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Research article

Lorentzian approximations for a Lorentzian α-Sasakian manifold and Gauss-Bonnet theorems

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Abstract: In this paper, we define the Lorentzian approximations of a 3-dimensional Lorentzian α-Sasakian manifold. Moreover, we define the notions of the intrinsic curvature for regular curves, the intrinsic geodesic curvature of regular curves on Lorentzian surfaces and spacelike surfaces and the intrinsic Gaussian curvature of Lorentzian surfaces and spacelike surfaces away from characteristic points. Furthermore, we derive the expressions of those curvatures and prove Gauss-Bonnet theorems for the Lorentzian surfaces and spacelike surfaces in the Lorentzian α-Sasakian manifold.

Keywords: Lorentzian α-Sasakian manifold; Gauss-Bonnet theorem; Lorentzian approximations; sub-Lorentzian geometry; intrinsic curvature
Mathematics Subject Classification: 53C40, 53C42

1. Introduction

Lorentzian α-Sasakian manifolds were introduced by Yildiz and Murathan in [1]. Then, many researchers began to study properties of α-Sasakian manifolds, such as second order parallel tensors [2], pseudosymmetric Lorentzian α-Sasakian manifolds [3], some special classes of Lorentzian α-Sasakian manifolds [4–6], certain derivations [7], Ricci solitons [8, 9], Lorentzian α-Sasakian manifolds admitting a quarter-symmetric metric connection [10], semi-symmetry type α-Sasakian manifolds [11] and $\mathcal{M}$-projectively semi-symmetric Lorentzian α-Sasakian manifolds [12]. Recently, Wang studied Gauss-Bonnet theorems in the BCV spaces and the twisted Heisenberg group [13], the affine group and the group of rigid motions of the Minkowski plane [14] by using the method of Riemannian approximations which first took by Balogh, Tyson and Vecchi to prove a Heisenberg version of the Gauss-Bonnet theorem [15, 16]. Riemannian approximations can be extended to the case for any Lie group equipped with left-invariant Lorentzian metric g, named Lorentzian approximations. Some typical works of Lorentzian approximations in a Lorentzian Heisenberg group are obtained in [17, 18]. Inspired by the above work, we proved Gauss-Bonnet theorems in the rototranslation group [19, 20], Lorentzian Sasakian space forms [21] and the group of rigid motions of the Minkowski plane with the general left-invariant metric [22]. However, very little is known about the Gauss-Bonnet theorem in 3-dimensional Lorentzian α-Sasakian manifolds. This paper attempts to solve this question by employing the method of the Lorentzian approximation scheme.

We restrict our attention to Lorentzian α-Sasakian manifolds. As we know, in [8], a differentiable manifold of dimension $(2n+1)$ is called a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, 1-form η and Lorentzian metric g which satisfy on M, respectively, that

$$\phi^2 = I + \eta\otimes\xi, \eta(\xi) = -1, \eta\circ\phi = 0, \phi\xi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), g(X, \xi) = \eta(X),$$

$$\nabla_X\xi = \alpha\phi X, (\nabla_X\eta)Y = \alpha g(\phi X, Y),$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g on M. Meanwhile, a Lorentzian α-Sasakian model of 3-dimensional Lorentzian α-Sasakian manifolds was constructed in [8]. In this paper, we focus on Gauss-Bonnet theorems for the Lorentzian surfaces and spacelike surfaces in the Lorentzian α-Sasakian model. We define the notions of the intrinsic curvature for regular curves, the intrinsic geodesic curvature of regular curves on Lorentzian surfaces and spacelike surfaces and the intrinsic Gaussian curvature of Lorentzian surfaces and spacelike surfaces away from characteristic points. Furthermore, we derive the expressions of those curvatures and prove Gauss-Bonnet theorems for the Lorentzian surfaces and spacelike surfaces in the 3-dimensional Lorentzian α-Sasakian manifold.

The paper is organized in the following way. Basic notions on (S_α, g) and the Lorentzian approximants (S_α, g_L) of the α-Sasakian manifold are given in Section 2. The sub-Lorentzian limit of curvature of curves in (S_α, g_L) will be computed. In Sections 3 and 4, we compute sub-Lorentzian limits of geodesic curvature of curves on Lorentzian surfaces and the intrinsic Gaussian curvature of Lorentzian surfaces in (S_α, g_L). In Section 5, we prove the Gauss-Bonnet theorem for Lorentzian surfaces. In Section 6, we prove the Gauss-Bonnet theorem for spacelike surfaces. Finally, we summarize the conclusions and add an appendix section on length measure and surface measure.

2. Lorentzian approximants and curvature tensor

In this section, some basic notions on a Lorentzian α-Sasakian manifold will be introduced. First, we recall the Lorentzian α-Sasakian model of 3-dimensional Lorentzian α-Sasakian manifolds constructed in [8]. Let α be some constant, and set $S_\alpha = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ equipped with a Lorentzian metric

$$g = e^{2z}dx^2 + e^{2z}(-dx + dy)^2 - \frac{1}{\alpha^2}dz^2.$$

Then, (S_α, g) was called the Lorentzian α-Sasakian model of 3-dimensional Lorentzian α-Sasakian manifold, where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. Let E_1, E_2 and E_3 be the vector fields on S_α given by

$$E_1 = \alpha\frac{\partial}{\partial z},\ E_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ E_3 = e^{-z}\frac{\partial}{\partial y}, \tag{2.1}$$

which are linearly independent at each point p of S_α. Then,

$$\frac{\partial}{\partial x} = e^z(E_2 - E_3),\ \frac{\partial}{\partial y} = e^z E_3,\ \frac{\partial}{\partial z} = \frac{1}{\alpha}E_1, \tag{2.2}$$

and $span\{E_1, E_2, E_3\} = T(S_\alpha)$. One can check the following brackets

$$[E_1, E_2] = -\alpha E_2,\ [E_2, E_3] = 0,\ [E_1, E_3] = -\alpha E_3. \tag{2.3}$$

Let $H = span\{E_1, E_2\}$ be the horizontal distribution on S_α. If we let

$$\theta_1 = \frac{1}{\alpha}dz, \theta_2 = e^z dx, \theta = e^z(-dx + dy),$$

then $H = \ker\theta$. To describe the Lorentzian approximants of S_α, let $L > 0$ and define a metric

$$g_L = -\theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + L\theta \otimes \theta,$$

so that $E_1, E_2, \widetilde{E_3} := L^{-\frac{1}{2}}E_3$ are a pseudo orthonormal basis on $T(S_\alpha)$ with respect to g_L. Hereafter, we denote the Lorentzian approximants to S_α by (S_α, g_L) and write S^L_α instead of (S_α, g_L). Note that $g = g_1$ is the Lorentzian metric on S_α. A non-zero vector $x \in S^L_\alpha$ is called spacelike, null or timelike if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. We define the norm of the vector $x \in S^L_\alpha$ by $\| x \| = \sqrt{|\langle x, x\rangle|}$. We assume that ∇^L is the Levi-civita connection on S^L_α with respect to g_L. Using the Koszul formula and (2.3), we have

Proposition 2.1. *The Levi-civita connection on S^L_α relative to the coordinate frame $E_1, E_2, \widetilde{E_3}$ is given by*

$$\begin{aligned} &\nabla^L_{E_1}E_1 = 0,\ \nabla^L_{E_1}E_2 = 0,\ \nabla^L_{E_1}E_3 = 0,\\ &\nabla^L_{E_2}E_1 = \alpha E_2,\ \nabla^L_{E_2}E_2 = \alpha E_1,\ \nabla^L_{E_2}E_3 = 0,\\ &\nabla^L_{E_3}E_1 = \alpha E_3,\ \nabla^L_{E_3}E_2 = 0,\ \nabla^L_{E_3}E_3 = \alpha L E_1. \end{aligned} \tag{2.4}$$

Proof. It follows from a direct application of the Koszul identity, which here simplifies

$$2\left\langle \nabla^L_{E_i}E_j, E_k\right\rangle_L = \left\langle \left[E_i, E_j\right], E_k\right\rangle_L - \left\langle \left[E_j, E_k\right], E_i\right\rangle_L + \left\langle [E_k, E_i], E_j\right\rangle_L, \tag{2.5}$$

where $i, j, k = 1, 2, 3$. □

For a Lorentzian α-Sasakian manifold M, one can compute the curvature tensor of the connection ∇^L by the formula $R^L(X, Y)Z = \nabla^L_X\nabla^L_Y Z - \nabla^L_Y\nabla^L_X Z - \nabla^L_{[X,Y]}Z$ or $R^L(X, Y)Z = \alpha^2[g(Y, Z)X - g(X, Z)Y]$. Then, we get the following proposition.

Proposition 2.2. *The curvature tensor of S^L_α is given by*

$$\begin{aligned} &R^L(E_1, E_2)E_1 = \alpha^2 E_2,\ R^L(E_1, E_2)E_2 = \alpha^2 E_1,\ R^L(E_1, E_2)E_3 = 0,\\ &R^L(E_1, E_3)E_1 = \alpha^2 E_3,\ R^L(E_1, E_3)E_2 = 0,\ R^L(E_1, E_3)E_3 = \alpha^2 L E_1,\\ &R^L(E_2, E_3)E_1 = 0,\ R^L(E_2, E_3)E_2 = -\alpha^2 E_3,\ R^L(E_2, E_3)E_3 = \alpha^2 L E_2. \end{aligned} \tag{2.6}$$

Proof. We take $R^L(X,Y)Z = \alpha^2[g(Y,Z)X - g(X,Z)Y]$ to compute curvature tensor of S^L_α. Taking

$$R^L(E_1,E_2)E_1 = \alpha^2[g(E_2,E_1)E_1 - g(E_1,E_1)E_2],$$

for example, we compute

$$g(E_2,E_1)E_1 = 0, g(E_1,E_1)E_2 = -E_2,$$

and hence

$$R^L(E_1,E_2)E_1 = \alpha^2 E_2.$$

□

3. Curvature for curves in S^L_α and sub-Lorentzian limit

In this section, we will compute the sub-Lorentzian limit of curvature for curves in S^L_α. Our approach is to define sub-Lorentzian objects as limits of horizontal objects in S^L_α, where a family of metrics g_L is essentially obtained as an anisotropic blow-up of the Lorentzian metric g. At the heart of this approach is the fact that the intrinsic horizontal geometry does not change with L. Let $\beta : I \to S^L_\alpha$ be a regular curve, where I is an open interval in R. The regular curve β is called a spacelike curve, timelike curve or null curve if $\dot\beta(t)$ is a spacelike vector, timelike vector or null vector at any $t \in I$, respectively.

Definition 3.1. *Let $\beta : I \to S^L_\alpha$ be a C^1 smooth curve, and we say that β is regular if $\dot\beta \neq 0$ for every $t \in I$. Moreover we say that $\beta(t)$ is a horizontal point of β if*

$$\theta(\dot\beta(t)) = e^{\beta_3}(\dot\beta_2(t) - \dot\beta_1(t)) = 0,$$

where $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$.

