Mathematics

AIMS Mathematics, 8(1): 484–500.
DOI: 10.3934/math.2023023
Received: 24 July 2022
Revised: 10 September 2022
Accepted: 13 September 2022
Published: 08 October 2022

http://www.aimspress.com/journal/Math

Research article

A modified global error minimization method for solving nonlinear Duffing-harmonic oscillators

Gamal M. Ismail[1,2,*], Maha M. El-Moshneb[1] and Mohra Zayed[3]

[1] Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524, Egypt
[2] Department of Mathematics, Faculty of Science, Islamic University of Madinah, 42351, Madinah, Saudi Arabia
[3] Mathematics Department, College of Science, King Khalid University, Abha, Saudi Arabia

* **Correspondence:** Email: gamalm2010@yahoo.com.

Abstract: In this paper, a third-order approximate solution of strongly nonlinear Duffing-harmonic oscillators is obtained by extending and improving an analytical technique called the global error minimization method (GEMM). We have made a comparison between our results, those obtained from the other analytical methods and the numerical solution. Consequently, we notice a better agreement with the numerical solution than other known analytical methods. The results are valid for both small and large oscillation amplitude. The obtained results demonstrate that the present method can be easily extended to strongly nonlinear problems, as indicated in the presented applications.

Keywords: global error minimization method; Duffing-harmonic oscillator; analytical and approximate solutions; periodic solution; numerical solution
Mathematics Subject Classification: 34A34, 34C15, 34C25, 35L65, 37N30

1. Introduction

Nonlinear oscillators have been widely used in various engineering and applied sciences, such as mathematics, physics, structural dynamics, mechanical engineering and other related fields of science [1–6]. Nonlinear differential equations (NDEs) can model many phenomena in various scientific aspects to present their effects and behaviors through mathematical principles. Perturbation methods are extremely beneficial when the nonlinear response is small [7–10]. In general, solving strongly nonlinear differential equations is very difficult. In [11], Mickens suggested an approximate expression for solving a truly nonlinear Duffing oscillator. Recently, various powerful analytical and numerical approximation techniques have been suggested for dealing with nonlinear oscillator differential equations. These include He's frequency-amplitude formulation [12], the harmonic balance method [13,14], the straightforward frequency prediction method [15], the modified harmonic balance method [16,17], the energy balance method [18–20], the homotopy perturbation method [21–25], the Hamiltonian approach [26,27], the weighted averaging method [28], the global residue harmonic balance method [29–31], the max-min approach [32,33], Newton's harmonic balance method [34], the variational iteration method [35], the parameter-expansion method [36], the Lindstedt-Poincaré method [37,38] and the global error minimization method [39–42].

The global error minimization method (GEMM) is one of the most frequently used techniques for dealing with the solutions of nonlinear oscillators, as it provides more accurate results valid for both weakly and strongly nonlinear oscillators than other known methods [39–42]. In the previous example, the solution up to the first approximation is calculated.

In this study, we extend and improve the global error minimization method up to a third order approximation to achieve the higher-order analytical solution of strongly nonlinear Duffing-harmonic oscillators. The present method is applied for two different problems, and the analytical results show that the modified global error minimization method (MGEMM) has better agreement with numerical solutions than the other analytical methods. Excellent agreement is observed between the approximate and exact solutions even for large amplitudes of the oscillation. Comparing the exact solutions with the approximate results has proved that the MGEMM is quite an accurate method in strongly nonlinear oscillator systems.

2. Basic idea of the modified global error minimization method (MGEMM)

To describe the proposed modified global error minimization method (MGEMM), we consider general second order nonlinear oscillator differential equations as follows:

$$\ddot{u} + F(\dot{u}, u, t) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{1}$$

We introducing $E(u)$ as a new function, defined as in [40,43], in the following form:

$$E(u) = \int_0^T \left(\ddot{u} + F(\dot{u}, u, t)\right)^2 dt, \qquad T = 2\pi\omega^{-1}. \tag{2}$$

By assuming that $F(u)$ is an odd function, a general n-th order trial function of Eq (1) can be expressed as a sum of trigonometric functions as follows:

$$u(t) = \sum_{n=0}^{\infty} (a_{2n+1)} \cos((2n+1)\omega t)), \tag{3}$$

where $a_{(2n+1)}$ are unknown constant values which satisfy the relation

$$A = \sum_{n=0}^{\infty} a_{(2n+1)}. \tag{4}$$

The following conditions were used to obtain the unknown parameters (i.e., $a_{(2n+1)}$ and ω):

$$\frac{\partial E(u)}{\partial \omega} = 0, \qquad \frac{\partial E(u)}{\partial a_{(2n+1)}} = 0, \quad n \geq 1. \tag{5}$$

By solving the *n* Eq (5) with the aid of Eq (4), the constants a_1, a_3, a_5 and the frequency of vibration ω are obtained.

3. Examples of two nonlinear Duffing-harmonic oscillators

In this section, two practical examples of nonlinear Duffing-harmonic oscillators are illustrated to show the effectiveness, accuracy and applicability of the proposed approach.

3.1. Example 1

In this application, we consider the following nonlinear Duffing-harmonic oscillator:

$$\ddot{u} + k_1 u + \frac{k_3 u}{1 + k_2 u^2} = 0, \qquad u(0) = A, \quad \dot{u}(0) = 0, \tag{6}$$

where dots denote differentiation with respect to t. Now, we shall study some different relevant cases considering Eq (6).

3.1.1. Case 1

First, we consider $k_1 = 1$, $k_2 = 1$ and $k_3 = 1$ in Eq (6). Then, we have a nonlinear oscillator system having an irrational elastic item [12, 16].

$$\ddot{u} + u + \frac{u}{1 + u^2} = 0, \qquad u(0) = A, \qquad \dot{u}(0) = 0. \tag{7}$$

According to the basic idea of the global error minimization method, the minimization problem of Eq (7) is

$$E(u) = \int_0^T \left(\ddot{u} + u + \frac{u}{1 + u^2} \right)^2 dt, \quad T = 2\pi / \omega. \tag{8}$$

The first-order approximate solution for Eq (7) can be represented as a trial function in the form

$$u_1(t) = a_1 \cos(\omega t). \tag{9}$$

Substituting Eq (9) into Eq (8) and choosing $a_1 = A$, it follows that

$$E(u_1) = \frac{4A^2\pi}{\omega} + \frac{3A^4\pi}{\omega} + \frac{5A^6\pi}{8\omega} - 4A^2\pi\omega - \frac{9}{2}A^4\pi\omega - \frac{5}{4}A^6\pi\omega + A^2\pi\omega^3 + \frac{3}{2}A^4\pi\omega^3 + \frac{5}{8}A^6\pi\omega^3 = 0. \tag{10}$$

Applying $\partial E(u_1)/\partial\omega = 0$, the frequency of the nonlinear oscillator is obtained as follows:

$$\omega = \omega_1 = \sqrt{\frac{16 + 18A^2 + 5A^4 + 2\sqrt{256 + 576A^2 + 487A^4 + 180A^6 + 25A^8}}{3(8 + 12A^2 + 5A^4)}}. \tag{11}$$

In order to illustrate the capacity of the global error minimization method, the second-order approximation is applied to the Duffing-harmonic oscillator by using the following new trial solution:

$$u_2(t) = a_1\cos(\omega t) + a_3\cos(3\omega t), \tag{12}$$

where $A = a_1 + a_3$. By substituting Eq (12) into Eq (8), we have

$$\begin{aligned} E(u_2) = \tfrac{\pi}{8\omega}\Big(&5(\omega^2-1)^2 a_1^6 + 5(11\omega^4 - 14\omega^2 + 3)a_1^5 a_3 + 2a_1^3 a_3 (8(2 - 9\omega^2 + 5\omega^4) \\ &+ 3(5 - 50\omega^2 + 109\omega^4)a_3^2\big) + 3a_1^4\big(4(2 - 3\omega^2 + \omega^4) + (15 - 110\omega^2 + 159\omega^4)a_3^2\big) \\ &+ a_3^2\big(8(2-9\omega^2)^2 + 12(2 - 27\omega^2 + 81\omega^4)a_3^2 + 5(1-9\omega^2)^2 a_3^4\big) \\ &+ a_1^2\big(8(-2+\omega^2)^2 + 16(6 - 45\omega^2 + 59\omega^4)a_3^2 + 3(15 - 190\omega^2 + 559\omega^4)a_3^4\big)\Big) = 0. \end{aligned} \tag{13}$$

