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Research article

Approximate solutions for a class of nonlinear Volterra-Fredholm integro-differential equations under Dirichlet boundary conditions

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Abstract: This paper studies the solvability of boundary value problems for a nonlinear integro-differential equation. Converting the problem to an equivalent nonlinear Volterra-Fredholm integral equation (NVFIE) is driven by using a suitable transformation. To investigate the existence and uniqueness of continuous solutions for the NVFIE under certain given conditions, the Krasnoselskii fixed point theorem and Banach contraction principle have been used. Finally, we numerically solve the NVFIE and study the rate of convergence using methods based on applying the modified Adomian decomposition method, and Liao's homotopy analysis method. As applications, some examples are provided to support our work.

Keywords: boundary value problem; integro-differential equations; existence; uniqueness; modified Adomian's decomposition method; homotopy analysis method
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1. Introduction

Nonlinear integro differential equations arise in various scientific phenomena in applied mathematics, mathematical physics, and biology. The delayed integro differential equations of the Volterra type are used to characterize the evolution of biological populations [1]. In physics, systems of integro differential equations are used to study continuous medium-nuclear reactors [2]. Further, some singular integral equations occur in the process of formulating mixed boundary value problems in mathematical physics [3]. Constructing different techniques to study the solutions of nonlinear integral equations dates back to the early 1980s [4].

Analytical solutions for the majority of nonlinear equations do not have a closed form. Consequently, there are many techniques such as the perturbation methods [5–7] and non-perturbation method [8] to find the solution to these types of equations. Perturbation methods are commonly based on transferring the nonlinear problem to an infinite number of linear sub-problems through the perturbation parameters which are introduced to get approximate solutions [9]. The modify Adomian decomposition method (MADM) [24] is a non-perturbation method that has became a remarkable technique to find the exact and approximate solutions for a large class of linear and nonlinear integral equations. Moreover, with this method, we provide a numerical algorithm based on the application of the so-called Adomian polynomial to solve nonlinear equations.

The focus has primarily been on obtaining approximate solutions to nonlinear integro-differential equations which are induced by converting initial and boundary value problems. In [10] Atkinson and Potra applied the discrete Galerkin method for solving nonlinear integral equations and gave a general framework and error analysis for the numerical method, while Yousefi and Razzaghi [11] used Legendre wavelets method together with Gaussian integration method to evaluated the unknown coefficients and found an approximate solution to nonlinear Volterra-Fredholm integral equations. The nonexistence of global solutions of a nonlinear integral equation was studied in [12]. Maleknejad et al. [13] proposed an orthogonal triangular function to approximate the solution of nonlinear Volterra-Fredholm integral equations, and they used a collocation method to reduce it to the solution of algebraic equations. In [14], the authors studied the mean square convergence of the series solution for a stochastic integro-differential equation and evaluated the truncation error by using the Adomain decomposition method (ADM). Mashayekhi et al. [15] proposed the hybrid of block-pulse functions and Bernoulli polynomial for solving the nonlinear Volterr-Fredholm integral equation (NVFIE). Deniz [16] presented an optimal perturbation iteration method and employed it for solving an NVFIE, and he used new algorithms that were constructed for integral equations. Comparing their new algorithms with those in some earlier papers proved the excellent accuracy of the newly proposed technique. Abdou and Youssef [17] discussed the solvability of a nonlinear Fredholm integro-differential equation (NFIE) with boundary conditions and they applied the MADM and Liaos homotopy analysis method (HAM) [25] for solving the NFIE numerically. Also, in [18] they have used the same methods for solving an NFIE of order n. Abed et al. [19] applied the MADM and variational iteration method to investigate the numerical solution for an NVFIE with initial conditions. In this paper, the existence and uniqueness of the analytical solutions of the NVFIE with boundary conditions are investigated. We consider the solvability of a two-point boundary value problem for a nonlinear integro-differential equation of the form

$$\begin{aligned} \omega\phi''(x) + A(x)\phi'(x) + B(x)\phi(x) = f(x) + \lambda_1 \int_a^x \psi(x,y)\,[\phi(y)]^p\,dy \\ + \lambda_2 \int_a^b \psi(x,y)\,[\phi(y)]^p\,dy, \qquad \text{for } x \in [a,b], \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\phi(a) = \eta_1, \quad \phi(b) = \eta_2, \quad \eta_1, \eta_2 \in \mathbb{R}, \tag{1.2}$$

where $p \geq 0$ is a non negative integer. The parameters $\{\omega, \lambda_1, \lambda_2\}$ are nonzero real numbers. The functions A, B and f and the kernel ψ are known functions satisfying certain conditions, as is to be stated in Section 3, while $x \mapsto \phi(x)$ is the required function to be found in the space $C^2(I, \mathbb{R})$.