As is well known, if β is a curve with arc length parametrization, then the standard definition of curvature for β in Riemannian geometry is $\kappa^L_\beta := \left\|\nabla^L_{\dot\beta}\dot\beta\right\|_L$. If β is a curve with an arbitrary parametrization, then we give the definitions as follows:

Definition 3.2. *Let $\beta : I \to S^L_\alpha$ be a C^2-smooth regular curve.*

(1) If $\nabla^L_{\dot\beta}\dot\beta$ is a spacelike vector, we define the curvature κ^L_β of β at $\beta(t)$ by

$$\kappa^L_\beta := \sqrt{\frac{\left\|\nabla^L_{\dot\beta}\dot\beta\right\|^2_L}{\left\|\dot\beta\right\|^4_L} - \frac{\left\langle\nabla^L_{\dot\beta}\dot\beta, \dot\beta\right\rangle^2_L}{\langle\dot\beta,\dot\beta\rangle^3_L}}. \tag{3.1}$$

(2) If $\nabla^L_{\dot\beta}\dot\beta$ is a timelike vector, we define the curvature κ^L_β of β at $\beta(t)$ by

$$\kappa^L_\beta := \sqrt{\frac{\left\|\nabla^L_{\dot\beta}\dot\beta\right\|^2_L}{\left\|\dot\beta\right\|^4_L} + \frac{\left\langle\nabla^L_{\dot\beta}\dot\beta, \dot\beta\right\rangle^2_L}{\langle\dot\beta,\dot\beta\rangle^3_L}}. \tag{3.2}$$

Proposition 3.3. *Suppose that $\beta : I \to S^L_\alpha$ is a C^2-smooth regular curve.*
(1) If $\nabla^L_{\dot\beta}\dot\beta$ is a spacelike vector, then

$$\begin{aligned}
\kappa^L_\beta =&\{\{-[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2]^2 + [2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]^2 \\
&+ L[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]^2\} \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2 + L(\theta(\dot\beta(t)))^2]^{-2} \\
&- \{-\frac{1}{\alpha}\dot\beta_3[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] + e^{\beta_3}\dot\beta_1[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1] \\
&+ L\theta(\dot\beta(t))[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\}^2 \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2 + L(\theta(\dot\beta(t)))^2]^{-3}\}^{\frac{1}{2}}.
\end{aligned} \tag{3.3}$$

In particular, if $\beta(t)$ is a horizontal point of β,

$$\begin{aligned}
\kappa^L_\beta =&\{\{-[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2]^2 + [2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]^2 + L[\frac{d}{dt}(\theta(\dot\beta(t)))]^2\} \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-2} \\
&- \{-\frac{1}{\alpha}\dot\beta_3[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] + e^{\beta_3}\dot\beta_1[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-3}\}^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

(2) If $\nabla^L_{\dot\beta}\dot\beta$ is a timelike vector, then

$$\begin{aligned}
\kappa^L_\beta =&\{-\{-[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2]^2 + [2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]^2 \\
&+ L[(\dot\beta_3\theta(\dot\beta(t))) + \frac{d}{dt}(\theta(\dot\beta(t)))]^2\} \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2 + L(\theta(\dot\beta(t)))^2]^{-2} \\
&+ \{-\frac{1}{\alpha}\dot\beta_3[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] + e^{\beta_3}\dot\beta_1[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1] \\
&+ L\theta(\dot\beta(t))[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\}^2 \\
&\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2 + L(\theta(\dot\beta(t)))^2]^{-3}\}^{\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

In particular, if $\beta(t)$ is a horizontal point of β,

$$\begin{aligned}\kappa_\beta^L =&\{-\{-[\frac{1}{\alpha}\ddot{\beta}_3+\alpha e^{2\beta_3}\dot{\beta}_1^2]^2+[2\dot{\beta}_3\dot{\beta}_1e^{\beta_3}+e^{\beta_3}\ddot{\beta}_1]^2+L[\frac{d}{dt}(\theta(\dot{\beta}(t)))]^2\}\\ &\times[-\frac{1}{\alpha^2}\dot{\beta}_3^2+e^{2\beta_3}\dot{\beta}_1^2]^{-2}\\ &+\{-\frac{1}{\alpha}\dot{\beta}_3[\frac{1}{\alpha}\ddot{\beta}_3+\alpha e^{2\beta_3}\dot{\beta}_1^2]+e^{\beta_3}\dot{\beta}_1[2\dot{\beta}_3\dot{\beta}_1e^{\beta_3}+e^{\beta_3}\ddot{\beta}_1]\}^2\\ &\times[-\frac{1}{\alpha^2}\dot{\beta}_3^2+e^{2\beta_3}\dot{\beta}_1^2]^{-3}\}^{\frac{1}{2}}.\end{aligned} \tag{3.6}$$

Proof. By (2.2), we have

$$\dot{\beta}(t)=\frac{1}{\alpha}\dot{\beta}_3E_1+e^{\beta_3}\dot{\beta}_1E_2+\theta\,(\dot{\beta}(t))E_3. \tag{3.7}$$

By Proposition 2.1 and (3.7), we obtain

$$\nabla^L_{\dot{\beta}}E_1=\alpha e^{\beta_3}\dot{\beta}_1E_2+\alpha\theta(\dot{\beta}(t))E_3,$$

$$\nabla^L_{\dot{\beta}}E_2=\alpha e^{\beta_3}\dot{\beta}_1E_1,$$

$$\nabla^L_{\dot{\beta}}E_3=\alpha L\theta(\dot{\beta}(t))E_1. \tag{3.8}$$

Coming by (3.7), we have

$$\begin{aligned}\nabla^L_{\dot{\beta}}\dot{\beta}=&\left[\frac{1}{\alpha}\ddot{\beta}_3+\alpha e^{2\beta_3}\dot{\beta}_1^2+\alpha L(\theta(\dot{\beta}(t)))^2\right]E_1\\ &+\left[2\dot{\beta}_3\dot{\beta}_1e^{\beta_3}+e^{\beta_3}\ddot{\beta}_1\right]E_2\\ &+\left[\dot{\beta}_3\theta(\dot{\beta}(t))+\frac{d}{dt}(\theta(\dot{\beta}(t)))\right]E_3.\end{aligned} \tag{3.9}$$

By (3.7), (3.9), and the definition of κ_β^L, we get Proposition 3.3. □

Definition 3.4. *Let $\beta : I \to S^L_\alpha$ be a C^2-smooth regular curve. We define the intrinsic curvature κ_β^∞ of β at $\beta(t)$ to be*

$$\kappa_\beta^\infty := \lim_{L\to\infty}\kappa_\beta^L,$$

if the limit exists.

We introduce the following notation : For continuous functions $f_1, f_2 : (0,+\infty) \to \mathbb{R}$,

$$f_1(L)\sim f_2(L),\ \text{as } L\to+\infty \Leftrightarrow \lim_{L\to\infty}\frac{f_1(L)}{f_2(L)}=1.$$

Proposition 3.5. *Suppose that $\beta : I \to S^L_\alpha$ is a C^2-smooth regular curve in the Lorentzian α-Sasakian manifold.*

(1) If $\nabla^L_{\dot\beta}\dot\beta$ *is a spacelike vector, then* κ^∞_β *does not exist, if* $\theta(\dot\beta(t)) \neq 0$.

$$\begin{aligned}\kappa^\infty_\beta =& \{\{-[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2]^2 + [2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]^2\} \\ &\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-2} \\ &- \{-\frac{1}{\alpha}\dot\beta_3[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] + e^{\beta_3}\dot\beta_1[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\ &\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-3}\}^{\frac{1}{2}},\end{aligned} \tag{3.10}$$

if $\theta(\dot\beta(t)) = 0$ *and* $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$.

$$\lim_{L\to\infty}\frac{\kappa^L_\beta}{\sqrt{L}} = \frac{\left|\frac{d}{dt}(\theta(\dot\beta(t)))\right|}{\left|-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2\right|}, if\ \theta(\dot\beta(t)) = 0 \ and\ \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{3.11}$$

(2) If $\nabla^L_{\dot\beta}\dot\beta$ *is a timelike vector, then*

$$\kappa^\infty_\beta = \mid \alpha \mid, if\ \theta(\dot\beta(t)) \neq 0. \tag{3.12}$$

$$\begin{aligned}\kappa^\infty_\beta =& \{-\{-[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2]^2 + [2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]^2\} \\ &\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-2} \\ &+ \{-\frac{1}{\alpha}\dot\beta_3[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] + e^{\beta_3}\dot\beta_1[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\ &\times [-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2]^{-3}\}^{\frac{1}{2}},\end{aligned} \tag{3.13}$$

if $\theta(\dot\beta(t)) = 0$ *and* $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$.

$$\lim_{L\to\infty}\frac{\kappa^L_\beta}{\sqrt{L}} = \frac{\sqrt{-[\frac{d}{dt}(\theta(\dot\beta(t)))]^2}}{\left|-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2\right|}, if\ \theta(\dot\beta(t)) = 0 \ and\ \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{3.14}$$

Therefore, this situation does not exist.

Proof. (1) If $\nabla^L_{\dot\beta}\dot\beta$ is a spacelike vector, we have

$$\langle\nabla^L_{\dot\beta}\beta, \nabla^L_{\dot\beta}\beta\rangle_L \sim -\alpha^2 L^2(\theta(\dot\beta(t)))^4 \text{ as } L \to +\infty,$$

$$\langle\dot\beta, \dot\beta\rangle_L \sim L[\theta(\dot\beta(t))]^2,\ \left\langle\nabla^L_{\dot\beta}\beta, \dot\beta\right\rangle_L^2 \sim O\left(L^2\right) \text{ as } L \to +\infty.$$

Thus,

$$\frac{\langle\nabla^L_{\dot\beta}\beta, \nabla^L_{\dot\beta}\beta\rangle_L}{\left\|\dot\beta\right\|_L^4} \to -\alpha^2 \text{ as } L \to +\infty,$$

$$\frac{\left\langle \nabla^L_{\dot\beta}\dot\beta,\dot\beta\right\rangle^2_L}{\langle\dot\beta,\dot\beta\rangle^3_L}\to 0 \text{ as } L\to+\infty,$$

$$\kappa^\infty_\beta = \sqrt{-\alpha^2}.$$

So, using (3.1), we know κ^∞_β does not exist, if $\theta(\dot\beta(t)) \neq 0$. If $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$, we get (3.10). If $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) \neq 0$, then

$$\langle\nabla^L_{\dot\beta}\beta,\nabla^L_{\dot\beta}\beta\rangle_L \sim L\left[\frac{d}{dt}(\theta(\dot\beta(t)))\right]^2 \text{ as } L\to+\infty,$$

$$\langle\dot\beta,\dot\beta\rangle_L = -\frac{1}{\alpha^2}\dot\beta_3{}^2 + e^{2\beta_3}\dot\beta_1{}^2,$$

$$\left\langle\nabla^L_{\dot\beta}\dot\beta,\dot\beta\right\rangle^2_L \sim O(1) \text{ as } L\to+\infty.$$

By (3.1), we get (3.11).