Setting $\partial E(u_2)/\partial\omega = 0$ and $\partial E(u_2)/\partial a_3 = 0$ leads to

$$\begin{aligned} &\tfrac{\pi}{8\omega}(20\omega(-1+\omega^2)a_1^6 + 5(-28\omega + 44\omega^3)a_1^5 a_3 + 2a_1^3 a_3(8(-18\omega + 20\omega^3) + 3(-100\omega + 436\omega^3)a_3^2 \\ &+ 3a_1^4(4(-6\omega + 4\omega^3) + (-220\omega + 636\omega^3)a_3^2) + a_3^2(-288\omega(2-9\omega^2) + 12(-54\omega + 324\omega^3)a_3^2 \\ &- 180\omega(1-9\omega^2)a_3^4) + a_1^2(32\omega(-2+\omega^2) + 16(-90\omega + 236\omega^3)a_3^2 + 3(-380\omega + 2236\omega^3)a_3^4)) \\ &- \tfrac{1}{8\omega^2}\pi(5(-1+\omega^2)^2 a_1^6 + 5(3 - 14\omega^2 + 11\omega^4)a_1^5 a_3 + 2a_1^3 a_3(8(2 - 9\omega^2 + 5\omega^4) + 3\big(5 - 50\omega^2 \\ &+ 109\omega^4\big)a_3^2) + 3a_1^4(4(2 - 3\omega^2 + \omega^4) + (15 - 110\omega^2 + 159\omega^4)a_3^2) + a_3^2(8(2-9\omega^2)^2 \\ &+ 12(2 - 27\omega^2 + 81\omega^4)a_3^2 + 5(1-9\omega^2)^2 a_3^4) + a_1^2(8(-2+\omega^2)^2 + 16(6 - 45\omega^2 + 59\omega^4)a_3^2 \\ &+ 3(15 - 190\omega^2 + 559\omega^4)a_3^4)) = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\tfrac{\pi}{8\omega}(5(3 - 14\omega^2 + 11\omega^4)a_1^5 + 6(15 - 110\omega^2 + 159\omega^4)a_1^4 a_3 + 12(5 - 50\omega^2 + 109\omega^4)a_1^3 a_3^2 \\ &+ 2a_1^3\big(8(2 - 9\omega^2 + 5\omega^4) + 3(5 - 50\omega^2 + 109\omega^4)a_3^2\big) + a_3^2\big(24(2 - 27\omega^2 81\omega^4)a_3 + 20(1-9\omega^2)^2 a_3^3\big) \\ &+ a_1^2\big(32(6 - 45\omega^2 + 59\omega^4)a_3 + 12(15 - 190\omega^2 + 559\omega^4)a_3^3\big) + 2a_3\big(8(2-9\omega^2)^2 \\ &+ 12(2 - 27\omega^2 + 81\omega^4)a_3^2 + 5(1-9\omega^2)^2 a_3^2\big)\big) = 0. \end{aligned} \tag{15}$$

For a known amplitude, the parameters of a_1, a_3 and angular frequency ω can be obtained by using the condition $A = a_1 + a_3$ and solving Eqs (14) and (15). The computations were performed using the Mathematica software program, version 9.

To illustrate the capacity of this method, the third order approximation is applied by using the following trial function:

$$u_3(t) = a_1\cos(\omega t) + a_3\cos(3\omega t) + a_5\cos(5\omega t), \tag{16}$$

where $A = a_1 + a_3 + a_5$. Bringing Eq (16) into Eq (8) results in

$$
\begin{aligned}
&E(u_3)=\tfrac{\pi}{8\omega}\Big(5a_1^6(\omega^2-1)^2+a_1^5(\omega^2-1)\big(5a_3(11\omega^2-3)+3a_5(9\omega^2-1)\big)+5a_3^6(1-9\omega^2)^2\\
&+2a_1^3a_3\big(3a_3^2(109\omega^4-50\omega^2+5)+3a_5a_3(663\omega^4-230\omega^2+15)+8(5\omega^4-9\omega^2+2)\\
&+10a_5^2(251\omega^4-62\omega^2+3)\big)+3a_3^4\big(a_5^2(2911\omega^4-430\omega^2+15)+4(81\omega^4-27\omega^2+2)\big)\\
&+2a_1a_3^2a_5\big(a_3^2(2894\omega^4-620\omega^2+30)+15a_5a_3(153\omega^4-26\omega^2+1)+8(287\omega^4-99\omega^2+6)\\
&+3a_5^2(3399\omega^4-470\omega^2+15)\big)+a_5^2\big(5a_5^4(1-25\omega^2)^2+12a_5^2(625\omega^4-75\omega^2+2)+8(2-25\omega^2)^2\big)\\
&+a_3^2\big(3a_5^4(5631\omega^4-590\omega^2+15)+16a_5^2(803\omega^4-153\omega^2+6)+8(2-9\omega^2)^2\big)\\
&+a_3^2\big(3a_5^4(5631\omega^4-590\omega^2+15)+16a_5^2(803\omega^4-153\omega^2+6)+8(2-9\omega^2)\omega^2\big)^2\\
&+3a_5^2(277\omega^4-90\omega^2+5)\big)\big)+a_1^2\big(3a_3^4(559\omega^4-190\omega^2+15)+18a_5a_3^3(341\omega^4-90\omega^2+5)\\
&+8(\omega^2-2)^2\big)+9a_5^4(1317\omega^4-170\omega^2+5)+48a_5^2(121\omega^4-39\omega^2+2)\\
&4a_3^2\big(3a_5^2(1743\omega^4-350\omega^2+15)+4(59\omega^4-45\omega^2+6)\big)\\
&+6a_3a_5\big(a_5^2(2719\omega^4-430\omega^2+15)+8(49\omega^4-27\omega^2+2)\big)\big)\big)\Big)=0.
\end{aligned}
\tag{17}
$$

Applying $\partial E(u_3)/\partial\omega=0$, $\partial E(u_3)/\partial a_3=0$ and $\partial E(u_3)/\partial a_5=0$ yields

$$
\begin{aligned}
&\tfrac{\pi}{8\omega}\Big(20a_1^6\omega(\omega^2-1)+a_1^5(\omega^2-1)(110a_3\omega+54a_5\omega)+2a_1^5\omega\big(5a_3(11\omega^2-3)+3a_5(9\omega^2-1)\big)\\
&+3a_5(9\omega^2-1)\big)+2a_1^3a_3\big(3a_3^2(436\omega^3-100\omega)+3a_5a_3(2652\omega^3-460\omega)+8(20\omega^3-18\omega)\\
&+10a_5^2(1004\omega^3-124\omega)\big)+3a_3^4\big(a_5^2(11644\omega^3-860\omega)+4(324\omega^3-54\omega)\big)\\
&+2a_1a_3^2a_5\big(a_3^2(11576\omega^3-1240\omega)+15a_5a_3(612\omega^3-52\omega)+8(1148\omega^3-198\omega)\\
&+3a_5^2(13596\omega^3-940\omega)\big)+a_5^2\big(12a_5^2(2500\omega^3-150\omega)-800\omega(2-25\omega^2)\\
&-500a_5^4\omega(1-25\omega^3)\big)+a_3^2\big(3a_5^4(22524\omega^3-1180\omega)+16a_5^2(3212\omega^3-306\omega)-288\omega(2-9\omega^2)\big)\\
&+a_1^4\big(a_3^2(1908\omega^3-660\omega)+4a_5a_3(1468\omega^3-380\omega)+3\big(3a_5^2(1108\omega^3-180\omega)+4(4\omega^3-6\omega)\big)\big)\\
&a_1^2\big(3a_3^4(2236\omega^3-380\omega)+18a_5a_3^3(1364\omega^3-180\omega)+48a_5^2(484\omega^3-78\omega)+32\omega(\omega^2-2)\\
&+9a_5^4(5268\omega^3-340\omega)+4a_3^2\big(3a_5^2(6972\omega^3-700\omega)+4(236\omega^3-90\omega)\big)\\
&+6a_3a_5\big(a_5^2(10876\omega^3-860\omega)+8(196\omega^3-54\omega)\big)\big)\big)\\
&-\tfrac{\pi}{8\omega^2}\Big(5a_1^6(\omega^2-1)^2+a_1^5(\omega^2-1)\big(5a_3(11\omega^2-3)+3a_5(9\omega^2-1)\big)+5a_3^6(1-9\omega^2)^2\\
&+2a_1^3a_3\big(3a_3^2(109\omega^4-50\omega^2+5)+3a_5a_3(663\omega^4-230\omega^2+15)+8(5\omega^4-9\omega^2+2)\\
&+10a_5^2(251\omega^4-62\omega^2+3)\big)+3a_3^4\big(a_5^2(2911\omega^4-430\omega^2+15)+4(81\omega^4-27\omega^2+2)\big)\\
&+2a_1a_3^2a_5\big(a_3^2(2894\omega^4-620\omega^2+30)+15a_5a_3(153\omega^4-26\omega^2+1)+8(287\omega^4-99\omega^2+6)\\
&+3a_5^2(3399\omega^4-470\omega^2+15)\big)+a_5^2\big(5a_3^4(1-25\omega^2)^2+12a_5^2(625\omega^4-75\omega^2+2)+8(2-25\omega^2)^2\big)\\
&+a_3^2\big(3a_5^4(5631\omega^4-590\omega^2+15)+16a_5^2(803\omega^4-153\omega^2+6)+8(2-9\omega^2)^2\big)\\
&+a_3^2\big(3a_5^4(5631\omega^4-590\omega^2+15)+16a_5^2(803\omega^4-153\omega^2+6)+8(2-9\omega^2)^2\big)\\
&+3a_5^2(277\omega^4-90\omega^2+5)\big)\big)\big)+a_1^2\big(3a_3^4(559\omega^4-190\omega^2+15)+8(\omega^2-2)^2\\
&+9a_5^4(1317\omega^4-170\omega^2+5)+48a_5^2(121\omega^4-39\omega^2+2)+18a_3^3a_5(341\omega^4-90\omega^2+5)\\
&+4a_3^2\big(3a_5^2(1743\omega^4-350\omega^2+15)+4(59\omega^4-45\omega^2+6)\big)\\
&+6a_3a_5\big(a_5^2(2719\omega^4-430\omega^2+15)+8(49\omega^4-27\omega^2+2)\big)\big)\big)\Big)=0,
\end{aligned}
\tag{18}
$$