The paper is structured as follows. In Section 2 some basic notations, definitions and theorems regarding the existence, uniqueness, and convergent results in Banach Space are recalled. In Section 3 we show that the NVFIE has at least one continuous solution. Then we provide sufficient conditions for which it has a unique solution. In Sections 4 and 5 the MADM and the HAM are applied respectively. Then some analytical and numerical examples are provided in Section 6. Finally, Section 7 concludes the paper.

2. Preliminaries

Before driving integro-differential equations to the Volterra-Fredholm integral equation, we review some basic definitions and theorems, which have been given in [20–23]

Definition 2.1. *(Contraction mapping) [20] Let (M, d) be a metric space and $f : M \to M$ be a function, which has the property that there is some nonnegative real number $0 \leq k < 1$ such that for all $x, y \in M$, $d(f(x), f(y)) \leq kd(x, y)$.*

Theorem 2.1. *(Banach contraction principle) [22] Let (M, d) be a metric space; then, each contraction mapping $\tau : M \to M$ has a unique fixed point x of τ in M.*

Theorem 2.2. *(Schauder fixed point Theorem) [23] Let X be a Banach space and A be a convex, closed subset of X. Let $T : A \longrightarrow A$ be a map such that the set $Tu : u \in A$ is relatively compact in X. Then T has at least one fixed point $u^* \in A$ i.e., $Tu^* = u^*$.*

Theorem 2.3. *(Arzela-Ascoli theorem) [20] If a sequence $\{f_n\}_{n=0}^{\infty}$ in a closed and bounded interval $[a, b]$ is a bounded and equicontinuous, then it has a uniformly convergent subsequence.*

Theorem 2.4. *(Krasnoselskii fixed point theorem) [21] Let μ be a closed convex non-empty subset of a Banach space X. Suppose that A and B map μ into X, and that*

(1) A is continuous and compact,

(2) $Ax + By \in \mu$ for all $x, y \in \mu$,

(3) B is a contraction mapping.

Then, there exists y in μ such that $Ay + By = y$.

3. Outcomes for existence and uniqueness

In order to prove all theorems we suppose the following postulates:

p.1 The functions A and B are elements in the space $C(I, \mathbb{R})$.

p.2 The known free function f belongs to the space $C^2(I, \mathbb{R})$.

p.3 For each $y \in I$ the kernel $(x, y) \mapsto \psi(x, y)$ is continuous in x, with x taking values in $\mathbb{R}$.

$$\left(\int_a^b (\psi(x,y))^2 dy\right)^{\frac{1}{2}} \leq \gamma, \quad \text{for all } x \in I, \ \gamma > 0. \tag{3.1}$$

p.4 $(\alpha + (k_1|\lambda_1| + |\lambda_2|)C^*(1)) \leq |\omega|$, where $\alpha = (b-a)\,(\|A\|_\infty + (b-a)\|B\|_\infty)$,

$$C^*(l) = \binom{p}{l}\frac{\gamma(b-a)^{2l+\frac{1}{2}}(d^*(l))^{\frac{1}{2}}}{(2p-2l+1)^{\frac{1}{2}}}, \tag{3.2}$$

and

$$d^*(l) = \left\{\eta_2^{2p-2l} + \eta_2^{2p-2l-1}\eta_1 + ... + \eta_1^{2p-2l}\right\}. \tag{3.3}$$

p.5 $(\alpha + (|\lambda_1| + |\lambda_2|)\Lambda) \leq |\omega|$, where

$$\Lambda = \sum_{l=1}^{p} \frac{e(l)C^*(l)}{(b-a)^{3l-3}}, \tag{3.4}$$

where $I = [a, b]$, and the bounded constant $k_1 = 1$.