(2) If $\nabla^L_{\dot\beta}\beta$ is a timelike vector, we have

$$\langle\nabla^L_{\dot\beta}\beta,\nabla^L_{\dot\beta}\beta\rangle_L \sim \alpha^2L^2(\theta(\dot\beta(t)))^4 \text{ as } L\to+\infty,$$

$$\langle\dot\beta,\dot\beta\rangle_L \sim L[\theta(\dot\beta(t))]^2,\ \left\langle\nabla^L_{\dot\beta}\dot\beta,\dot\beta\right\rangle^2_L \sim O\left(L^2\right) \text{ as } L\to+\infty,$$

$$\kappa^\infty_\beta = \sqrt{\alpha^2} =|\ \alpha\ |\ .$$

We get (3.12). If $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$, we get (3.13). If $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) \neq 0$,

$$\lim_{L\to\infty}\frac{\kappa^L_\beta}{\sqrt{L}} = \frac{\sqrt{-[\frac{d}{dt}(\theta(\dot\beta(t)))]^2}}{\left|-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2\right|},$$

and then, the situation does not exist. □

4. Geodesic curvatures of curves on Lorentzian surfaces in S^L_α

In this section, we will compute the expressions of intrinsic geodesic curvatures of curves on Lorentzian surfaces in S^L_α. We will say that a surface S^L_α is regular if S is a C^2-smooth compact and oriented surface. In particular we will assume that there exists a C^2-smooth function $h : S^L_\alpha \to \mathbb{R}$ such that

$$S = \{(x_1, x_2, x_3) \in S^L_\alpha : h(x_1, x_2, x_3) = 0\},$$

and $\nabla_{S^L_\alpha}h = h_{x_1}\partial_{x_1} + h_{x_2}\partial_{x_2} + h_{x_3}\partial_{x_3} \neq 0$. Let $\nabla_H h = E_1(h)E_1 + E_2(h)E_2$. A point $x \in S$ is called characteristic if $\nabla_H h(x) = (0, 0)$. Our computations will be local and away from characteristic points of S. Let us define first $p := E_1h, q := E_2h,$ and $r := \widetilde{E_3}h$. Since $-p^2 + q^2 > 0$, we say $S \subset S^L_\alpha$ is a horizontal spacelike surface. When $L \to +\infty$, $-p^2 + q^2 + r^2 > 0$. Then, we define

$$\begin{aligned} &l := \sqrt{-p^2 + q^2}, l_L := \sqrt{-p^2 + q^2 + r^2}, \bar p := \frac{p}{l},\\ &\bar q := \frac{q}{l}, \bar p_L := \frac{p}{l_L}, \bar q_L := \frac{q}{l_L}, \bar r_L := \frac{r}{l_L}. \end{aligned} \tag{4.1}$$

In particular, $-\bar{p}^2 + \bar{q}^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$N_L = -\bar{p}_L E_1 + \bar{q}_L E_2 + \bar{r}_L \widetilde{E_3}, F_1 = \bar{q}E_1 - \bar{p}E_2, F_2 = \bar{r}_L\bar{p}E_1 - \bar{r}_L\bar{q}E_2 + \frac{l}{l_L}\widetilde{E_3}. \tag{4.2}$$

Then, N_L is the unit spacelike normal vector to S, and F_1 is a unit timelike vector, while F_2 is a unit spacelike vector of S. $\{F_1, F_2\}$ is the orthonormal basis of S. Let $\dot{\beta} = aF_1 + bF_2$. We define $J_L(\dot{\beta}) = aF_2 + bF_1$ if β is a C^2-smooth spacelike curve, and we define $J_L(\dot{\beta}) = -aF_2 - bF_1$ if β is a C^2-smooth timelike curve. Then, $g_L(\dot{\beta}, J_L(\dot{\beta})) = 0$ and $(\dot{\beta}, J_L(\dot{\beta}))$ have the same orientation with $\{F_1, F_2\}$.

For every $U, V \in TS$, we define $\nabla^{S,L}_U V = \pi\nabla^L_U V$ where $\pi : TS^L_\alpha \to TS$ is the projection. Then, $\nabla^{S,L}$ is the Levi-Civita connection on S with respect to the metric g_L. By (3.9), (4.2) and

$$\nabla^{S,L}_{\dot{\beta}}\dot{\beta} = -\langle\nabla^L_{\dot{\beta}}\dot{\beta}, F_1\rangle_L F_1 + \langle\nabla^L_{\dot{\beta}}\dot{\beta}, F_2\rangle_L F_2, \tag{4.3}$$

we have

$$\begin{aligned}\nabla^{S,L}_{\dot{\beta}}\dot{\beta} =& - \{-\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2 + \alpha L(\theta(\dot{\beta}(t)))^2] - \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}F_1 \\ &+ \{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2 + \alpha L(\theta(\dot{\beta}(t)))^2] - \overline{r_L}\bar{q}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1] \\ &+ \frac{l}{l_L}L^{\frac{1}{2}}[\dot{\beta}_3\theta(\dot{\beta}(t)) + \frac{d}{dt}(\theta(\dot{\beta}(t)))]\}F_2.\end{aligned} \tag{4.4}$$

Therefore, when $\theta(\dot{\beta}(t)) = 0$, we have

$$\begin{aligned}\nabla^{S,L}_{\dot{\beta}}\dot{\beta} =& \{\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] + \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}F_1 \\ &+ \{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \overline{r_L}\bar{q}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1] \\ &+ \frac{l}{l_L}L^{\frac{1}{2}}\frac{d}{dt}(\theta(\dot{\beta}(t)))\}F_2.\end{aligned} \tag{4.5}$$

Definition 4.1. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface, $\beta : I \to S$ be a C^2-smooth regular curve.*
(1) If $\nabla^{S,L}_{\dot{\beta}}\dot{\beta}$ is a spacelike vector, the geodesic curvature $\kappa^L_{\beta,S}$ of β at $\beta(t)$ is defined as

$$\kappa^L_{\beta,S} := \sqrt{\frac{\|\nabla^{S,L}_{\dot{\beta}}\dot{\beta}\|^2_{S,L}}{\|\dot{\beta}\|^4_{S,L}} - \frac{\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta}, \dot{\beta}\rangle^2_{S,L}}{\langle\dot{\beta}, \dot{\beta}\rangle^3_{S,L}}}. \tag{4.6}$$

(2) If $\nabla^{S,L}_{\dot{\beta}}\dot{\beta}$ is a timelike vector, the geodesic curvature $\kappa^L_{\beta,S}$ of β at $\beta(t)$ is defined as

$$\kappa^L_{\beta,S} := \sqrt{\frac{\|\nabla^{S,L}_{\dot{\beta}}\dot{\beta}\|^2_{S,L}}{\|\dot{\beta}\|^4_{S,L}} + \frac{\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta}, \dot{\beta}\rangle^2_{S,L}}{\langle\dot{\beta}, \dot{\beta}\rangle^3_{S,L}}}. \tag{4.7}$$

Definition 4.2. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface, $\beta : I \to S$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $\kappa^\infty_{\beta,S}$ of β at $\beta(t)$ to be*

$$\kappa^\infty_{\beta,S} := \lim_{L\to+\infty} \kappa^L_{\beta,S},$$

if the limit exists.

Proposition 4.3. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface, $\beta : I \to S$ be a C^2-smooth regular curve.*

(1) If $\nabla^{S,L}_{\dot\beta}\dot\beta$ is a spacelike vector, then $\kappa^\infty_{\beta,S}$ does not exist, if $\theta(\dot\beta(t)) \neq 0$.

$$\kappa^\infty_{\beta,S} = 0, \ \textit{if}\ \theta(\dot\beta(t)) = 0 \ \textit{and}\ \frac{d}{dt}(\theta(\dot\beta(t))) = 0,$$

$$\lim_{L\to+\infty} \frac{\kappa^L_{\beta,S}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\theta(\dot\beta(t)))|}{(\frac{1}{\alpha}\bar q\dot\beta_3 + e^{\beta_3}\bar p\dot\beta_1)^2}, \ \textit{if}\ \theta(\dot\beta(t)) = 0 \ \textit{and}\ \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{4.8}$$

(2) If $\nabla^{S,L}_{\dot\beta}\dot\beta$ is a timelike vector, then

$$\kappa^\infty_{\beta,S} = |\alpha\bar q|, \ \textit{if}\ \theta(\dot\beta(t)) \neq 0, \tag{4.9}$$

$$\kappa^\infty_{\beta,S} = 0, \ \textit{if}\ \theta(\dot\beta(t)) = 0 \ \textit{and}\ \frac{d}{dt}(\theta(\dot\beta(t))) = 0,$$

$\lim_{L\to+\infty} \frac{\kappa^L_{\beta,S}}{\sqrt{L}}$ *does not exist, if $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) \neq 0$.*

Proof. (1) If $\nabla^{S,L}_{\dot\beta}\dot\beta$ is a spacelike vector, by (3.7) and $\dot\beta \in TS$, we have

$$\begin{aligned}\dot\beta(t) &= aF_1 + bF_2\\ &= (a\bar q + b\overline{r_L}\bar p)E_1 + (-a\bar p - b\overline{r_L}\bar q)E_2 + \frac{bl}{l_L}L^{-\frac{1}{2}}E_3.\end{aligned}$$

Thus,

$$\begin{cases} a\bar q + b\overline{r_L}\bar p = \frac{1}{\alpha}\dot\beta_3,\\ -a\bar p - b\overline{r_L}\bar q = e^{\beta_3}\dot\beta_1,\\ \frac{bl}{l_L}L^{-\frac{1}{2}} = \theta(\dot\beta(t)),\end{cases}$$

and we have

$$\begin{cases} a = \frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p,\\ b = \frac{l_L}{l}L^{\frac{1}{2}}\theta(\dot\beta(t)).\end{cases}$$

Thus,

$$\dot\beta(t) = (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)F_1 + \frac{l_L}{l}L^{\frac{1}{2}}\theta(\dot\beta(t))F_2. \tag{4.10}$$

By (4.4), we have

$$\begin{aligned}\langle\nabla^{S,L}_{\dot\beta}\dot\beta, \nabla^{S,L}_{\dot\beta}\dot\beta\rangle_{S,L} = &-\{-\bar q[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \bar p[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2\\ &+ \{-\overline{r_L}\bar p[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \overline{r_L}\bar q[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\\ &+ \frac{l}{l_L}L^{\frac{1}{2}}[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\}^2.\end{aligned} \tag{4.11}$$

Similarly, we have that when $\theta(\dot{\beta}(t)) \neq 0$,

$$\begin{aligned}\langle\dot{\beta},\dot{\beta}\rangle_{S,L} &= -(\frac{1}{\alpha}\dot{\beta}_3\bar{q} + e^{\beta_3}\dot{\beta}_1\bar{p})^2 + (\frac{l_L}{l})^2 L(\theta(\dot{\beta}(t)))^2 \\ &\sim L(\theta(\dot{\beta}(t)))^2 \text{ as } L \to +\infty.\end{aligned} \tag{4.12}$$

By (4.4) and (4.9), we have

$$\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta},\dot{\beta}\rangle_{S,L} \sim M_0 L, \tag{4.13}$$

where M_0 does not depend on L. By Definition 4.1, (4.11)–(4.13), $\kappa^{\infty}_{\beta,S}$ does not exist, if $\theta(\dot{\beta}(t)) \neq 0$. When $\theta(\dot{\beta}(t)) = 0$ and $\frac{d}{dt}(\theta(\dot{\beta}(t))) = 0$, then

$$\begin{aligned}\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta}, \nabla^{S,L}_{\dot{\beta}}\dot{\beta}\rangle_{S,L} &= -\{-\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}^2 \\ &\quad + \{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \overline{r_L}\bar{q}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}^2 \\ &\sim -\{-\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}^2 \text{ as } L \to +\infty,\end{aligned} \tag{4.14}$$

and

$$\langle\dot{\beta},\dot{\beta}\rangle_{S,L} = -(\frac{1}{\alpha}\dot{\beta}_3\bar{q} + e^{\beta_3}\dot{\beta}_1\bar{p})^2 \text{ as } L \to +\infty, \tag{4.15}$$

$$\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta},\dot{\beta}\rangle_{S,L} = (\frac{1}{\alpha}\dot{\beta}_3\bar{q} + e^{\beta_3}\dot{\beta}_1\bar{p})\{-\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]\}, \tag{4.16}$$

where $\bar{A} = -\bar{q}[\frac{1}{\alpha}\ddot{\beta}_3 + \alpha e^{2\beta_3}\dot{\beta}_1^2] - \bar{p}[2\dot{\beta}_3\dot{\beta}_1 e^{\beta_3} + e^{\beta_3}\ddot{\beta}_1]$ and $\bar{B} = \frac{1}{\alpha}\dot{\beta}_3\bar{q} + e^{\beta_3}\dot{\beta}_1\bar{p}$. By (4.14)–(4.16) and (4.6), we get

$$\kappa^{\infty}_{\beta,S} = \sqrt{\frac{-\bar{A}^2}{\bar{B}^4} + \frac{\bar{A}^2\bar{B}^2}{\bar{B}^6}} = 0.$$

When $\theta(\dot{\beta}(t)) = 0$, and $\frac{d}{dt}(\theta(\dot{\beta}(t))) \neq 0$, we have

$$\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta}, \nabla^{S,L}_{\dot{\beta}}\dot{\beta}\rangle_{S,L} \sim L[\frac{d}{dt}(\theta(\dot{\beta}(t)))]^2,$$

$$\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta},\dot{\beta}\rangle_{S,L} \sim O(1).$$

Therefore, (4.8) holds.

