$

which implies to

$$
\|\Omega_1(x_2,y_2)-\Omega_1(x_1,y_1)\|\leq\Big(\mathcal{K}_1 l_1+\mathcal{K}_2 l_2\Big)\Big(\|x_2-x_1\|+\|y_2-y_1\|\Big). \tag{3.31}
$$

Similarly, we obtain

$$
\|\Omega_2(x_2,y_2)-\Omega_2(x_1,y_1)\|\leq\Big(\bar{\mathcal{K}}_1 l_1+\bar{\mathcal{K}}_2 l_2\Big)\Big(\|x_2-x_1\|+\|y_2-y_1\|\Big). \tag{3.32}
$$

Hence , using (3.31) and (3.32) we can get

$$
\|\Omega(x_2,y_2)-\Omega(x_1,y_1)\|\leq\Big[\Big(\mathcal{K}_1+\bar{\mathcal{K}}_1\Big)l_1+\Big(\mathcal{K}_2+\bar{\mathcal{K}}_2\Big)l_2\Big]\Big[\|x_2-x_1\|+\|y_2-y_1\|\Big].
$$

As a consequence of the condition (3.28), Ω is a contraction operator. As an outcome of the Banach fixed point theorem, we can conclude that the operator Ω has a unique fixed point, which is the unique solution of the problem (1.1), and (1.2). □

4. Illustrative example

In this section, we study an example to verify our results.

Example 4.1. *Consider the following coupled system:*

$$
\begin{cases}
{}^{CF}\mathcal{D}^{\frac{3}{2}}_{0+}x(t)=f(t,x(t),y(t)),\ t\in\mathcal{I}:=[0,5],\\
{}^{CF}\mathcal{D}^{\frac{5}{4}}_{0+}y(t)=g(t,x(t),y(t)),
\end{cases} \tag{4.1}
$$

supplemented with boundary conditions:

$$
\begin{cases}
x(0)=0, \qquad x(5)=\frac{1}{3}y(1),\\
y(0)=0, \qquad y(5)=\frac{1}{2}x(3).
\end{cases} \tag{4.2}
$$

Here $p=\frac{3}{2},\ q=\frac{5}{4},\ T=5,\ \alpha=\frac{1}{3},\ \beta=\frac{1}{2},\ \gamma=1,\ \eta=3.$
In order to illustrate our results, we choose the functions f *and* g *as:*

$$
f(t,x(t),y(t))=\frac{1}{45+t}\Big[\sin x(t)+|y(t)|\Big]+\cos t, \tag{4.3}
$$

$$g(t, x(t), y(t)) = \frac{1}{15e}\Big[\cos x(t) + |y(t)|\Big] + 5t. \tag{4.4}$$

Clearly, the functions f and g satisfy the $(\mathcal{H}_1)$ condition with $l_1 = \frac{1}{45}$ and $l_2 = \frac{1}{15e}$.

Now we have $\bar{\rho}_1 \approxeq 6.5625$, $\bar{\rho}_1 \approxeq 6.01563$, and then $l_1\bar{\rho}_1 + l_2\bar{\rho}_2 \approxeq 0.29337 < 1$. As a result, the condition (3.15) *of Theorem 3.1 is satisfied and then the problem (4.1), and (4.2) has at least one solution on $\mathcal{I}$.*

Furthermore, we find that, $\mathcal{K}_1 \approxeq 13.3546$, $\mathcal{K}_2 \approxeq 0.6696$, $\bar{\mathcal{K}}_1 \approxeq 3.4439$, $\bar{\mathcal{K}}_2 \approxeq 12.2321$. Consequently, $(\mathcal{K}_1 + \bar{\mathcal{K}}_1)l_1 + (\mathcal{K}_2 + \bar{\mathcal{K}}_2)l_2 \approxeq 0.6897 < 1$, so the conditions of Theorem 3.2 are satisfied. Therefore, the problem (4.1) and (4.2) has a unique solution on $[0, 5]$ *with f and g given by (4.3) and (4.4) respectively.*

5. Conclusions

In this paper, we have studied a new coupled system of nonlinear fractional differential equations with Caputo-Fabrizio derivatives of order of $(1, 2)$ supplemented with nonlocal boundary conditions. We successfully attained several essential conditions consistent to the existence and uniqueness of the solutions for this boundary value problem on an arbitrary domain. The proposed problem is transformed into an equivalent fractional integral equation, which is solved by using some standard fixed-point theorems like Krasnoselskii's fixed-point and Banach's contraction.

We emphasize that our results are novel and also provide certain new results. For instance, considering a different type of boundary conditions or studying the problem with mixed fractional orders. Therefore, the present study will contribute to the existing works of such coupled systems and establish new studies on the topic.

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Conflict of interests

No conflicts of interest exist.

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