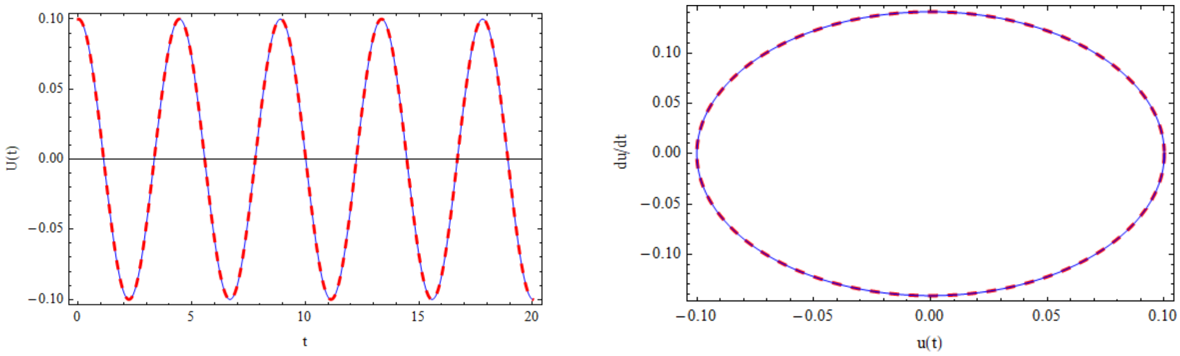
$$
\begin{aligned}
&\tfrac{\pi}{8\omega}\Big(5a_1^5(\omega^2-1)(11\omega^2-3)+30a_3^5(1-9\omega^2)^2+2a_1a_3^2a_5\big(2a_3(2894\omega^4-620\omega^2+30)\\
&+15a_5(153\omega^4-26\omega^2+1)\big)+a_1^4\big(2a_3(477\omega^4-330\omega^2+45)+4a_5(367\omega^4-190\omega^2+15)\big)\\
&+2a_1^3a_3\big(6a_3(109\omega^4-50\omega^2+5)+3a_5(663\omega^4-230\omega^2+15)\big)+2a_1^3\big(8(5\omega^4-9\omega^2+2)\\
&+3a_3^2(109\omega^4-50\omega^2+5)+3a_5a_3(663\omega^4-230\omega^2+15)+10a_5^2(251\omega^4-62\omega^2+3)\big)\\
&+12a_3^3\big(a_5^2(2911\omega^4-430\omega^2+15)+4(81\omega^4-27\omega^2+2)\big)+4a_1a_3a_5\big(8(287\omega^4-99\omega^2+6)\\
&+a_3^2(2894\omega^4-620\omega^2+30)+15a_5a_3(153\omega^4-26\omega^2+1)+3a_5^2(3399\omega^4-470\omega^2+15)\big)\\
&+2a_3\big(3a_5^4(5631\omega^4-590\omega^2+15)+16a_5^2(803\omega^4-153\omega^2+6)+8(2-9\omega^2)^2\big)\\
&+a_1^2\big(12a_3^3(559\omega^4-190\omega^2+15)+54a_3^2a_5(341\omega^4-90\omega^2+5)+8a_3\big(4(59\omega^4-45\omega^2+6)\\
&+3a_5^2(1743\omega^4-350\omega^2+15)\big)+6a_5\big(a_5^2(2719\omega^4-430\omega^2+15)+8(49\omega^4-27\omega^2+2)\big)\big)\Big)=0,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\tfrac{\pi}{8\omega}\Big(3a_1^5(\omega^2-1)(9\omega^2-1)+6a_3^4a_5(2911\omega^4-430\omega^2+15)+2a_1^3a_3\big(3a_3(663\omega^4-230\omega^2+15)\\
&+20a_5(251\omega^4-62\omega^2+3)\big)+a_1^4\big(4a_3(367\omega^4-190\omega^2+15)+18a_5(277\omega^4-90\omega^2+5)\big)\\
&+2a_1a_3^2a_5\big(15a_3(153\omega^4-26\omega^2+1)+6a_5(3399\omega^4-470\omega^2+15)\big)+2a_1a_3^2\big(8(287\omega^4-99\omega^2+6)\\
&+a_3^2(2894\omega^4-620\omega^2+30)+15a_5a_3(153\omega^4-26\omega^2+1)+3a_5^2(3399\omega^4-470\omega^2+15)\big)\\
&+a_5^2\big(20a_5^3(1-25\omega^2)^2+24a_5(625\omega^4-75\omega^2+2)\big)+a_3^2\big(32a_5(803\omega^4-153\omega^2+6)\\
&+12a_5^3(5631\omega^4-590\omega^2+15)\big)+2a_5\big(12a_5^2(625\omega^4-75\omega^2+2)+8(2-25\omega^2)^2\\
&+5a_5^4(1-25\omega^2)^2\big)+a_1^2\big(18a_3^3(341\omega^4-90\omega^2+5)+96a_5(121\omega^4-39\omega^2+2)\\
&+36a_5^3(1317\omega^4-170\omega^2+5)+12a_3a_5^2(2719\omega^4-430\omega^2+15)+24a_3^2a_5(1743\omega^4-350\omega^2+15)\\
&+6a_3\big(a_5^2(2719\omega^4-430\omega^2+15)+8(49\omega^4-27\omega^2+2)\big)\big)\Big)=0.
\end{aligned}
\tag{20}
$$

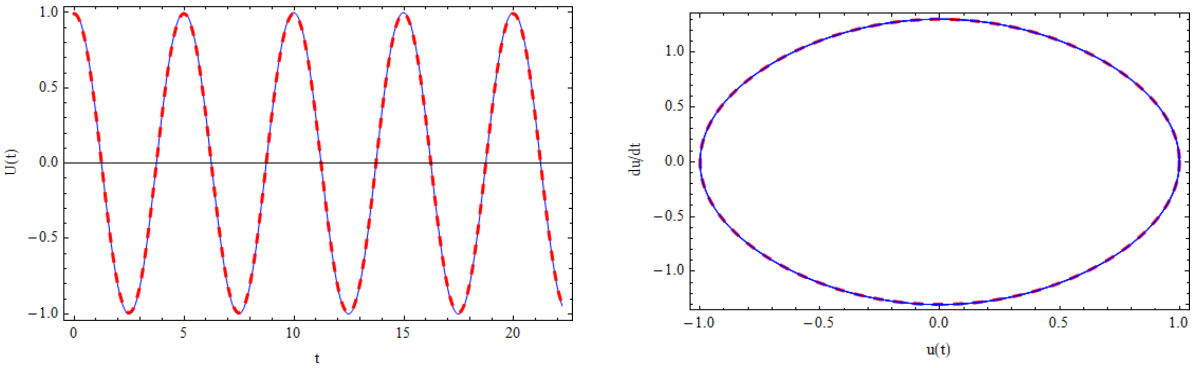
Now, by solving Eqs (18)–(20) and applying the condition $A = a_1 + a_3 + a_5$, the parameters a_1, a_3, a_5 and the angular frequency ω can be obtained for the known amplitude *A*, using the Mathematica software program, version 9. To examine the accuracy of the MGEMM solutions, the obtained results are compared with the frequency-amplitude formulation (FAF) [12], the energy balance method (EBM) [19], the modified harmonic balance method (MHBM) [16] and the exact solutions, as presented in Table 1 and Figure 1. We conclude that the third order approximation provides an excellent accuracy with respect to the exact numerical solutions.

Table 1. Comparison of the approximate analytical frequencies with the exact solutions.