(2) If $\nabla^{S,L}_{\dot{\beta}}\dot{\beta}$ is a timelike vector, by similar calculation, we get (2). □

Definition 4.4. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface, $\beta : I \to S$ be a C^2-smooth regular curve. The signed geodesic curvature $\kappa^{L,c}_{\beta,S}$ of β at $\beta(t)$ is defined as*

$$\kappa^{L,c}_{\beta,S} := \frac{\langle\nabla^{S,L}_{\dot{\beta}}\dot{\beta}, J_L(\dot{\beta})\rangle_{S,L}}{||\dot{\beta}||^3_{S,L}}.$$

Definition 4.5. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface, $\beta : I \to S$ be a C^2-smooth regular curve. We define the intrinsic geodesic curvature $\kappa^{\infty}_{\beta,S}$ of β at the non-characteristic point $\beta(t)$ to be*

$$\kappa^{\infty,c}_{\beta,S} := \lim_{L\to+\infty} \kappa^{L,c}_{\beta,S},$$

if the limit exists.

Proposition 4.6. *Let $S \subset S^L_\alpha$ be a Lorentzian regular surface.*

(1) If $\beta : I \to S$ is a C^2-smooth regular spacelike curve, then

$$\kappa^{\infty,c}_{\beta,S} = -\alpha\bar{q}, \ \textit{if} \ \ \theta(\dot\beta(t)) \neq 0, \tag{4.17}$$

$$\kappa^{\infty,c}_{\beta,S} = 0, \ \textit{if} \ \ \theta(\dot\beta(t)) = 0 \ \textit{and} \ \frac{d}{dt}(\theta(\dot\beta(t))) = 0,$$

$$\lim_{L\to+\infty} \frac{k^{L,c}_{\beta,S}}{\sqrt{L}} = \frac{\frac{d}{dt}(\theta(\dot\beta(t)))}{-(\frac{1}{\alpha}\bar{q}\dot\beta_3 + e^{\beta_3}\bar{p}\dot\beta_1)^2}, \ \textit{if} \ \ \theta(\dot\beta(t)) = 0 \ \textit{and} \ \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{4.18}$$

(2) If $\beta : I \to S$ is a C^2-smooth regular timelike curve, then

$$\kappa^{\infty,c}_{\beta,S} = \alpha\bar{q}, \ \textit{if} \ \ \theta(\dot\beta(t)) \neq 0, \tag{4.19}$$

$$\kappa^{\infty,c}_{\beta,S} = 0, \ \textit{if} \ \ \theta(\dot\beta(t)) = 0 \ \textit{and} \ \frac{d}{dt}(\theta(\dot\beta(t))) = 0,$$

$$\lim_{L\to+\infty} \frac{k^{L,c}_{\beta,S}}{\sqrt{L}} = \frac{\frac{d}{dt}(\theta(\dot\beta(t)))}{(\frac{1}{\alpha}\bar{q}\dot\beta_3 + e^{\beta_3}\bar{p}\dot\beta_1)^2}, \ \textit{if} \ \ \theta(\dot\beta(t)) = 0 \ \textit{and} \ \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{4.20}$$

Proof. For (4.1), by (4.10), we have

$$J_L(\dot\beta) = \frac{l_L}{l} L^{\frac{1}{2}} \theta(\dot\beta(t)) F_1 + (\frac{1}{\alpha}\dot\beta_3\bar{q} + e^{\beta_3}\dot\beta_1\bar{p}) F_2. \tag{4.21}$$

By (4.4) and (4.21), we have

$$\begin{aligned}
\langle \nabla^{S,L}_{\dot\beta}\dot\beta, J_L(\dot\beta)\rangle_{S,L} =& \frac{l_L}{l} L^{\frac{1}{2}} \theta(\dot\beta(t))\{-\bar{q}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \bar{p}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\} \\
&+ (\frac{1}{\alpha}\dot\beta_3\bar{q} + e^{\beta_3}\dot\beta_1\bar{p})\{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] \\
&- \overline{r_L}\bar{q}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1] + \frac{l}{l_L} L^{\frac{1}{2}}[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\} \\
\sim& -\alpha L^{\frac{3}{2}}(\theta(\dot\beta(t)))^3\bar{q} \text{ as } L \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\| \dot\beta \|^2_{S,L} =& -(\frac{1}{\alpha}\dot\beta_3\bar{q} + e^{\beta_3}\dot\beta_1\bar{p})^2 + [\frac{l_L}{l} L^{\frac{1}{2}}\theta(\dot\beta(t))]^2 \\
\sim& L(\theta(\dot\beta(t)))^2 \text{ as } L \to +\infty.
\end{aligned}$$

Thus, if $\theta(\dot\beta(t)) \neq 0$, (4.17) holds. When $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$, we get

$$\begin{aligned}\langle \nabla^{S,L}_{\dot\beta}\dot\beta, J_L(\dot\beta)\rangle_{L,S} &= (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)\{-\overline{r_L}\bar p[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1] \\ &\quad - \overline{r_L}\bar q[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\} \\ &\sim O(L^{-\frac{1}{2}}) \text{ as } L \to +\infty.\end{aligned}$$

So, $\kappa^{\infty,c}_{\beta,S} = 0$. When $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) \neq 0$, we have

$$\langle \nabla^{S,L}_{\dot\beta}\dot\beta, J_L(\dot\beta)\rangle_{L,S} \sim L^{\frac{1}{2}}(\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)\frac{d}{dt}(\theta(\dot\beta(t))) \text{ as } L \to +\infty.$$

We get

$$\lim_{L\to+\infty}\frac{k^{L,c}_{\beta,S}}{\sqrt{L}} = \frac{\frac{d}{dt}(\theta(\dot\beta(t)))}{-(\frac{1}{\alpha}\bar q\dot\beta_3 + e^{\beta_3}\bar p\dot\beta_1)^2}.$$

(2) If $\beta : I \to S$ is a C^2-smooth regular timelike curve, by similar calculation, we get (2). □

5. Lorentzian surface and a Gauss-Bonnet theorem in S^L_α

In this section, we will compute the expression of the sub-Lorentzian limit of the Gaussian curvature of Lorentzian surfaces in S^L_α. Then, we will prove a Gauss-Bonnet theorem for Lorentzian surfaces in S^L_α. To do this, we define the second fundamental form II^L of the embedding of S into S^L_α by

$$II^L = \begin{pmatrix}\langle \nabla^L_{F_1}N_L, F_1\rangle_L & \langle \nabla^L_{F_1}N_L, F_2\rangle_L \\ \langle \nabla^L_{F_2}N_L, F_1\rangle_L & \langle \nabla^L_{F_2}N_L, F_2\rangle_L\end{pmatrix}.$$

We have the following theorem.

Theorem 5.1. *For the embedding of S into S^L_α, the second fundamental form II^L of the embedding of S is given by*

$$II^L = \begin{pmatrix}h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix},$$

where

$$h_{11} = \frac{l}{l_L}[E_1(\bar p) - E_2(\bar q)] + \alpha\bar p_L,$$

$$h_{12} = \frac{l_L}{l}\langle F_1, \nabla_H \bar{r_L}\rangle_L,$$

$$h_{21} = \frac{l_L}{l}\langle F_1, \nabla_H \bar{r_L}\rangle_L + \bar{r_L}^2\sqrt{L},$$

$$h_{22} = \frac{l^2}{l_L^2}\langle F_2, \nabla_H(\frac{r}{l})\rangle_L - \frac{l}{l_L}\bar{r_L}\widetilde{E_3}(\frac{l}{l_L}) + (\frac{l}{l_L})^2\widetilde{E_3}(\bar{r_L}) - \alpha\bar p_L.$$

Proof. Since $\langle F_1, N_L\rangle_L = 0, \langle F_2, N_L\rangle_L = 0$, we have

$$\langle \nabla^L_{F_1} N_L, F_1\rangle_L = -\langle \nabla^L_{F_1} F_1, N_L\rangle_L, \langle \nabla^L_{F_2} N_L, F_2\rangle_L = -\langle \nabla^L_{F_2} F_2, N_L\rangle_L.$$

Using the definition of the connection, the identities in (2.4) and grouping terms, we have

$$\begin{aligned}\nabla^L_{F_1} F_1 =& \nabla^L_{\bar{q}E_1 - \bar{p}E_2} \bar{q}E_1 - \bar{p}E_2\\ =& [\bar{q}E_1(\bar{q}) - \bar{p}E_2(\bar{q}) + \alpha\bar{p}^2]E_1 - [\bar{q}E_1(\bar{p}) - \bar{p}E_2(\bar{p}) + \alpha\bar{p}\bar{q}]E_2.\end{aligned}$$

Since $-\bar{p}^2 + \bar{q}^2 = 1$, we have $-\bar{p}E_i\bar{p} + \bar{q}E_i\bar{q} = 0, i = 1, 2, 3$. Thus, $\bar{q}E_1\bar{q} = \bar{p}E_1\bar{p}$, $\bar{q}E_2\bar{q} = \bar{p}E_2\bar{p}$. Next, we compute the inner product of this with N_L, and we have

$$\langle \nabla^L_{F_1} F_1, N_L\rangle = -\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] - \alpha\bar{p}_L.$$

We get

$$h_{11} = -\langle \nabla^L_{F_1} F_1, N_L\rangle_L = \frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] + \alpha\bar{p}_L.$$

To compute h_{12}, using the definition of the connection, we obtain

$$\begin{aligned}\nabla^L_{F_1} F_2 =& \nabla^L_{\bar{q}E_1 - \bar{p}E_2} \bar{r_L}\bar{p}E_1 - \bar{r_L}\bar{q}E_2 + \frac{l}{l_L}\widetilde{E_3}\\ =& [\bar{q}E_1(\bar{r_L}\bar{p}) - \bar{p}E_2(\bar{r_L}\bar{p}) + \alpha\bar{r_L}\bar{p}\bar{q}]E_1 - [\bar{q}E_1(\bar{r_L}\bar{q}) - \bar{p}E_2(\bar{r_L}\bar{q}) + \alpha\bar{r_L}\bar{p}^2]E_2\\ &+ [\bar{q}E_1(\frac{l}{l_L}) - \bar{p}E_2(\frac{l}{l_L})]\widetilde{E_3}.\end{aligned}$$