Theorem 3.1. *Let the conditions (**p.1**) to (**p.3**) be satisfied. Then the boundary value problems (1.1) and (1.2) are equivalent to the following NVFIE,*

$$\begin{aligned}\omega u(x) &+ \int_a^b [W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy - \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy]u(t)dt \\ &= F(x) + \lambda_1 \int_a^x \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy \\ &+ \lambda_2 \int_a^b \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy,\end{aligned} \tag{3.5}$$

where

$$u(x) := \phi''(x), \tag{3.6}$$

$$W(x,t) := \frac{1}{(b-a)} \times \begin{cases} W_1(x,t) = (t-a)(A(x) - (b-x)B(x)), & a \leq t \leq x, \\ W_2(x,t) = (t-b)(A(x) - (a-x)B(x)), & x \leq t \leq b, \end{cases} \tag{3.7}$$

$$R(x,y;l) := \binom{p}{l}\frac{\psi(x,y)}{(b-a)^p}\left[\eta_1(b-y) + \eta_2(y-a)\right]^{p-l}, \tag{3.8}$$

$$H_2(y,t) := \begin{cases} (b-y)(a-t), & a \leq t \leq y, \\ (a-y)(b-t), & y \leq t \leq b, \end{cases} \tag{3.9}$$

$$\mu(x) := \frac{1}{(b-a)}\left(\eta_1[-A(x) + (b-x)B(x)] + \eta_2[A(x) + (x-a)B(x)]\right), \tag{3.10}$$

$$F(x) \quad := \quad f(x) - \mu(x) + \lambda_1 \int_a^x R(x, y; 0)dy + \lambda_2 \int_a^b R(x, y; 0)dy. \tag{3.11}$$

Proof. Let $\phi^{''}(x) = u(x)$, where the function $x \mapsto u(x)$ is an element in the space $C(I, \mathbb{R})$. So, we have

$$\phi^{'}(x) = \phi^{'}(a) + \int_a^x u(t)dt \tag{3.12}$$

and

$$\phi(x) = \eta_1 + (x - a)\phi^{'}(a) + \int_a^x (x - t)u(t)dt. \tag{3.13}$$

Putting $x = b$ in Eq (3.13) and then using the results of Eqs (3.12) and (3.13) gives

$$\phi^{'}(x) = \frac{1}{(b-a)} \left[(\eta_2 - \eta_1) + \int_a^b H_1(x, t)u(t)dt \right], \tag{3.14}$$

$$\phi(x) = \frac{1}{(b-a)} \left[\eta_1(b - x) + \eta_2(x - a) + \int_a^b H_2(x, t)u(t)dt \right], \tag{3.15}$$

where

$$H_1(x, t) \quad := \quad \begin{cases} (t - a), & a \le t \le x, \\ (t - b), & x \le t \le b, \end{cases}$$
$$H_2(x, t) \quad := \quad \begin{cases} (b - x)(a - t), & a \le t \le x, \\ (a - x)(b - t), & x \le t \le b, \end{cases}$$

and

$$[\phi(x)]^p = \frac{1}{(b-a)^p} \sum_{l=0}^{p} \binom{p}{l} [\eta_1(b - x) + \eta_2(x - a)]^{p-l} \left(\int_a^b H_2(x, t)u(t)dt \right)^l. \tag{3.16}$$

Substitution of Eqs (3.14)–(3.16) into Eq (1.1) gives

$$\begin{aligned} \omega u(x) + \frac{1}{(b-a)} \int_a^b [A(x)H_1(x, t) \quad &+ \quad B(x)H_2(x, t)]u(t)dt - \lambda_1 \int_a^x R(x, y; 1)H_2(y, t)dy \\ &- \quad \lambda_2 \int_a^b R(x, y; 1)H_2(y, t)dy]u(t)dt \\ &= \quad F(x) + \lambda_1 \int_a^x \sum_{l=2}^{p} R(x, y; l) \left(\int_a^b H_2(y, t)u(t)dt \right)^l dy \\ &+ \quad \lambda_2 \int_a^b \sum_{l=2}^{p} R(x, y; l) \left(\int_a^b H_2(y, t)u(t)dt \right)^l dy. \end{aligned}$$

$$
\begin{aligned}
\omega u(x) + \int_a^b [W(x,t) - \lambda_1 \int_a^x & R(x,y;1)H_2(y,t)dy \\
& - \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy]u(t)dt \\
& = F(x) + \lambda_1 \int_a^x \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy \\
& + \lambda_2 \int_a^b \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy,
\end{aligned} \tag{3.17}
$$

where $W(x,t)$, $R(x,y;l)$, $\mu(x)$ and $F(x)$ are defined as shown in Eqs (3.7), (3.8), (3.10) and (3.11) respectively.

The following theorem tells if the NVFIE (3.17) satisfies the conditions (**p.1**)–(**p.4**); then, it has a continuous solution.