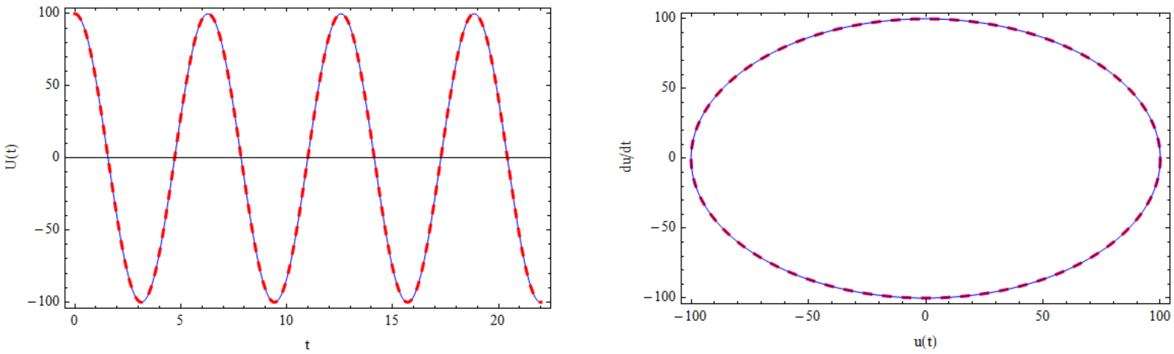
A	ω_{FAF}	ω_{EBM}	ω_{MHBM}	$\omega_{3rd\,GEMM}$	ω_{exact}
	[12]	[19]	[16]	present	Exact
0.01	1.41419	1.41419	1.41419	1.41419	1.41419
0.1	1.41158	1.41158	1.41158	1.41158	1.41158
0.2	1.40388	1.40389	1.40390	1.40390	1.40390
0.4	1.37581	1.37595	1.37616	1.37616	1.37616
0.6	1.33694	1.33743	1.33827	1.33827	1.33827
0.8	1.29448	1.29550	1.29744	1.29743	1.29743
1	1.25375	1.25514	1.25845	1.25840	1.25842
5	1.02500	1.02588	1.03148	1.02945	1.03139
10	1.00656	1.00681	1.00895	1.00790	1.00893
100	1.00007	1.0000	1.00010	1.00010	1.00010
1000	1.00000	1.00000	1.00000	1.00000	1.00000



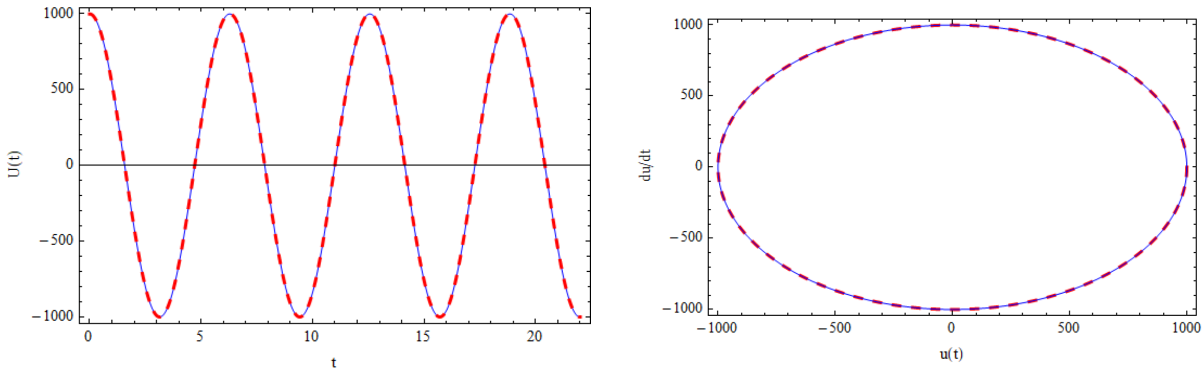
(a) $A = 0.1$



(b) $A = 1$



(c) $A = 100$



(d) $A = 1000$

Figure 1. Comparison of the approximate solution (red line) with the numerical solution (blue line).

3.1.2. Case 2

Now, if we put $k_1 = 0$, $k_2 = 1$ and $k_3 = 1$ in Eq (6), we obtain the following nonlinear oscillator, in which the restoring force has a rational expression [21,26].

$$\ddot{u} + \frac{u}{1+u^2} = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{21}$$

Using the previously mentioned procedure, the solution up to a third-order approximation is calculated. Depending on the analytical approximation, first, second or third, the approximate solution is assumed in the forms of (9), (12) and (16), respectively.

Finally, as in Case 1, the third order approximate solutions are compared with the homotopy perturbation method (HPM) [21], the Hamiltonian approach (HA) [26] and the exact solutions, as displayed in Table 2.

Table 2. Comparison of the approximate analytical frequencies with the exact solutions.

A	ω_{HPM}	ω_{HA}	$\omega_{3rd\,GEMM}$	ω_{exact}
	[21]	[26]	present	Exact
0.01	0.999963	0.9999625	0.999963	0.99999
0.1	0.996271	0.99627403	0.996273	0.9991208
1	0.755929	0.765366864	0.761539	0.76157808
10	0.114708	0.13420106	0.11948	0.123322
100	0.0115462	0.01407125	0.0120357	0.0125265

3.2. Example 2

In the second application, we will consider the following nonlinear Duffing-harmonic oscillator [18]:

$$\ddot{u} + k_1 u + \frac{k_3 u^3}{1+k_2 u^2} = 0, \quad u(0) = A, \quad \dot{u}(0) = 0, \tag{22}$$

where dots denote differentiation with respect to t. Now, we consider some following cases to compare the present solutions with published solutions using different approximate analytical methods.

3.2.1. Case 1

First, we consider $k_1 = 1$, $k_2 = 1$ and $k_3 = 1$ in Eq (22). Then, we have the following nonlinear Duffing-harmonic equation [18, 28]:

$$\ddot{u} + u + \frac{u^3}{1+u^2} = 0, \quad u(0) = A, \ \dot{u}(0) = 0. \tag{23}$$

The minimization problem is

$$E(u)=\int_0^T\left(\ddot{u}+u+\frac{u^3}{1+u^2}\right)^2dt,\ \ T=\frac{2\pi}{\omega}. \tag{24}$$

For the first-order approximation, assume that the trial function is given by

$$u_1(t)=a_1\cos(\omega t), \tag{25}$$

where $A=a_1$. By inserting Eq (25) into Eq (24), we obtain

$$E(u_1)=\frac{A^2\pi}{\omega}+\frac{3A^4\pi}{\omega}+\frac{5A^6\pi}{2\omega}-2A^2\pi\omega-\frac{9}{2}A^4\pi\omega-\frac{5}{2}A^6\pi\omega+A^2\pi\omega^3+\frac{3}{2}A^4\pi\omega^3+\frac{5}{8}A^6\pi\omega^3=0. \tag{26}$$

The frequency can be found through the condition $\partial E(u_1)/\partial\omega=0$, as follows:

$$\omega=\omega_1=\sqrt{\frac{8+18A^2+10A^4+2\sqrt{64+288A^2+487A^4+360A^6+100A^8}}{3(8+12A^2+5A^4)}}. \tag{27}$$

To improve the analytical approximation, we add additional terms to the trial function:

$$u_2(t)=a_1\cos(\omega t)+a_3\cos(3\omega t). \tag{28}$$

The constraint of this minimization is $A=a_1+a_3$. Substituting the above new trial function into Eq (24), we obtain

$$\begin{aligned}E(u_2)=\tfrac{\pi}{8\omega}\Big(&5a_1^6(\omega^2-2)^2+5a_1^5a_3(11\omega^4-28\omega^2+12)+3a_1^4(4(\omega^4-3\omega^2+2)\\&+a_3^2(159\omega^4-220\omega^2+60))+2a_1^3a_3(a_3^2(327\omega^4-300\omega^2+60)+8(5\omega^4-9\omega^2+2))\\&+a_3^2(5a_3^4(2-9\omega^2)^2+12a_3^2(81\omega^4-27\omega^2+2)+8(1-9\omega^2)^2)\\&+a_1^2(3a_3^4(559\omega^4-380\omega^2+60)+16a_3^2(59\omega^4-45\omega^2+6)+8(\omega^2-1)^2))=0.\Big)\end{aligned} \tag{29}$$