We get

$$\langle \nabla^L_{F_1} F_2, N_L\rangle_L = -\frac{l_L}{l}\langle F_1, \nabla^L_H \bar{r_L}\rangle_L,$$

and therefore

$$h_{12} = -\langle \nabla^L_{F_1} F_2, N_L\rangle_L = \frac{l_L}{l}\langle F_1, \nabla^L_H \bar{r_L}\rangle_L.$$

To compute h_{21}, using the definition of the connection, we obtain

$$\nabla^L_{F_2} F_1 = \nabla^L_{\bar{r_L}\bar{p}E_1 - \bar{r_L}\bar{q}E_2 + \frac{l}{l_L}\widetilde{E_3}} \bar{q}E_1 - \bar{p}E_2.$$

We get

$$\langle \nabla^L_{F_2} F_1, N_L\rangle_L = -\frac{l_L}{l}\langle F_1, \nabla_H \bar{r_L}\rangle_L - \bar{r_L}^2\sqrt{L}.$$

Therefore,

$$h_{21} = \frac{l_L}{l}\langle F_1, \nabla_H \bar{r_L}\rangle_L + \bar{r_L}^2\sqrt{L}.$$

Since $\langle \nabla_{F_2} N_L, F_2 \rangle_L = -\langle \nabla_{F_2} F_2, N_L \rangle_L$, we use the definition of connection, the identities in (2.4) and grouping terms. Taking the inner product with N_L and under some simplifications similar to Theorem 4.3 in [23], we have

$$\langle \nabla^L_{F_2} F_2, N_L \rangle_L = -\frac{l^2}{l_L^2}\langle F_2, \nabla_H(\frac{r}{l})\rangle_L + \frac{l}{l_L}\bar{r_L}\widetilde{E_3}(\frac{l}{l_L}) - (\frac{l}{l_L})^2\widetilde{E_3}(\bar{r_L}) + \alpha\bar{p}_L,$$

and then we get

$$\begin{aligned} h_{22} &= -\langle \nabla_{F_2} F_2, N_L \rangle_L \\ &= \frac{l^2}{l_L^2}\langle F_2, \nabla_H(\frac{r}{l})\rangle_L - \frac{l}{l_L}\bar{r_L}\widetilde{E_3}(\frac{l}{l_L}) + (\frac{l}{l_L})^2\widetilde{E_3}(\bar{r_L}) - \alpha\bar{p}_L. \end{aligned}$$

□

We define the mean curvature $\mathcal{H}_L$ of S by

$$\mathcal{H}_L := tr(II^L) = h_{11} + h_{22}.$$

Let

$$\mathcal{K}^{S,L}(F_1, F_2) = -\langle R^{S,L}(F_1, F_2) F_1, F_2 \rangle_{S,L},\ \mathcal{K}^L(F_1, F_2) = -\langle R^L(F_1, F_2) F_1, F_2 \rangle_L.$$

By the Gauss equation, we have

$$\mathcal{K}^{S,L}(F_1, F_2) = \mathcal{K}^L(F_1, F_2) + det(II^L). \tag{5.1}$$

Proposition 5.2. *The horizontal mean curvature $\mathcal{H}_\infty$ of $S \subset S_\alpha$ away from characteristic point is given in the following form:*

$$\mathcal{H}_\infty = \lim_{L\to+\infty} \mathcal{H}_L = E_1(\bar{p}) - E_2(\bar{q}). \tag{5.2}$$

Proof. By

$$\begin{aligned} \frac{l_L}{l}\langle F_2, \nabla_H \bar{r_L}\rangle &= \bar{r_L}\bar{p}E_1(\bar{r_L}) - \bar{r_L}\bar{q}E_2(\bar{r_L}) \\ &= \frac{\bar{p}r}{l}E_1(\bar{r_L}) - \frac{\bar{q}r}{l}E_2(\bar{r_L}) \\ &\sim O(L^{-\frac{1}{2}}), \end{aligned}$$

$$\widetilde{E_3}(\bar{r_L}) \to 0, \bar{p}_L \to \bar{p},$$

$$\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] \to E_1(\bar{p}) - E_2(\bar{q}),$$

we get (5.2). □

Proposition 5.3. *Away from characteristic points, we have*

$$\mathcal{K}^{S,L}(F_1, F_2) \longrightarrow A + O(L^{-1}) \text{ as } L \to +\infty, \tag{5.3}$$

where

$$A := -\alpha^2(\frac{l}{l_L})^2 - \frac{l}{l_L}\alpha\bar{p}_L[E_1(\bar{p}) - E_2(\bar{q})] - \alpha^2\bar{p}_L{}^2. \tag{5.4}$$

Proof. We compute

$$R^L(F_1,F_2)F_1 = \alpha^2\bar{r}_L\bar{p}E_1 - \alpha^2\bar{r}_L\bar{q}E_2 + \alpha^2\frac{l}{l_L}\widetilde{E_3},$$

and then

$$\langle R^L(F_1,F_2)F_1,F_2\rangle_L = -\alpha^2\bar{r}_L{}^2\bar{p}^2 + \alpha^2\bar{r}_L{}^2\bar{q}^2 + \alpha^2(\frac{l}{l_L})^2. \tag{5.5}$$

By Proposition 2.2, we find

$$\begin{aligned}\mathcal{K}^L(F_1,F_2) &= \alpha^2\bar{r}_L{}^2\bar{p}^2 - \alpha^2\bar{r}_L{}^2\bar{q}^2 - \alpha^2(\frac{l}{l_L})^2\\ &= -\alpha^2(\frac{l}{l_L})^2 \text{ as } L\to\infty.\end{aligned} \tag{5.6}$$

By the second fundamental form and $\nabla_H(\bar{r}_L) = L^{-\frac{1}{2}}\nabla_H\left(\frac{E_3h}{|\nabla_H h|}\right) + O\left(L^{-1}\right)$ as $L\to+\infty$, we get

$$\begin{aligned}\det\left(II^L\right) &= h_{11}h_{22} - h_{12}h_{21} \\ &= \{\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] + \alpha\bar{p}_L\}\{\frac{l^2}{l_L^2}\langle F_2,\nabla_H(\frac{r}{l})\rangle_L - \frac{l}{l_L}\widetilde{E_3}(\frac{l}{l_L}) + (\frac{l}{l_L})^2\widetilde{E_3}(\bar{r}_L) - \alpha\bar{p}_L\}\\ &\quad - \frac{l_L}{l}\langle F_1,\nabla_H\bar{r}_L\rangle_L\{\frac{l_L}{l}\langle F_1,\nabla_H\bar{r}_L\rangle_L + \bar{r}_L{}^2\sqrt{L}\}\\ &\sim -\frac{l}{l_L}\alpha\bar{p}_L[E_1(\bar{p}) - E_2(\bar{q})] - \alpha^2\bar{p}_L{}^2 \text{ as } L\to+\infty.\end{aligned} \tag{5.7}$$

By (5.1), (5.6) and (5.7), we get the desired equation. □

We get a Gauss-Bonnet theorem for Lorentzian surface in S^L_α as follows.

Theorem 5.4. *Let $S\subset S^L_\alpha$ be a regular Lorentzian surface with finitely many boundary components $(\partial S)_i, i\in\{1,\cdots,n\}$, given by Euclidean C^2-smooth regular and closed spacelike curves $\beta_i:[0,2\pi]\to(\partial S)_i$. Let A be Gaussian curvature of Σ in Proposition 5.3 and $\kappa^{\infty,c}_{\beta_i,S}$ the sub-Lorentzian signed geodesic curvature of β_i relative to Σ in Proposition 4.6. Supposing that the characteristic set $C(S)$ be the empty set, $d\sigma_S$ is defined by (8.5), and ds is defined by (8.1) in Appendix A. Then,*

$$\int_S A\,d\sigma_S + \sum_{i=1}^n\int_{\beta_i}\kappa^{\infty,c}_{\beta_i,S}ds = 0.$$

Proof. By the discussions in [15], suppose that all points satisfy $\theta(\dot{\beta(t)})\neq 0$ on β_i. Therefore, using Proposition 4.6, we obtain

$$\kappa^{L,c}_{\beta_i,S} = \kappa^{\infty,c}_{\beta_i,S} + O\left(L^{-1}\right). \tag{5.8}$$

According to the Gauss-Bonnet theorem (see [4], page 90 Theorem 1.4), we get

$$\int_S \mathcal{K}^{S,L}\frac{1}{\sqrt{L}}d\sigma_{S,L} + \sum_{i=1}^n\int_{\beta_i}\kappa^{L,c}_{\beta_i,S}\frac{1}{\sqrt{L}}ds_L = 0. \tag{5.9}$$

Therefore, by (5.8), (5.9), (8.6), (5.3) and (8.4), we get

$$\left(\int_S A\,d\sigma_S + \sum_{i=1}^n\int_{\beta_i}\kappa^{\infty,c}_{\beta_i,S}ds\right) + O\left(L^{-\frac{1}{2}}\right) = 0. \tag{5.10}$$

Let L go to infinity and use the dominated convergence theorem, and we get the desired result. □

6. Spacelike surface and a Gauss-Bonet theorem in S^L_α

In this section, we will prove a Gauss-Bonet theorem for spacelike surface in S^L_α. Let

$$p := E_1 h, q := E_2 h, \text{ and } r := \widetilde{E_3} h.$$

Let $p^2 - q^2 > 0$, when $L \to +\infty$, and we have $p^2 - q^2 - r^2 > 0$. We define

$$\begin{aligned} l &:= \sqrt{p^2 - q^2}, l_L := \sqrt{p^2 - q^2 - r^2}, \bar{p} := \frac{p}{l}, \\ \bar{q} &:= \frac{q}{l}, \bar{p}_L := \frac{p}{l_L}, \bar{q}_L := \frac{q}{l_L}, \bar{r}_L := \frac{r}{l_L}. \end{aligned} \tag{6.1}$$

In particular, $\bar{p}^2 - \bar{q}^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$N_L = -\bar{p}_L E_1 + \bar{q}_L E_2 + \bar{r}_L \widetilde{E_3}, F_1 = \bar{q} E_1 - \bar{p} E_2, F_2 = \bar{r}_L \bar{p} E_1 - \bar{r}_L \bar{q} E_2 + \frac{l}{l_L} \widetilde{E_3}, \tag{6.2}$$

and then N_L is the unit timelike normal vector to S, F_1 and F_2 are the unit spacelike vector of S. $\{F_1, F_2\}$ is the orthonormal basis of S. We call S a spacelike surface in Lorentzian α-Sasakian space. We define a linear transformation on TS by $J_L : TS \to TS$, and the transformation is well defined:

$$J_L(F_1) = F_2, J_L(F_2) = -F_1. \tag{6.3}$$

For every $U, V \in TS$, we define $\nabla^{S,L}_U V = \pi \nabla^L_U V$ where $\pi : TS^L_\alpha \to TS$ is the projection. Then, $\nabla^{S,L}$ is the Levi-Civita connection on S with respect to the metric g_L. By (3.9), (6.2) and

$$\nabla^{S,L}_{\dot{\beta}} \dot{\beta} = \langle \nabla^L_{\dot{\beta}} \dot{\beta}, F_1 \rangle_L F_1 + \langle \nabla^L_{\dot{\beta}} \dot{\beta}, F_2 \rangle_L F_2, \tag{6.4}$$

we have

$$\begin{aligned} \nabla^{S,L}_{\dot{\beta}} \dot{\beta} =& \{-\bar{q}[\frac{1}{\alpha} \ddot{\beta}_3 + \alpha e^{2\beta_3} \dot{\beta}_1^2 + \alpha L(\theta(\dot{\beta}(t)))^2] - \bar{p}[2\dot{\beta}_3 \dot{\beta}_1 e^{\beta_3} + e^{\beta_3} \ddot{\beta}_1]\} F_1 \\ &+ \{-\overline{r_L} \bar{p}[\frac{1}{\alpha} \ddot{\beta}_3 + \alpha e^{2\beta_3} \dot{\beta}_1^2 + \alpha L(\theta(\dot{\beta}(t)))^2] - \overline{r_L} \bar{q}[2\dot{\beta}_3 \dot{\beta}_1 e^{\beta_3} + e^{\beta_3} \ddot{\beta}_1] \\ &+ \frac{l}{l_L} L^{\frac{1}{2}} [\dot{\beta}_3 \theta(\dot{\beta}(t)) + \frac{d}{dt}(\theta(\dot{\beta}(t)))]\} F_2. \end{aligned} \tag{6.5}$$