Theorem 3.2. *If the NVFIE (3.17) satisfies the conditions (**p.1**) to (**p.4**), then it has a continuous solution.*

Proof. Let $\Gamma_r = \{u \in C(I,\mathbb{R}) : \|u\|_\infty = \sup_{x\in I}|u(x)| \le r\}$. The radius r is a finite positive solution for the equation

$$
\sum_{l=2}^{p} (|\lambda_1|k_1^l + |\lambda_2|)C^*(l)r^l) + [(\alpha + k_1|\lambda_1| + |\lambda_2|)\, C^*(1) - |\omega|]\, r + \|F\|_\infty = 0, \tag{3.18}
$$

where k_1 is an upper bound of $|H_2(x,t)|$.

For $u_1, u_2 \in \Gamma_r$, we define the following two operators from Eq (3.5)

$$
\begin{aligned}
(Tu_1)(x) &= \frac{1}{\omega}F(x) - \frac{1}{\omega}\int_a^b \Big[W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy \\
&\qquad -\lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy\Big]u(t)dt.
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
(Wu_2)(x) &= \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy \\
&\qquad + \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy.
\end{aligned} \tag{3.20}
$$

Now,

$$\begin{aligned}
|(Tu_1)(x)| &\leq \frac{1}{|\omega|}|F(x)| + \frac{r}{|\omega|}\int\limits_a^b |W(x,t)|dt \\
&+\frac{|\lambda_1|r}{|\omega|}\int\limits_a^x\int\limits_a^b |R(x,y;1)||H_2(y,t)|dy \\
&+\tfrac{|\lambda_2|r}{|\omega|}\int\limits_a^b\int\limits_a^b |R(x,y;1)||H_2(y,t)|dy \\
&\leq \tfrac{1}{|\omega|}|F(x)| + \tfrac{\alpha r}{|\omega|} + \frac{k_1|\lambda_1|pr}{|\omega|(b-a)^{p-3}}\int\limits_a^b \frac{|\psi(x,y)|}{|(\eta_1-\eta_2)y+(\eta_2 b-\eta_1 a)|^{1-p}}dy \\
&+\frac{|\lambda_2|pr}{|\omega|(b-a)^{p-3}}\int\limits_a^b \tfrac{|\psi(x,y)|}{|(\eta_1-\eta_2)y+(\eta_2 b-\eta_1 a)|^{1-p}}dy \\
&\leq \frac{1}{|\omega|}|F(x)| + \tfrac{\alpha r}{|\omega|} + (k_1|\lambda_1|+|\lambda_2|)\frac{p(b-a)^{\frac{5}{2}}(d^*(1))^{\frac{1}{2}}r}{|\omega|(2p-1)^{\frac{1}{2}}}\left(\int\limits_a^b(\psi(x,y))^2dy)\right)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\|(Tu_1)(x)\|_\infty \leq \frac{1}{|\omega|}\|F(x)\|_\infty + \frac{1}{|\omega|}(\alpha + (k_1|\lambda_1| + |\lambda_2|)C^*(1))r. \tag{3.21}$$

Using similar arguments as we used above implies

$$\begin{aligned}
|(Wu_2)(x)| &\leq \frac{|\lambda_1|}{|\omega|}\int\limits_a^x\sum_{l=2}^p |R(x,y;l)|\left(\int\limits_a^b |H_2(y,t)u(t)|dt\right)^l dy \\
&+\tfrac{|\lambda_2|}{|\omega|}\int\limits_a^b\sum_{l=2}^p |R(x,y;l|)\left(\int\limits_a^b |H_2(y,t)u(t)|dt\right)^l dy \\
&\leq |\lambda_1|\sum_{l=2}^p\binom{p}{l}\tfrac{(b-a)^{2l+\frac{1}{2}}(d^*(l))^{\frac{1}{2}}(k_1r)^l}{|\omega|(2p-2l+1)^{\frac{1}{2}}}\left(\int\limits_a^b(\psi(x,y))^2dy\right)^{\frac{1}{2}} \\
&+|\lambda_2|\sum_{l=2}^p\binom{p}{l}\tfrac{(b-a)^{2l+\frac{1}{2}}(d^*(l))^{\frac{1}{2}}r^l}{|\omega|(2p-2l+1)^{\frac{1}{2}}}\left(\int\limits_a^b(\psi(x,y))^2dy\right)^{\frac{1}{2}}.
\end{aligned}$$