By using $\partial E(u_2)/\partial\omega=0$ and $\partial E(u_2)/\partial a_3=0$, it follows that

$$\begin{aligned}&\tfrac{\pi}{8\omega}(5a_3a_1^5(44\omega^3-56\omega)+3a_1^4(a_3^2(636\omega^3-440\omega)+4(4\omega^3-6\omega))+20a_1^6\omega(\omega^2-2)\\&+2a_1^3a_3(a_3^2(1308\omega^3-600\omega)+8(20\omega^3-18\omega))+a_3^2(12a_3^2(324\omega^3-54\omega)-288\omega(1-9\omega^3)\\&-180a_3^4\omega(2-9\omega^3))+a_1^2(3a_3^4(2236\omega^3-760\omega)+16a_3^2(236\omega^3-90\omega)+32\omega(\omega^2-1)))\\&-\tfrac{\pi}{8\omega^2}(5a_1^6(\omega^2-2)^2+5a_3a_1^5(11\omega^4-28\omega^2+12)+3a_1^4(a_3^2(159\omega^4-220\omega^2+60)+4(\omega^4-3\omega^2+2))\\&+2a_1^3a_3(a_3^2(327\omega^4-300\omega^2+60)+8(5\omega^4-9\omega^2+2))+a_3^2(12a_3^2(81\omega^4-27\omega^2+2)+8(1-9\omega^2)^2\\&+5a_3^4(2-9\omega^2)^2)+a_1^2(3a_3^4(559\omega^4-380\omega^2+60)+16a_3^2(59\omega^4-45\omega^2+6)+8(\omega^2-1)^2))=0,\end{aligned}$$

$$\begin{aligned}&\tfrac{\pi}{8\omega}(5a_1^5(11\omega^4-28\omega^2+12)+6a_3a_1^4(159\omega^4-220\omega^2+60)+4a_3^2a_1^3(327\omega^4-300\omega^2+60))\\&+2a_1^3(a_3^2(327\omega^4-300\omega^2+60)+8(5\omega^4-9\omega^2+2))+a_3^2(24a_3(81\omega^4-27\omega^2+2)\\&+20a_3^3(2-9\omega^2)^2)+a_1^2(12a_3^3(559\omega^4-380\omega^2+60)+32a_3(59\omega^4-45\omega^2+6))\\&+2a_3(5a_3^4(2-9\omega^2)^2+12a_3^2(81\omega^4-27\omega^2+2)+8(1-9\omega^2)^2))=0.\end{aligned} \tag{31}$$

(30)

The minimization problem's conditions can be easily achieved by replacing $a_1=A-a_3$, and the parameters a_1, a_3 and angular frequency ω can be obtained for a known amplitude A.

To show the accuracy of the MGEM method in higher order approximations, we apply the third order approximation and consider the following trial function:

$$u_3(t)=a_1\cos(\omega t)+a_3\cos(3\omega t)+a_5\cos(5\omega t). \tag{32}$$

Using Eq (32) as the trial function in Eq (24), where $A=a_1+a_3+a_5$, leads to

$$\begin{array}{l}
E(u_3)=\frac{\pi\omega}{8}(5a_1^6(\omega^2-4)+a_1^5(5a_3(11\omega^2-28)+3a_5(9\omega^2-20))+45a_3^6(9\omega^2-4)\\
+2a_1^3a_3(a_3^2(327\omega^2-300)+10a_5^2(251\omega^2-124)+3a_3a_5(663\omega^2-460)+40\omega^2)\\
+a_1^4(a_3^2(477\omega^2-660)+9a_5^2(277\omega^2-180)+4a_3a_5(367\omega^2-380)+12\omega^2)\\
+3a_3^4(a_5^2(2911\omega^2-860)+324\omega^2)+2a_1a_3^2a_5(2a_3^2(1447\omega^2-620)+2296\omega^2\\
+3a_5^2(3399\omega^2-940)+15a_3a_5(153\omega^2-52))+125a_5^2(60a_5^2\omega^2+40\omega^2\\
+a_5^4(25\omega^2-4))+a_1^2(3a_3^4(559\omega^2-380)+944a_3^2\omega^2+8\omega^2\\
+6a_3a_5(3a_3^2(341\omega^2-180)+392\omega^2)+12a_5^2(7a_3^2(249\omega^2-100)+484\omega^2)\\
+9a_5^4(1317\omega^2-340)+6a_3a_5^3(2719\omega^2-860))+a_3^2(12848a_5^2\omega^2+648\omega^2\\
+3a_5^4(5631\omega^2-1180)))=0.
\end{array} \tag{33}$$

By setting $\partial E(u_3)/\partial\omega=0$, $\partial E(u_3)/\partial a_3=0$ and $\partial E(u_3)/\partial a_5=0$, we obtain

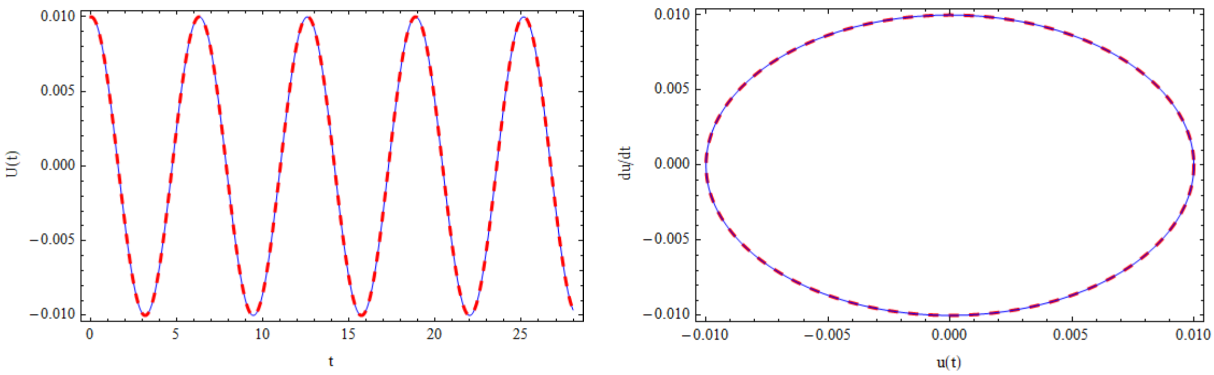
$$\begin{array}{l}
\frac{1}{8\omega}\pi(20a_1^6\omega(\omega^2-2)-180a_3^6\omega(2-9\omega^2)+a_1^5(\omega^2-2)(110a_3\omega+54a_5\omega)+2\omega_3a_1^5(5a_3(11\omega^2-6)\\
+3a_5(9\omega^2-2))+3a_3^4(a_5^2(11644\omega^3-1720\omega)+4(324\omega^3-54\omega))+2a_1a_3^2a_5(8(1148\omega^3-198\omega)\\
+2a_3^2(5788\omega^3-1240\omega)+15a_5a_3(612\omega^3-104\omega)+3a_5^2(13596\omega^3-1880\omega))+a_5^2(-800\omega(1-25\omega^2)\\
+12a_5^2(2500\omega^3-150\omega)-500a_5^4\omega(2-25\omega^2))+a_3^2(16a_5^2(3212\omega^3-306\omega)-288\omega(1-9\omega^2)\\
+3a_5^4(22524\omega^3-2360\omega))+2a_1^3a_3(a_3^2(1308\omega^3-600\omega)+3a_5a_3(2652\omega^3-920\omega)\\
+2(5a_5^2(1004\omega^3-248\omega)+4(20\omega^3-18\omega)))+a_1^4(3a_3^2(636\omega^3-440\omega)+4a_5a_3(1468\omega^3-760\omega)\\
+3(a_5^2(3324\omega^3-1080\omega)+4(4\omega^3-6\omega)))+a_1^2(3a_3^4(2236\omega^3-760\omega)+32\omega(\omega^2-1)\\
+18a_3^3a_5(1364\omega^3-360\omega)+48a_5^2(484\omega^3-78\omega)+9a_5^4(5268\omega^3-680\omega)+4a_3^2(4(236\omega^3-90\omega)\\
+18a_3^3a_5(1364\omega^3-360\omega)+48a_5^2(484\omega^3-78\omega)+9a_5^4(5268\omega^3-680\omega)+4a_3^2(4(236\omega^3-90\omega)\\
-\frac{1-\pi}{8\omega^2})5a_1^6(\omega^2-2)^2+a_1^5(\omega^2-2)(5a_3(11\omega^2-6)+3a_5(9\omega^2-2))+5a_3^6(2-9\omega^2)^2\\
+3a_3^4(a_5^2(2911\omega^4-860\omega^2+60)+4(81\omega^4-27\omega^2+2))+2a_1a_3^2a_5(8(287\omega^4-99\omega^2+6)\\
+2a_3^2(1447\omega^4-620\omega^2+60)+15a_5a_3(153\omega^4-52\omega^2+4)+3a_5^2(3399\omega^4-940\omega^2+60))\\
+a_5^2(5a_5^4(2-25\omega^2)^2+12a_5^2(625\omega^4-75\omega^2+2)+8(1-25\omega^2)^2)+a_3^2(8(1-9\omega^2)^2+\\
3a_5^4(5631\omega^4-1180\omega^2+60)+16a_5^2(803\omega^4-153\omega^2+6))+2a_1^3a_3(a_3^2(327\omega^4-300\omega^2+60)\\
+3a_3a_5(663\omega^4-460\omega^2+60)+2(5a_5^2(251\omega^4-124\omega^2+12)+4(5\omega^4-9\omega^2+2)))\\
+a_1^4(3a_3^2(159\omega^4-220\omega^2+60)+4a_5a_3(367\omega^4-380\omega^2+60)+3(4(\omega^4-3\omega^2+2)\\
a_5^2+(831\omega^4-540\omega^2+60)))+a_1^2(3a_3^4(559\omega^4-380\omega^2+60)+8(\omega^2-1)^2\\
+9a_5^4(1317\omega^4-340\omega^2+20)+48a_5^2(121\omega^4-39\omega^2+2)+18a_3^3a_5(341\omega^4-180\omega^2+20)\\
+4a_3^2(3a_5^2(1743\omega^4-700\omega^2+60)+4(59\omega^4-45\omega^2+6))+6a_3a_5(8(49\omega^4-27\omega^2+2)\\
+a_5^2+(2719\omega^4-860\omega^2+60))))=0,
\end{array} \tag{34}$$