Therefore, when $\theta(\dot{\beta}(t)) = 0$, we have

$$\begin{aligned} \nabla^{S,L}_{\dot{\beta}} \dot{\beta} =& \{-\bar{q}[\frac{1}{\alpha} \ddot{\beta}_3 + \alpha e^{2\beta_3} \dot{\beta}_1^2] - \bar{p}[2\dot{\beta}_3 \dot{\beta}_1 e^{\beta_3} + e^{\beta_3} \ddot{\beta}_1]\} F_1 \\ &+ \{-\overline{r_L} \bar{p}[\frac{1}{\alpha} \ddot{\beta}_3 + \alpha e^{2\beta_3} \dot{\beta}_1^2] - \overline{r_L} \bar{q}[2\dot{\beta}_3 \dot{\beta}_1 e^{\beta_3} + e^{\beta_3} \ddot{\beta}_1] \\ &+ \frac{l}{l_L} L^{\frac{1}{2}} \frac{d}{dt}(\theta(\dot{\beta}(t)))\} F_2. \end{aligned} \tag{6.6}$$

Definition 6.1. *Let $S \subset S^L_\alpha$ be a regular spacelike surface, $\beta : I \to S$ be a C^2-smooth regular curve. We define the geodesic curvature $\kappa^L_{\beta,S}$ of β at $\beta(t)$ by*

$$\kappa^L_{\beta,S} := \sqrt{\frac{||\nabla^{S,L}_{\dot{\beta}} \dot{\beta}||^2_{S,L}}{||\dot{\beta}||^4_{S,L}} - \frac{\langle \nabla^{S,L}_{\dot{\beta}} \dot{\beta}, \dot{\beta} \rangle^2_{S,L}}{\langle \dot{\beta}, \dot{\beta} \rangle^3_{S,L}}}. \tag{6.7}$$

Definition 6.2. *Let $S \subset S^L_\alpha$ be a regular spacelike surface, $\beta : I \to S$ be a C^2-smooth regular curve. The intrinsic geodesic curvature $\kappa^\infty_{\beta,S}$ of β at $\beta(t)$ is defined as*

$$\kappa^\infty_{\beta,S} := \lim_{L\to+\infty} \kappa^L_{\beta,S},$$

if the limit exists.

Proposition 6.3. *Let $S \subset S^L_\alpha$ be a regular spacelike surface, $\beta : I \to S$ be a C^2-smooth spacelike curve, and then we have the following assertions:*

$$\kappa^\infty_{\beta,S} =| \alpha\bar{q} |, \text{ if } \theta(\dot\beta(t)) \neq 0, \tag{6.8}$$

$$\kappa^\infty_{\beta,S} = 0, \text{ if } \theta(\dot\beta(t)) = 0 \text{ and } \frac{d}{dt}(\theta(\dot\beta(t))) = 0,$$

$$\lim_{L\to+\infty} \frac{\kappa^L_{\beta,S}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\theta(\dot\beta(t)))|}{(\frac{1}{\alpha}\bar{q}\dot\beta_3 + e^{\beta_3}\bar{p}\dot\beta_1)^2}, \text{ if } \theta(\dot\beta(t)) = 0 \text{ and } \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{6.9}$$

Proof. By (3.7) and $\dot\beta \in TS$, we have

$$\dot\beta(t) = -(\frac{1}{\alpha}\dot\beta_3\bar{q} + e^{\beta_3}\dot\beta_1\bar{p})F_1 + \frac{l_L}{l}L^{\frac{1}{2}}\theta(\dot\beta(t))F_2. \tag{6.10}$$

By (6.5), we have

$$\begin{aligned}\langle\nabla^{S,L}_{\dot\beta}\dot\beta, \nabla^{S,L}_{\dot\beta}\dot\beta\rangle_{S,L} =& \{-\bar{q}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \bar{p}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\ &+ \{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \overline{r_L}\bar{q}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1] \\ &+ \frac{l}{l_L}L^{\frac{1}{2}}[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\}^2.\end{aligned} \tag{6.11}$$

Similarly, if $\theta(\dot\beta(t)) \neq 0$,

$$\begin{aligned}\langle\dot\beta, \dot\beta\rangle_{S,L} &= (\frac{1}{\alpha}\dot\beta_3\bar{q} + e^{\beta_3}\dot\beta_1\bar{p})^2 + (\frac{l_L}{l})^2 L(\theta(\dot\beta(t)))^2 \\ &\sim L(\theta(\dot\beta(t)))^2 \text{ as } L \to +\infty.\end{aligned} \tag{6.12}$$

By (6.5) and (6.10), we have

$$\langle\nabla^{S,L}_{\dot\beta}\dot\beta, \dot\beta\rangle_{S,L} \sim M_0 L, \tag{6.13}$$

where M_0 does not depend on L. By (6.7) and (6.11)-(6.13), (6.8) holds. When $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$,

$$\begin{aligned}\langle\nabla^{S,L}_{\dot\beta}\dot\beta, \nabla^{S,L}_{\dot\beta}\dot\beta\rangle_{S,L} &= \{-\bar{q}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] - \bar{p}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\ &+ \{-\overline{r_L}\bar{p}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] - \overline{r_L}\bar{q}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \\ &\sim \{-\bar{q}[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] - \bar{p}[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}^2 \text{ as } L \to +\infty\end{aligned} \tag{6.14}$$

and

$$\langle\dot\beta,\dot\beta\rangle_{S,L} = (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)^2 \text{ as } L \to +\infty. \tag{6.15}$$

$$\langle\nabla^{S,L}_{\dot\beta}\dot\beta,\dot\beta\rangle_{S,L} = -(\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)\{-\bar q[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2] - \bar p[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}. \tag{6.16}$$

By (6.14)-(6.16) and (6.7), we get

$$\kappa^\infty_{\beta,S} = 0.$$

When $\theta(\dot\beta(t)) = 0$, and $\frac{d}{dt}(\theta(\dot\beta(t))) \neq 0$, we have

$$\langle\nabla^{S,L}_{\dot\beta}\dot\beta, \nabla^{S,L}_{\dot\beta}\dot\beta\rangle_{S,L} \sim L[\frac{d}{dt}(\theta(\dot\beta(t)))]^2,$$

$$\langle\nabla^{S,L}_{\dot\beta}\dot\beta,\dot\beta\rangle_{S,L} \sim O(1).$$

Therefore, (6.9) holds. □

Proposition 6.4. *Let $S \subset S^L_\alpha$ be a regular spacelike surface. $\beta : I \to S$ is a C^2-smooth regular spacelike curve, and then*

$$\kappa^{\infty,c}_{\beta,S} = \alpha\bar q, \textit{ if } \theta(\dot\beta(t)) = 0; \tag{6.17}$$

$$\kappa^{\infty,c}_{\beta,S} = 0, \textit{ if } \theta(\dot\beta(t)) = 0 \textit{ and } \frac{d}{dt}(\theta(\dot\beta(t))) = 0;$$

$$\lim_{L\to+\infty}\frac{k^{L,c}_{\beta,S}}{\sqrt{L}} = \frac{-\frac{d}{dt}(\theta(\dot\beta(t)))}{(\frac{1}{\alpha}\bar q\dot\beta_3 + e^{\beta_3}\bar p\dot\beta_1)^2}, \textit{ if } \theta(\dot\beta(t)) = 0 \textit{ and } \frac{d}{dt}(\theta(\dot\beta(t))) \neq 0. \tag{6.18}$$

Proof. By (6.3) and (6.10), we have

$$J_L(\dot\beta) = -\frac{l_L}{l}L^{\frac12}\theta(\dot\beta(t))F_1 - (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)F_2. \tag{6.19}$$

By (6.5) and (6.19), we have

$$\begin{aligned}\langle\nabla^{S,L}_{\dot\beta}\dot\beta, J_L(\dot\beta)\rangle_{S,L} =& -\frac{l_L}{l}L^{\frac12}\theta(\dot\beta(t))\{-\bar q[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2] - \bar p[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}\\ &- (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)\{-\overline{r_L}\bar p[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1^2 + \alpha L(\theta(\dot\beta(t)))^2]\\ &- \overline{r_L}\bar q[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1] + \frac{l}{l_L}L^{\frac12}[\dot\beta_3\theta(\dot\beta(t)) + \frac{d}{dt}(\theta(\dot\beta(t)))]\}\\ \sim& \alpha L^{\frac32}(\theta(\dot\beta(t)))^3\bar q \text{ as } L \to +\infty,\end{aligned}$$

$$\begin{aligned}\|\dot\beta\|^2_{S,L} &= (\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)^2 + [\frac{l_L}{l}L^{\frac12}\theta(\dot\beta(t))]^2\\ &\sim L(\theta(\dot\beta(t)))^2 \text{ as } L \to +\infty.\end{aligned}$$

Therefore, if $\theta(\dot\beta(t)) \neq 0$, (6.17) holds. When $\theta(\dot\beta(t)) = 0$ and $\frac{d}{dt}(\theta(\dot\beta(t))) = 0$, we get

$$\begin{aligned}\langle\nabla^{S,L}_{\dot\beta}\dot\beta, J_L(\dot\beta)\rangle_{L,S} &= -(\frac{1}{\alpha}\dot\beta_3\bar q + e^{\beta_3}\dot\beta_1\bar p)\{-\overline{r_L}\bar p[\frac{1}{\alpha}\ddot\beta_3 + \alpha e^{2\beta_3}\dot\beta_1]\\ &\quad - \overline{r_L}\bar q[2\dot\beta_3\dot\beta_1 e^{\beta_3} + e^{\beta_3}\ddot\beta_1]\}\\ &\sim O(L^{-\frac12}) \text{ as } L \to +\infty.\end{aligned}$$

So, $\kappa_{\beta,S}^{\infty,c} = 0$. When $\theta(\dot{\beta}(t)) = 0$ and $\frac{d}{dt}(\theta(\dot{\beta}(t))) \neq 0$, we have

$$\langle \nabla_{\dot{\beta}}^{S,L}\dot{\beta}, J_L(\dot{\beta})\rangle_{L,S} \sim -L^{\frac{1}{2}}(\frac{1}{\alpha}\dot{\beta}_3\bar{q} + e^{\beta_3}\dot{\beta}_1\bar{p})\frac{d}{dt}(\theta(\dot{\beta}(t))) \text{ as } L \to +\infty.$$

We get

$$\lim_{L\to+\infty} \frac{k_{\beta,S}^{L,c}}{\sqrt{L}} = \frac{-\frac{d}{dt}(\theta(\dot{\beta}(t)))}{(\frac{1}{\alpha}\bar{q}\dot{\beta}_3 + e^{\beta_3}\bar{p}\dot{\beta}_1)^2}.$$

□

In the following, we investigate the sub-Lorentzian limit of the Gaussian curvature of spacelike surfaces in S_α^L. The second fundamental form II^L of the embedding of S into S_α^L is defined as

$$II^L = \begin{pmatrix} \langle \nabla_{F_1}^L N_L, F_1\rangle_L & \langle \nabla_{F_1}^L N_L, F_2\rangle_L \\ \langle \nabla_{F_2}^L N_L, F_1\rangle_L & \langle \nabla_{F_2}^L N_L, F_2\rangle_L \end{pmatrix}.$$

Similar to Theorem 4.3 in [23], we have the following.