That is,

$$\|(Wu_2)(x)\|_\infty \leq \frac{1}{|\omega|}\sum_{l=2}^p\left(|\lambda_1|k_1^l + |\lambda_2|\right)C^*(l)r^l. \tag{3.22}$$

Using Eqs (3.21) and (3.22) gives

$$\begin{aligned}\|T(u_1) - W(u_2)\|_\infty &\le \|T(u_1)\|_\infty + \|W(u_2)\|_\infty \\ &\le \frac{1}{|\omega|}\|F(x)\|_\infty + \frac{r}{|\omega|}\left(\alpha + (k_1|\lambda_1| + |\lambda_2|)C^*(1)\right) r \\ &+ \frac{1}{|\omega|}\sum_{l=2}^{p}(|\lambda_1|k_1^l + |\lambda_2|)C^*(l)r^l = r.\end{aligned} \tag{3.23}$$

Therefore,

$$T(u_1) + W(u_2) \in \Gamma_r, \ \forall u_1, u_2 \in \Gamma_r.$$

Now, suppose $x_1 < x_2$ constitute two elements in $[a, b]$. The functions F, W_1 and W_2 are continuous in x from applying the conditions (**p.1**)–(**p.3**) ; therefore, we have

$$\begin{aligned}|(Tu_1)(x_2) - (Tu_1)(x_1)| \le &\frac{1}{|\omega|}|F(x_2) - F(x_1)| \\ &+ \frac{r}{|\omega|(b-a)}\int_a^{x_1} |W_1(x_2, t) - W_1(x_1, t)|dt \\ &+ \frac{r}{|\omega|(b-a)}\int_a^{x_1} |W_2(x_2, t) - W_2(x_1, t)|dt \\ &+ \int_a^{x_1} |W_1(x_2, t) - W_2(x_1, t)|dt \\ &+ \frac{(k_1|\lambda_1| + |\lambda_2|)pr}{|\omega|(b-a)^{p-3}} \\ &\times \int_a^b \frac{|\psi(x_2, y)| - \psi(x_1, y)|}{|(\eta_1 - \eta_2)y + (\eta_2 b - \eta_1 a)|^{1-p}}dy,\end{aligned} \tag{3.24}$$

dy approaches zero whereas x_2 approaches x_1. Also,

$$\begin{aligned}|(Wu_2)(x_2) - (Wu_2)(x_1)| \le &\sum_{l=2}^{p} \frac{k_1^l|\lambda_1| + |\lambda_2|}{|\omega|}\binom{p}{l}(b-a)^{3l-p}r^l \\ &\times \int_a^b \frac{|\psi(x_2, y)| - \psi(x_1, y)|}{|(\eta_1 - \eta_2)y + (\eta_2 b - \eta_1 a)|^{2l-2p}}dy,\end{aligned} \tag{3.25}$$

dy approaches zero whereas x_2 approaches x_1.

Hence, Tu_1 and Wu_2 are elements in the space $C([a, b], \mathbb{R})$. Consequently, the operator $T + W$ is a self-operator on Γ_r. Let u and u^* be any two functions in the set Γ_r. So,

$$\|T(u) - T(u^*)\|_\infty \le \frac{1}{|\omega|}(\alpha + (k_1|\lambda_1| + |\lambda_2|)C^*(1))\|u - u^*\|_\infty. \tag{3.26}$$

Therefore, the operator T is a contraction operator on Γ_r according to the condition (**p.4**). Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in \Gamma_r$ such that u_n approaches u when n aproaches ∞. It is clear that $u \in \Gamma_r$ and $\sup\limits_{x\in[a,b]} |u_n(x)| \leq r, \forall n \in \mathbb{N}$. Applying the Arzela convergence theorem implies

$$\begin{aligned}
\lim_{n\to\infty} |(Wu_n)(x) - (Wu)(x)| &\leq \sum_{l=2}^{p} \frac{(k_1^l|\lambda_1| + |\lambda_2|)}{|\omega|} \lim_{n\to\infty} \int_a^b |R(x,y;l)| \\
&\quad \times \left| \left(\int_a^b H_2(y,t)u_n(t)dt \right)^l - \left(\int_a^b H_2(y,t)u(t)dt \right)^l \right| \\
&\leq \sum_{l=2}^{p} \frac{(k_1^l|\lambda_1| + |\lambda_2|)}{|\omega|} e(l) \int_a^b |R(x,y;l)| \\
&\quad \times \left(\int_a^b |H_2(y,t)| \lim_{n\to\infty} |u_n - u| dt \right) dy = 0,
\end{aligned} \tag{3.27}$$