$$
\begin{aligned}
&\tfrac{\pi}{8\omega}(3a_1^5(\omega^2-2)(9\omega^2-2)+6a_3^4a_5(2911\omega^4-860\omega^2+60)+2a_1^3a_3(3a_3(663\omega^4-460\omega^2+60)\\
&+20a_5(251\omega^4-124\omega^2+12))+a_1^4(4a_3(367\omega^4-380\omega^2+60)+6a_5(831\omega^4-540\omega^2+60))\\
&+2a_1a_3^2a_5(15a_3(153\omega^4-52\omega^2+4)+6a_5(3399\omega^4-940\omega^2+60))+2a_1a_3^2(8(287\omega^4-99\omega^2+6)\\
&+2a_3^2(1447\omega^4-620\omega^2+60)+15a_5a_3(153\omega^4-52\omega^2+4)+3a_5^2(3399\omega^4-940\omega^2+60))\\
&+a_5^2(20a_5^3(2-25\omega^2)^2+24a_5(625\omega^4-75\omega^2+2))+a_3^2(32a_5(803\omega^4-153\omega^2+6)\\
&+12a_5^3(5631\omega^4-1180\omega^2+60))+2a_5(12a_5^2(625\omega^4-75\omega^2+2)+8(1-25\omega^2)^2\\
&+5a_5^4(2-25\omega^2)^2)+a_1^2(18a_3^3(341\omega^4-180\omega^2+20)+96a_5(121\omega^4-39\omega^2+2)\\
&+24a_5a_3^2(1743\omega^4-700\omega^2+60)+12a_5^2a_3(2719\omega^4-860\omega^2+60)\\
&+36a_5^3(1317\omega^4-340\omega^2+20)+6a_3(a_5^2(2719\omega^4-860\omega^2+60)+8(49\omega^4-27\omega^2+2))))=0,
\end{aligned}
\tag{35}
$$

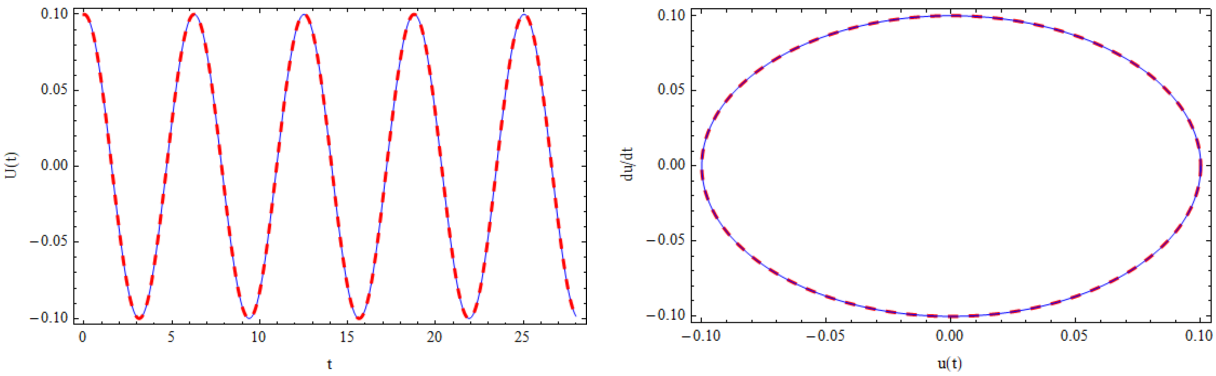
$$
\begin{aligned}
&\tfrac{1}{8\omega}\pi(5a_1^5(\omega^2-2)(11\omega^2-6)+30a_3^5(2-9\omega^2)^2+2a_1a_3^2a_5(4a_3(1447\omega^4-620\omega^2+60)\\
&+15a_5(153\omega^4-52\omega^2+4))+a_1^4(6a_3(159\omega^4-220\omega^2+60)+4a_5(367\omega^4-380\omega^2+60))\\
&+2a_1^3a_3(2a_3(327\omega^4-300\omega^2+60)+3a_5(663\omega^4-460\omega^2+60))+12a_3^3(4(81\omega^4-27\omega^2+2)\\
&+a_5^2(2911\omega^4-860\omega^2+60))+4a_1a_3a_5(2a_3^2(1447\omega^4-620\omega^2+60)+8(287\omega^4-99\omega^2+6)\\
&+3a_5^2(3399\omega^4-940\omega^2+60)+15a_3a_5(153\omega^4-52\omega^2+4))+2a_3(8(1-9\omega^2)^2\\
&+3a_5^4(5631\omega^4-1180\omega^2+60)+16a_5^2(803\omega^4-153\omega^2+6))+2a_1^3(a_3^2(327\omega_3^4-300\omega^2+60)\\
&+3a_3a_5(663\omega^4-460\omega^2+60)+2(5a_5^2(251\omega^4-124\omega^2+12)+4(5\omega^4-9\omega^2+2)))\\
&+a_1^2(12a_3^3(559\omega^4-380\omega^2+60)+54a_3^2a_5(341\omega^4-180\omega^2+20)+8a_3(4(59\omega^4-45\omega^2+6)\\
&+3a_5^2(1743\omega^4-700\omega^2+60))+6a_5(a_5^2(2719\omega^4-860\omega^2+60)+8(49\omega^4-27\omega^2+2))))=0.
\end{aligned}
\tag{36}
$$

Putting $a_1 = A - a_3 - a_5$ in the minimization problem changes the constraint minimization problem to an unconstrained minimization problem, which is easier to find the solution of Eq (33) by using the conditions $\partial E(u_3)/\partial\omega = 0$, $\partial E(u_3)/\partial a_3 = 0$ and $\partial E(u_3)/\partial a_5 = 0$.

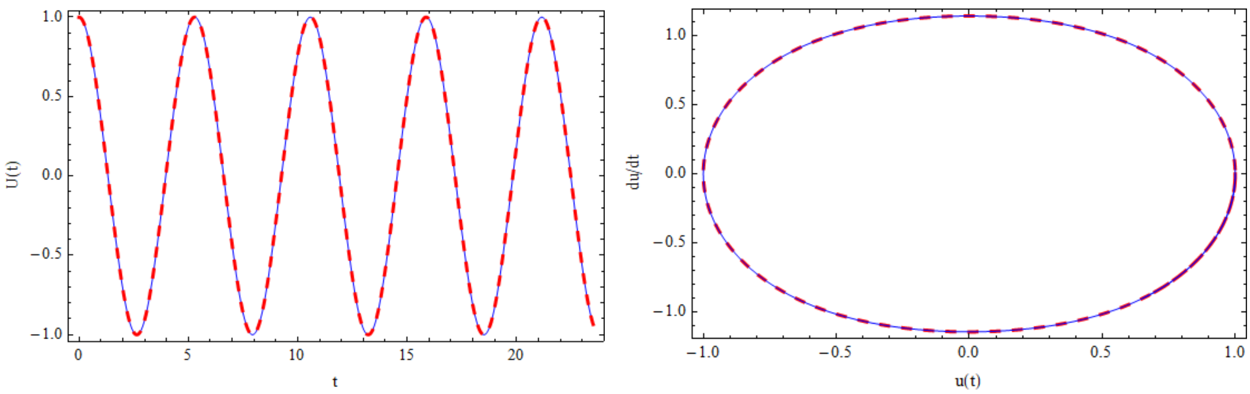
We plot the analytical solutions obtained from MGEMM (red line) Eq (32) and compare them with numerical solutions of Eq (23) obtained using the fourth order Runge-Kutta method (blue line). It is observed that for all different values of the amplitude A, the approximate solutions match extremely well with the numerical solutions (see Figure 2).