Theorem 6.5. *For the embedding of S into S_α^L, the second fundamental form II^L of the embedding of S is given by*

$$II^L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

where

$$h_{11} = -\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] - \alpha\bar{p}_L,$$

$$h_{12} = (\frac{l_L}{l} - \frac{2l}{l_L})\langle F_1, \nabla_H \bar{r}_L\rangle_L,$$

$$h_{21} = (\frac{2l}{l_L} - \frac{l_L}{l})\langle F_1, \nabla_H \bar{r}_L\rangle_L - \bar{r}_L{}^2\sqrt{L} - 2\alpha\bar{r}_L\bar{q}_L,$$

$$h_{22} = (\frac{l_L}{l} - \frac{2l}{l_L})\langle F_2, \nabla_H \bar{r}_L\rangle_L - (\frac{l}{l_L})^2\widetilde{E_3}(\frac{r}{l}) - \frac{2l}{l_L}\bar{r}_L\widetilde{E_3}(\frac{l}{l_L}) + \alpha\bar{p}_L.$$

Proof. We combine

$$N_L = -\bar{p}_L E_1 + \bar{q}_L E_2 + \bar{r}_L\widetilde{E_3}, F_1 = \bar{q}E_1 - \bar{p}E_2, F_2 = \bar{r}_L\bar{p}E_1 - \bar{r}_L\bar{q}E_2 + \frac{l}{l_L}\widetilde{E_3}, \tag{6.20}$$

and $\langle \nabla_{F_i}^L N_L, F_j\rangle_L = -\langle \nabla_{F_i}^L F_j, N_L\rangle_L, i, j = 1, 2$. By direct calculation, we obtain

$$\nabla_{F_1}^L F_1 = [\bar{q}E_1(\bar{q}) - \bar{p}E_2(\bar{q}) - \alpha\bar{p}^2]E_1 - [\bar{q}E_1(\bar{p}) - \bar{p}E_2(\bar{p}) + \alpha\bar{p}\bar{q}]E_2.$$

Since $\bar{p}^2 - \bar{q}^2 = 1$, we have $\bar{p}E_i\bar{p} - \bar{q}E_i\bar{q} = 0, i = 1, 2$. Then, $\bar{q}E_1\bar{q} = \bar{p}E_1\bar{p}$, $\bar{q}E_2\bar{q} = \bar{p}E_2\bar{p}$. Next, we compute the inner product of this with N_L, and we have

$$\langle \nabla_{F_1}^L F_1, N_L\rangle = \frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] + \alpha\bar{p}_L.$$

We obtain

$$h_{11} = -\langle \nabla^L_{F_1} F_1, N_L \rangle_L = -\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] - \alpha \bar{p}_L.$$

Similarly, we have

$$h_{12} = -\langle \nabla^L_{F_1} F_2, N_L \rangle_L = (\frac{l_L}{l} - \frac{2l}{l_L})\langle F_1, \nabla_H \bar{r_L} \rangle_L,$$

$$h_{21} = -\langle \nabla^L_{F_2} F_1, N_L \rangle_L = (\frac{2l}{l_L} - \frac{l_L}{l})\langle F_1, \nabla_H \bar{r_L} \rangle_L - \bar{r_L}^2 \sqrt{L} - 2\alpha \bar{r_L} \bar{q_L},$$

$$h_{22} = -\langle \nabla^L_{F_2} F_2, N_L \rangle_L = (\frac{l_L}{l} - \frac{2l}{l_L})\langle F_2, \nabla^L_H \bar{r_L} \rangle_L - (\frac{l}{l_L})^2 \widetilde{E_3}(\frac{r}{l}) - \frac{2l}{l_L} \bar{r_L} \widetilde{E_3}(\frac{l}{l_L}) + \alpha \bar{p}_L.$$

Thus, Theorem 6.5 holds. □

By the Gauss equation, we have

$$\mathcal{K}^{S,L}(F_1, F_2) = \mathcal{K}^L(F_1, F_2) - det(II^L). \tag{6.21}$$

Proposition 6.6. *The horizontal mean curvature $\mathcal{H}_\infty$ of $S \subset S_\alpha$ away from characteristic point is given in the following form:*

$$\mathcal{H}_\infty = \lim_{L \to +\infty} \mathcal{H}_L = -E_1(\bar{p}) + E_2(\bar{q}). \tag{6.22}$$

Proof. By

$$\begin{aligned}(\frac{l_L}{l} - \frac{2l}{l_L})\langle F_2, \nabla_H \bar{r_L} \rangle &= \bar{r_L}\bar{p}E_1(\bar{r_L}) - \bar{r_L}\bar{q}E_2(\bar{r_L}) \\ &= \frac{\bar{p}r}{l}E_1(\bar{r_L}) - \frac{\bar{q}r}{l}E_2(\bar{r_L}) \\ &\sim O(L^{-\frac{1}{2}}),\end{aligned}$$

$$\widetilde{E_3}(\bar{r_L}) \to 0, \bar{p}_L \to \bar{p},$$

$$\frac{l}{l_L}[E_1(\bar{p}) - E_2(\bar{q})] \to E_1(\bar{p}) - E_2(\bar{q}),$$

we get (6.22). □

Proposition 6.7. *Away from characteristic points, we have*

$$\mathcal{K}^{S,L}(F_1, F_2) \longrightarrow C + O(L^{-1}) \text{ as } L \to +\infty, \tag{6.23}$$

where

$$C := \alpha^2(\frac{l}{l_L})^2 + \frac{l}{l_L}\alpha \bar{p}_L[E_1(\bar{p}) - E_2(\bar{q})] + \alpha^2 \bar{p_L}^2. \tag{6.24}$$

Proof. We compute

$$R^L(F_1,F_2)F_1 = -\alpha^2\bar{r}_L\bar{p}E_1 + \alpha^2\bar{r}_L\bar{q}E_2 - \alpha^2\frac{l}{l_L}\widetilde{E_3},$$

and then

$$\langle R^L(F_1,F_2)F_1,F_2\rangle_L = \alpha^2\bar{r}_L{}^2\bar{p}^2 - \alpha^2\bar{r}_L{}^2\bar{q}^2 - \alpha^2(\frac{l}{l_L})^2. \tag{6.25}$$

So,

$$\begin{aligned}\mathcal{K}^L(F_1,F_2) &= -\langle R^L(F_1,F_2)F_1,F_2\rangle_L \\ &= -\alpha^2\bar{r}_L{}^2\bar{p}^2 + \alpha^2\bar{r}_L{}^2\bar{q}^2 + \alpha^2(\frac{l}{l_L})^2 \text{ as } L\to+\infty.\end{aligned} \tag{6.26}$$

By the second fundamental form and $\nabla_H(\bar{r}_L) = L^{-\frac{1}{2}}\nabla_H\left(\frac{E_3h}{|\nabla_H h|}\right) + O\left(L^{-1}\right)$ as $L\to+\infty$, we get

$$\begin{aligned}\det\left(II^L\right) &= h_{11}h_{22} - h_{12}h_{21} \\ &\sim -\frac{l}{l_L}\alpha\bar{p}_L[E_1(\bar{p}) - E_2(\bar{q})] - \alpha^2\bar{p}_L{}^2 \text{ as } L\to+\infty.\end{aligned} \tag{6.27}$$

By (6.21), (6.26) and (6.27), we get the desired equation. □

Theorem 6.8. *Let $S\subset S^L_\alpha$ be a regular spacelike surface with finitely many boundary components $(\partial S)_i, i\in\{1,\cdots,n\}$, given by Euclidean C^2-smooth regular and closed spacelike curves $\beta_i:[0,2\pi]\to(\partial S)_i$. Suppose that C is defined by (6.24), $d\sigma_S$ is defined by (8.8) and $\kappa^{\infty,c}_{\beta_i,S}$ is the sub-Lorentzian signed geodesic curvature of β_i relative to S. If the characteristic set $C(S)$ is the empty set, then*

$$\int_S C d\sigma_S + \sum_{i=1}^n\int_{\beta_i}\kappa^{\infty,c}_{\beta_i,S}ds = 0.$$

Proof. By the discussions in [15], we may assume that there are no points satisfying $\theta(\dot{\beta(t)}) = 0$ and $\frac{d}{dt}(\theta(\dot{\beta(t)}))\neq 0$ on β_i. Therefore, using Proposition 6.3, we obtain

$$\kappa^{L,c}_{\beta_i,S} = \kappa^{\infty,c}_{\beta_i,S} + O\left(L^{-1}\right). \tag{6.28}$$

According to the Gauss-Bonnet theorem , we get

$$\int_S\mathcal{K}^{S,L}\frac{1}{\sqrt{L}}d\sigma_{S,L} + \sum_{i=1}^n\int_{\beta_i}\kappa^{L,c}_{\beta_i,S}\frac{1}{\sqrt{L}}ds_L = 2\pi\frac{\mathcal{X}(S)}{\sqrt{L}}. \tag{6.29}$$

Therefore, by (6.28), (6.29), (8.9), (6.23), (8.3) and (8.4), we get

$$\left(\int_S C d\sigma_S + \sum_{i=1}^n\int_{\beta_i}\kappa^{\infty,c}_{\beta_i,S}d\bar{s}\right) + O\left(L^{-\frac{1}{2}}\right) = 2\pi\frac{\mathcal{X}(S)}{\sqrt{L}}. \tag{6.30}$$

Let L go to infinity and use the dominated convergence theorem, and we get the desired result. □

7. Conclusions

This paper proved two Gauss-Bonnet theorems for the Lorentzian surfaces and spacelike surfaces in a Lorentzian α-Sasakian manifold by using the method of the Lorentzian approximation scheme. For Lorentzian surfaces, we derive the expressions of the intrinsic curvature for regular curves, the intrinsic geodesic curvature of regular curves on Lorentzian surfaces and the intrinsic Gaussian curvature of Lorentzian surfaces away from characteristic points in the Lorentzian α-Sasakian manifold. Similarly, we get the corresponding results for the spacelike surface.

Appendix

To prove Gauss-Bonnet theorems, we need to define the Lorentzian length measure and the Lorentzian surface measure. Let us first consider the case of a regular spacelike curve $\beta : I \to S^L_\alpha$, and we define the length measure $ds_L = ||\dot{\beta}||_L dt$.