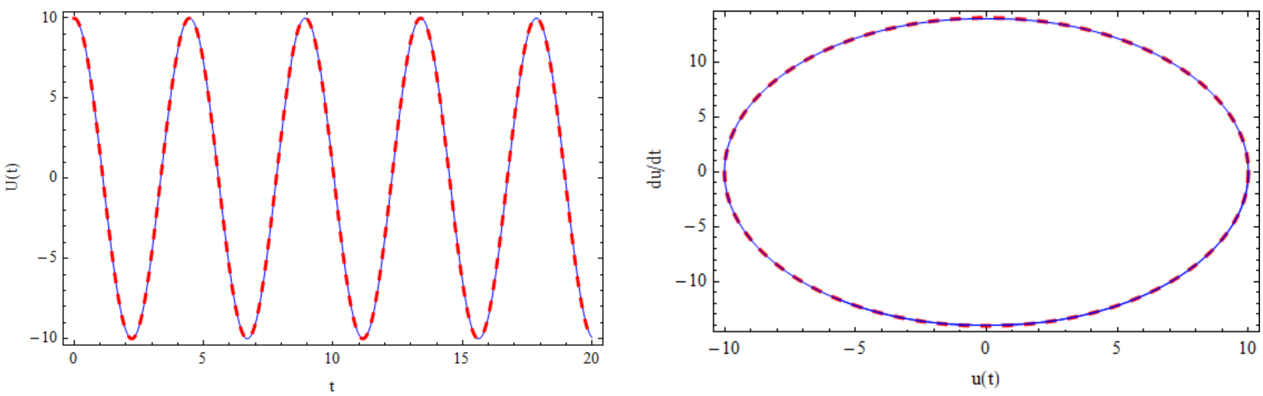
(a) $A = 0.01$



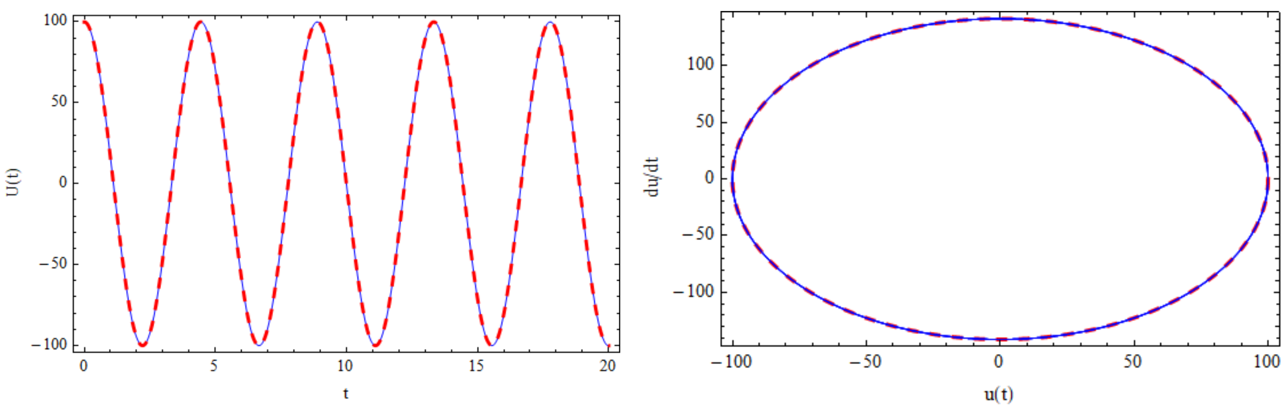
(b) $A = 0.1$



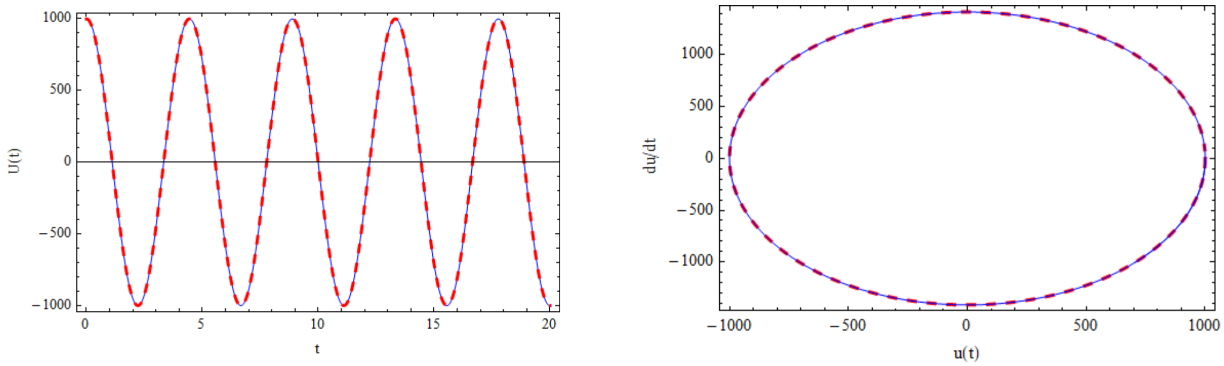
(c) $A = 1$



(d) $A = 10$



(e) $A = 100$



(f) $A = 1000$

Figure 2. Comparison of the approximate solution (red line) with the numerical solution (blue line).

3.2.2. Case 2

Second, we consider $k_1 = 0$, $k_2 = 1$ and $k_3 = 1$ in Eq (6). Hence, we have the one-dimensional nonlinear oscillator governed by [21–26]

$$\ddot{u} + \frac{u^3}{1+u^2} = 0, \qquad u(0) = A, \quad \dot{u}(0) = 0. \tag{37}$$

Finally, we can obtain the first and second-order approximations to Eq (37) given by Eqs (25) and (28), respectively. We remark that the third-order approximation in Eq (32) can be given by MGEMM in a similar manner. Generally, after three steps of MGEMM, one can obtain the approximated solutions to Eq (37) with sufficient accuracy. The analytical results of Eq (37) are compared with the iterative homotopy harmonic balance method (IHHBM) [34], the energy balance method (EBM) [20], the max-min approach (MMA) [32], the global residue harmonic balance method (GRHBM) [29], the Hamiltonian approach (HA) [26] and the exact solutions, as shown in Table 3.

Table 3. Comparison of the approximate analytical frequencies with the exact solutions.

A	ω_{EBM}	ω_{IHHBM}	ω_{MMA}	ω_{GRHBM}	ω_{HA}	$\omega_{3rd\,GEMM}$	ω_{exact}
	[20]	[34]	[32]	[29]	[26]	present	Exact
0.01	0.00866	0.008478	0.00866	0.008472	0.00865	0.00847	0.00847
0.1	0.08627	0.084418	0.08627	0.084394	0.08624	0.08439	0.08439
1	0.65164	0.63136	0.65465	0.636795	0.64359	0.636783	0.63678
5	0.97343	0.96667	0.97435	0.968107	0.96731	0.969202	0.96698
10	0.99314	0.99090	0.99340	0.991591	0.99095	0.992005	0.99092
50	0.99973	0.99961	0.99973	0.999657	0.999608	0.999676	0.99961
100	0.99999	0.999901	0.99993	0.999914	0.99990	0.999919	0.99990

4. Results and discussion

In this paper, we test the analytical solutions of strongly nonlinear Duffing-harmonic oscillators to show the effectiveness of MGEMM. Comparisons of the analytical solutions and the exact numerical solutions of the Duffing-harmonic oscillators for small and large values of the amplitude have been illustrated in Figures 1 and 2 and Tables 1–3. In these Figures, the analytical solution is indicated by a red line, while the numerical solution is represented by a blue line. There is good compatibility between the analytical and numerical solutions, which confirms the accuracy of our results, and excellent matching is observed in these calculations. Moreover, as shown in Tables 1–3, the results produced using the MGEMM are in better agreement with those obtained using exact solutions than other existing ones in the literature. The calculations of the above applications were done using Mathematica software.

5. Conclusions

In this paper, the modified global error minimization method has been presented successfully to obtain higher order approximate periodic solutions of strongly nonlinear Duffing-harmonic oscillators. The present method, which is proved to be a powerful mathematical tool to study nonlinear oscillators, can be easily extended to any nonlinear equation because of its efficiency and convenient applicability. We demonstrated the accuracy and efficiency of the proposed method by solving some examples. We showed that the obtained solutions are valid for the whole domain. Comparisons of the obtained solutions, whether numerical or analytical, revealed a clear match, highlighting the precision of the modified GEMM.