Lemma 7.1. *Let $\beta : I \to S^L_\alpha$ be a C^2-smooth spacelike curve. Let*

$$ds := |\theta(\dot{\beta}(t))|dt,\ d\bar{s} := \frac{1}{2}\frac{1}{|\theta(\dot{\beta}(t))|}(-\frac{1}{\alpha^2}\dot{\beta}_3^2 + e^{2\beta_3}\dot{\beta}_1^2)dt. \tag{7.1}$$

Then,

$$\lim_{L\to+\infty}\frac{1}{\sqrt{L}}\int_\beta ds_L = \int_a^b ds. \tag{7.2}$$

When $\theta(\dot{\beta}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L = ds + d\bar{s}L^{-1} + O\left(L^{-2}\right)\ as\ L \to +\infty. \tag{7.3}$$

With the situation of $\theta(\dot{\beta}(t)) = 0$, we have

$$\frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\frac{1}{\alpha^2}\dot{\beta}_3^2 + e^{2\beta_3}\dot{\beta}_1^2}dt. \tag{7.4}$$

Proof. We know that

$$||\dot{\beta}(t)||_L = \sqrt{-\frac{1}{\alpha^2}\dot{\beta}_3^2 + e^{2\beta_3}\dot{\beta}_1^2 + L(\theta(\dot{\beta}(t)))},$$

and similar to the proof of Lemma 6.1 in [15], we can prove

$$\begin{aligned}\lim_{L\to+\infty}\frac{1}{\sqrt{L}}\int_\beta ds_L &= \int_a^b \lim_{L\to+\infty}\frac{1}{\sqrt{L}}||\dot{\beta}(t)||_L dt\\ &= \int_a^b \lim_{L\to+\infty}\frac{1}{\sqrt{L}}\sqrt{-\frac{1}{\alpha^2}\dot{\beta}_3^2 + e^{2\beta_3}\dot{\beta}_1^2 + L(\theta(\dot{\beta}(t)))}dt\\ &= \int_a^b ds,\end{aligned}$$

so we get (8.2). When $\theta(\dot\beta(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L = \sqrt{L^{-1}(-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2) + \theta(\dot\beta(t))}dt.$$

Using the Taylor expansion, we can prove

$$\frac{1}{\sqrt{L}}ds_L = ds + d\bar{s}L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty.$$

From the definition of ds_L and $\theta(\dot\beta(t)) = 0$, we get

$$\frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\frac{1}{\alpha^2}\dot\beta_3^2 + e^{2\beta_3}\dot\beta_1^2}dt.$$

□

Proposition 7.2. *Let $S \subset S_\alpha^L$ be a regular Lorentzian C^2-smooth surface. Let $d\sigma_{S,L}$ denote the surface measure on S with respect to the Lorentzian metric g_L. Let*

$$d\sigma_S := -(\bar p\theta_2 - \bar q\theta_1) \wedge \theta,\ d\bar\sigma_S := -\frac{E_3h}{l}\theta_1 \wedge \theta_2 + \frac{(E_3h)^2}{2l^2}(\bar p\theta_2 - \bar q\theta_1) \wedge \theta. \tag{7.5}$$

Then,

$$\frac{1}{\sqrt{L}}d\sigma_{S,L} = d\sigma_S + d\bar\sigma_S L^{-1} + O(L^{-2}),\ \text{as } L \to +\infty. \tag{7.6}$$

If $S = f(D)$ with $f = f(h_1, h_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to S_\alpha^L$, then

$$\begin{aligned}\lim_{L\to+\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_{S,L} = \int_D &\{-e^{4z}[(f_1)_{h_1}(f_2)_{h_2} - (f_1)_{h_1}(f_1)_{h_2} - (f_1)_{h_2}(f_2)_{h_1} + (f_1)_{h_2}(f_1)_{h_2}]^2 \\ &- \frac{1}{\alpha^2}e^{2z}[(f_3)_{h_1}(f_2)_{h_2} - (f_3)_{h_1}(f_1)_{h_2} - (f_3)_{h_2}(f_2)_{h_1} + (f_3)_{h_2}(f_1)_{h_1}]^2\}^{\frac{1}{2}}dh_1dh_2.\end{aligned}$$

Proof. It is well known that

$$g_L(E_1, \cdot) = -\theta_1,\ g_L(E_2, \cdot) = \theta_2,\ g_L(E_3, \cdot) = L\theta.$$

We define ${F_1}^* := g_L(F_1, \cdot)$, ${F_2}^* := g_L(F_2, \cdot)$, and then

$$F_1^* = -\bar q\theta_1 - \bar p\theta_2,\ F_2^* = -\bar r_L\bar p\theta_1 - \bar r_L\bar q\theta_2 + \frac{l}{l_L}L^{\frac{1}{2}}\theta.$$

Therefore,

$$\frac{1}{\sqrt{L}}d\sigma_{S,L} = \frac{1}{\sqrt{L}}F_1^* \wedge F_2^* = -\frac{l}{l_L}(\bar p\theta_2 + \bar q\theta_1) \wedge \theta + \frac{1}{\sqrt{L}}\bar r_L\theta_1 \wedge \theta_2.$$

Recall

$$\bar r_L = \frac{(E_3h)\,L^{-\frac{1}{2}}}{\sqrt{-p^2 + q^2 + L^{-1}(E_3h)^2}}$$

and the Taylor expansion

$$\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3}(E_3 h)^2 L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty,$$

and we get (8.6). By (2.2), we have

$$\begin{aligned} f_{h_1} &= (f_1)_{h_1}\partial x + (f_2)_{h_1}\partial y + (f_3)_{h_1}\partial z \\ &= (f_1)_{h_1}[e^z(E_2 - E_3)] + (f_2)_{h_1} e^z E_3 + \frac{1}{\alpha}(f_3)_{h_1} E_1 \\ &= \frac{1}{\alpha}(f_3)_{h_1} E_1 + e^z (f_1)_{h_1} E_2 + \sqrt{L}e^z(-(f_1)_{h_1} + (f_2)_{h_1})\widetilde{E_3}, \end{aligned}$$

and

$$\begin{aligned} f_{h_2} &= (f_1)_{h_2}\partial x + (f_2)_{h_2}\partial y + (f_3)_{h_2}\partial z \\ &= \frac{1}{\alpha}(f_3)_{h_2} E_1 + e^z (f_1)_{h_2} E_2 + \sqrt{L}e^z\left(-(f_1)_{h_2} + (f_2)_{h_2}\right)\widetilde{E_3}. \end{aligned}$$

Let

$$\begin{aligned} \bar{N}_L &= \begin{vmatrix} -E_1 & E_2 & \widetilde{E_3} \\ \frac{1}{\alpha}(f_3)_{h_1} & e^z(f_1)_{h_1} & \sqrt{L}e^z\left(-(f_1)_{h_1} + (f_2)_{h_1}\right) \\ \frac{1}{\alpha}(f_3)_{h_2} & e^z(f_1)_{h_2} & \sqrt{L}e^z\left(-(f_1)_{h_1} + (f_2)_{h_1}\right) \end{vmatrix} \qquad (7.7) \\ &= -\sqrt{L}e^{2z}[(f_1)_{h_1}(f_2)_{h_2} - (f_1)_{h_1}(f_1)_{h_2} - (f_1)_{h_2}(f_2)_{h_1} + (f_1)_{h_2}(f_1)_{h_1}]E_1 \\ &\quad - \frac{\sqrt{L}}{\alpha}e^z[(f_3)_{h_1}(f_2)_{h_2} - (f_3)_{h_1}(f_1)_{h_2} - (f_3)_{h_2}(f_2)_{h_1} + (f_3)_{h_2}(f_1)_{h_1}]E_2 \\ &\quad + \frac{1}{\alpha}e^z[(f_3)_{h_1}(f_1)_{h_2} - (f_1)_{h_1}(f_3)_{h_2}]\widetilde{E_3}. \end{aligned}$$

We know that $d\sigma_{S,L} = \sqrt{\det\left(g_{ij}\right)}dh_1 dh_2,\ g_{ij} = g_L\left(f_{h_i}, f_{h_j}\right)$, and

$$\begin{aligned} \det\left(g_{ij}\right) &= \langle \bar{N}_L, \bar{N}_L\rangle_L \\ &= -Le^{4z}[(f_1)_{h_1}(f_2)_{h_2} - (f_1)_{h_1}(f_1)_{h_2} - (f_1)_{h_2}(f_2)_{h_1} + (f_1)_{h_2}(f_1)_{h_1}]^2 \\ &\quad + \frac{L}{\alpha^2}e^{2z}[(f_3)_{h_1}(f_2)_{h_2} - (f_3)_{h_1}(f_1)_{h_2} - (f_3)_{h_2}(f_2)_{h_1} + (f_3)_{h_2}(f_1)_{h_1})]^2 \\ &\quad + \frac{1}{\alpha^2}e^{2z}[(f_3)_{h_1}(f_1)_{h_2} - (f_1)_{h_1}(f_3)_{h_2}]^2, \end{aligned}$$

so by the dominated convergence theorem, we get

$$\begin{aligned} \lim_{L\to+\infty}\frac{1}{\sqrt{L}}\int_S d\sigma_{S,L} &= \int_D\{-e^{4z}[(f_1)_{h_1}(f_2)_{h_2} - (f_1)_{h_1}(f_1)_{h_2}) - (f_1)_{h_2}(f_2)_{h_1} + (f_1)_{h_2}(f_1)_{h_2})]^2 \\ &\quad + \frac{1}{\alpha^2}e^{2z}[(f_3)_{h_1}(f_2)_{h_2} - (f_3)_{h_1}(f_1)_{h_2}) - (f_3)_{h_2}(f_2)_{h_1} + (f_3)_{h_2}(f_1)_{h_1}]^2\}^{\frac{1}{2}}dh_1 dh_2. \end{aligned}$$

□

Proposition 7.3. *Let $S \subset S^L_\alpha$ be a spacelike C^2-smooth surface. Let $d\sigma_{S,L}$ denote the surface measure on S with respect to the metric g_L. Suppose that*

$$d\sigma_S := (\bar{p}\theta_2 - \bar{q}\theta_1) \wedge \theta,\ d\bar{\sigma}_S := \frac{E_3 h}{l}\theta_1 \wedge \theta_2 + \frac{(E_3 h)^2}{2l^2}(\bar{p}\theta_2 - \bar{q}\theta_1) \wedge \theta. \tag{7.8}$$

Then,

$$\frac{1}{\sqrt{L}} d\sigma_{S,L} = d\sigma_S + d\bar{\sigma}_S L^{-1} + O(L^{-2}),\ as\ L \to +\infty. \tag{7.9}$$

Proof. It is well known that

$$g_L(E_1, \cdot) = -\theta_1,\ g_L(E_2, \cdot) = \theta_2,\ g_L(E_3, \cdot) = L\theta.$$

Then,

$$F_1^* = -\bar{q}\theta_1 - \bar{p}\theta_2,\ F_2^* = -\bar{r}_L \bar{p}\theta_1 - \bar{r}_L \bar{q}\theta_2 + \frac{l}{l_L} L^{\frac{1}{2}}\theta.$$

Therefore,

$$\frac{1}{\sqrt{L}} d\sigma_{S,L} = \frac{1}{\sqrt{L}} F_1^* \wedge F_2^* = -\frac{l}{l_L}(\bar{p}\theta_2 + \bar{q}\theta_1) \wedge \theta - \frac{1}{\sqrt{L}} \bar{r}_L \theta_1 \wedge \theta_2.$$

Recall

$$\bar{r}_L = \frac{(E_3 h) L^{-\frac{1}{2}}}{\sqrt{p^2 - q^2 - L^{-1}(E_3 h)^2}}$$

and the Taylor expansion

$$\frac{1}{l_L} = \frac{1}{l} + \frac{1}{2l^3}(E_3 h)^2 L^{-1} + O\left(L^{-2}\right) \text{ as } L \to +\infty,$$

and we get (8.9). □

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Conflict of interest

The authors declare that there are no conflicts of interests in this work.

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