Acknowledgments

M. Zayed and G. M. Ismail extend their appreciation to the Deanship of Scientific Research at King Khalid University, Saudi Arabia, for funding this work through the research groups program under grant R.G.P.2/207/43.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. H. He, The simplest approach to nonlinear oscillators, *Results Phys.*, **15** (2019), 102546. https://doi.org/10.1016/j.rinp.2019.102546
2. S. Li, J. Niu, X. Li, Primary resonance of fractional-order Duffing-Van der Pol oscillator by harmonic balance method, *Chinese Phys. B*, **27** (2018), 120502. https://doi.org/10.1088/1674-1056/27/12/120502
3. B. Wu, W. Liu, X. Chen, C. W. Lim, Asymptotic analysis and accurate approximate solutions for strongly nonlinear conservative symmetric oscillators, *Appl. Math. Model.*, **49** (2017), 243–254. https://doi.org/10.1016/j.apm.2017.05.004
4. J. H. He, T. S. Amer, S. Elnaggar, A. A. Galal, Periodic property and instability of a rotating pendulum system, *Axioms.*, **10** (2021), 191. https://doi.org/10.3390/axioms10030191
5. N. Qie, W. F. Houa, J. H. He, The fastest insight into the large amplitude vibration of a string, *Rep. Mech. Eng.*, **2** (2021), 1–5. https://doi.org/10.31181/rme200102001q
6. Y. Tian, Frequency formula for a class of fractal vibration system, *Rep. Mech. Eng.*, **3** (2022), 55–61. https://doi.org/10.31181/rme200103055y

7. J. B. Marion, *Classical dynamics of particles and system*, Harcourt Brace Jovanovich, San Diego, CA, 1970.
8. N. N. Krylov, N. N. Bogoliubov, *Introduction to nonlinear mechanics*, Princeton University Press, New Jersey, 1947.
9. N. N. Bogoliubov, Y. A. Mitropolskii, *Asymptotic methods in the theory of nonlinear oscillations*, Gordan and Breach, New York, 1961.
10. A. H. Nayfeh, D. T. Mook, *Nonlinear oscillations*, John Wiley & Sons, New York, 1979.
11. R. E. Mickens, Mathematical and numerical study of the Duffing-harmonic oscillator, *J. Sound Vib.*, **244** (2000), 563–567. https://doi.org/10.1006/jsvi.2000.3502
12. J. Fan, He's frequency-amplitude formulation for the Duffing harmonic oscillator, *Comput. Math. Appl.*, **58** (2009), 2473–2476. https://doi.org/10.1016/j.camwa.2009.03.049
13. C. W. Lim, B. S. Wu, A new analytical approach to the Duffing-harmonic oscillator, *Phys. Lett. A*, **311** (2003), 365–373. https://doi.org/10.1016/S0375-9601(03)00513-9
14. A. Beléndez, D. I. Mendez, T. Beléndez, A. Hernández, M. L. Álvarez, Harmonic balance approaches to the nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable, *J. Sound Vib.*, **314** (2008), 773–782. https://doi.org/10.1016/j.jsv.2008.01.021
15. J. H. He, Q. Yang, C. H. He, Y. Khan, A simple frequency formulation for the tangent oscillator, *Axioms*, **10** (2021), 320. https://doi.org/10.3390/axioms10040320
16. N. Sharif, A. Razzak, M. Z. Alam, Modified harmonic balance method for solving strongly nonlinear oscillators, *J. Interdiscipl. Math.*, **22** (2019), 353–375. https://doi.org/10.1080/09720502.2019.1624304
17. M. A. Hosen, M. S. H. Chowdhury, G. M. Ismail, A. Yildirim, A modified harmonic balance method to obtain higher-order approximations to strongly nonlinear oscillators, *J. Interdiscipl. Math.*, **23** (2020), 1325–1345. https://doi.org/10.1080/09720502.2020.1745385
18. M. Momeni, N. Jamshidi, A. Barari, D. D. Ganji, Application of He's energy balance method to Duffing-harmonic oscillators, Int. J. Comput. Math., **88** (2011), 135–144. https://doi.org/10.1080/00207160903337239
19. A. M. El-Naggar, G. M. Ismail, Periodic solutions of the Duffing harmonic oscillator by He's energy balance method, *J. Appl. Comput. Mech.*, **2** (2016), 35–41.
20. D. D. Ganji, N. R. Malidarreh, M. Akbarzade, Comparison of energy balance period with exact period for arising nonlinear oscillator equations, *Acta Appl. Math.*, **108** (2009), 353–362. https://doi.org/10.1007/s10440-008-9315-2
21. D. H. Shou, The homotopy perturbation method for nonlinear oscillators, *Comput. Math. Appl.*, **58** (2009), 2456–2459. https://doi.org/10.1016/j.camwa.2009.03.034
22. J. H. He, Y. O. El-Dib, A. A. Mady, Homotopy perturbation method for the fractal toda oscillator, *Fractal Fract.*, **5** (2021), 93. https://doi.org/10.3390/fractalfract5030093
23. J. H. He, T. S. Amer, S. Elnaggar, A. A. Galal, Periodic property and instability of a rotating pendulum system, *Axioms*, **10** (2021), 191. https://doi.org/10.3390/axioms10030191
24. S. Noeiaghdam, A. Dreglea, J. H. He, Z. Avazzadeh, M. Suleman, M. A. F. Araghi, et al., Error estimation of the homotopy perturbation method to solve second kind Volterra integral equations with piecewise smooth kernels: Application of the CADNA Library, *Symmetry*, **12** (2020), 1730. https://doi.org/10.3390/sym12101730

25. M. Bayat, M. Head, L. Cveticanin, P. Ziehl, Nonlinear analysis of two-degree of freedom system with nonlinear springs, *Mech. Syst. Signal Pr.*, **171** (2022), 108891. https://doi.org/10.1016/j.ymssp.2022.108891
26. A. Yildirim, Z. Saadatnia, H. Askari, Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms, *Math. Comput. Model.*, **54** (2011), 697–703. https://doi.org/10.1016/j.mcm.2011.03.012
27. G. M. Ismail, L. Cveticanin, Higher order Hamiltonian approach for solving doubly clamped beam type N/MEMS subjected to the van der Waals attraction, *Chinese J. Phys.*, **72** (2021), 69–77. https://doi.org/10.1016/j.cjph.2021.04.016
28. V. H. Dang, An approximate solution for a nonlinear Duffing-harmonic oscillator, *Asian Res. J. Math.*, **15** (2019), 1–14. https://doi.org/10.9734/arjom/2019/v15i430154
29. P. Ju, Global residue harmonic balance method for Helmholtz-Duffing oscillator, *Appl. Math. Model.*, **39** (2015), 2172–2179. https://doi.org/10.1016/j.apm.2014.10.029
30. J. Lu, Global residue harmonic balance method for strongly nonlinear oscillator with cubic and harmonic restoring force, *J. Low Freq. Noise V. A.*, 2022. https://doi.org/10.1177/14613484221097465
31. G. M. Ismail, M. Abul-Ez, N. M. Farea, N. Saad, Analytical approximations to nonlinear oscillation of nanoelectro-mechanical resonators, *Eur. Phys. J. Plus*, **134** (2019), 47. https://doi.org/10.1140/epjp/i2019-12399-2
32. S. S. Ganji, D. D. Ganji, A. G. Davodi, S. Karimpour, Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max-min approach, *Appl. Math. Model.*, **34** (2010), 2676–2684. https://doi.org/10.1016/j.apm.2009.12.002
33. M. Bayat, M. Bayat, M. Kia, H. R. Ahmadi, I. Pakar, Nonlinear frequency analysis of beams resting on elastic foundation using max-min approach, Geomech. Eng., **16** (2018), 355–361.
34. C. W. Lim, B. S. Wub, W. P. Sun, Higher accuracy analytical approximations to the Duffing-harmonic oscillator, *J. Sound Vib.*, **296** (2006), 1039–1045. https://doi.org/10.1016/j.jsv.2006.02.020
35. M. Fesanghary, T. Pirbodaghi, M. Asghari, H. Sojoudi, A new analytical approximation to the Duffing-harmonic oscillator, *Chaos Soliton. Fract.*, **42** (2009), 571–576. https://doi.org/10.1016/j.chaos.2009.01.024
36. L. Xu, Application of He's parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire, *Phys. Lett. A*, **368** (2007), 259–262. https://doi.org/10.1016/j.physleta.2007.04.004
37. C. S. Liu, Y. W. Chen, A Simplified Lindstedt-Poincaré method for saving computational cost to determine higher order nonlinear free vibrations, *Mathematics*, **9** (2021) 3070. https://doi.org/10.3390/math9233070
38. Z. Li, J. Tang, High accurate homo-heteroclinic solutions of certain strongly nonlinear oscillators based on generalized Padé-Lindstedt-Poincaré method, *J. Vib. Eng. Technol.*, **10** (2022), 1291–1308. https://doi.org/10.1007/s42417-022-00446-7
39. M. K. Yazdi, P. H. Tehrani, Frequency analysis of nonlinear oscillations via the global error minimization, *Nonlinear Eng.*, **5** (2016). https://doi.org/10.1515/nleng-2015-0036
40. Y. Farzaneh, A. A. Tootoonchi, Global error minimization method for solving strongly nonlinear oscillator differential equations, *Comput. Math. Appl.*, **59** (2010), 2887–2895. https://doi.org/10.1016/j.camwa.2010.02.006

41. A. Mirzabeigy, M. K. Yazdi, A. Yildirim, Analytical approximations for a conservative nonlinear singular oscillator in plasma physics, *J. Egypt. Math. Soc.*, **20** (2012), 163–166. https://doi.org/10.1016/j.joems.2012.05.001
42. G. M. Ismail, H. Abu-Zinadah, Analytic approximations to non-linear third order jerk equations via modified global error minimization method, *J. King Saud Univ. Sci.*, **33** (2021), 101219. https://doi.org/10.1016/j.jksus.2020.10.016
43. K. P. Badakhshan, A. V. Kamyad, A. Azemi, Using AVK method to solve nonlinear problems with uncertain parameters, *Appl. Math. Comput.*, **189** (2007), 27–34. https://doi.org/10.1016/j.amc.2006.11.172