where $e(l)$ is a finite positive constant dependent on l. Therefore, the operator W is a sequentially continuous operator on Γ_r; hence, it is continuous on Γ_r. It is clear from Eq (3.22) that

$$\forall W_{u_n} \in \sup_{x\in[a,b]} |(Wu_2)(x)| \leq \sum_{l=2}^{p} \frac{(k_1^l|\lambda_1| + |\lambda_2|)}{|\omega|} C^*(l) r^l,$$

hence, the set $W\Gamma_r$ is uniformly bounded. Consider the sequence $(W_{u_n})_{n\in\mathbb{N}}$ with $(W_{u_n}) \in W\Gamma_r$.

Using similar steps as we followed in Eq (3.25) implies

$$|(Wu_2)(x_2) - (Wu_2)(x_1)| < \epsilon, \quad \forall n \in \mathbb{N} \text{ when } |x_2 - x_1| < \delta.$$

Therefore, there exists a sub-sequence $\{W_{u_n k}\}_{k\in\mathbb{N}}$ which converges uniformly in $W\Gamma_r$ as a result of applying the Arzela-Ascoli theorem; consequently, the set $W\Gamma_r$ is compact. As a result, the operator W is completely continuous. Now all conditions of the Krasnoselskii theorem are satisfied; therefore, the operator $T + W$ has at least one fixed point in the set Γ_r which is a solution for the NVFIE (3.17). The proof is completed.

Theorem 3.3. *If the NVFIE (3.17) satisfies the conditions (**p.1**), (**p.2**), (**p.3**) and (**p.5**), then it has a unique solution.*

Proof. It is clear that the operator $T + W$ is a self-operator on Γ_r. Using similar steps as we have used in Eq (3.26) leads to

$$||W(v) - W(v^*)||_\infty \leq \frac{1}{|\omega|}(\alpha + (|\lambda_1| + |\lambda_2|) \sum_{l=2}^{p} \frac{e(l)C^*(l)}{(b-a)^{3l-3}})||v - v^*||_\infty, \qquad \forall v, v^* \in \Gamma_r. \tag{3.28}$$

Using Eq (3.26) with the constant $k_1 = 1$ and Eq (3.28) leads to

$$\begin{aligned}\|(T+W)(v)-(T+W)(v^*)\|_\infty &\leq \|(T)(v)-(T)(v^*)\|_\infty + \|(W)(v)-(W)(v^*)\|_\infty \\ &\leq \frac{1}{|\omega|}(\alpha + (|\lambda_1| + |\lambda_2|)C^*(1))\|v-v^*\|_\infty \\ &\quad + \sum_{l=2}^{p} \frac{1}{|\omega|}(\alpha + (|\lambda_1| + |\lambda_2|)\tfrac{e(l)C^*(l)}{(b-a)^{3l-3}})\|v-v^*\|_\infty \\ &\leq \frac{1}{|\omega|}(\alpha + (|\lambda_1| + |\lambda_2|)\Lambda)\|v-v^*\|_\infty.\end{aligned}$$

$$\|(T+W)(v)-(T+W)(v^*)\|_\infty \leq \|v-v^*\|_\infty. \tag{3.29}$$

So, the operator $T + W$ is a contraction on Γ_r according to the condition (**p.5**), consequently, the NVFIE (3.17) possesses a unique continuous solution in Γ_r based on application of the Banach fixed point theorem. The proof is completed.

4. MADM for the NVFIE

This section is devoted to using the MADM to find an approximate solution to the NVFIE (3.17) which is subject to satisfying the conditions of Theorem 3.3. Assume the unknown function $u(x)$ of Eq (3.17) can be approximated by using the formula

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{4.1}$$

Let $F(x) = F_1(x) + F_2(x)$. Then

$$u_0(x) = \frac{1}{\omega} F_1(x), \tag{4.2}$$

$$\begin{aligned}u_1(x) =& \frac{1}{\omega}F_2(x) - \frac{1}{\omega}\left\{\int_a^b \left[W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy - \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy\right] u_0(t)dt\right\} \\ &+ \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)A_0(y,t)dy + \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)A_0(y,t)dy,\end{aligned} \tag{4.3}$$

$$\begin{aligned}u_n(x) =& -\frac{1}{\omega}\left\{\int_a^b \left[W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy - \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy\right] u_{n-1}(t)dt\right\} \\ &+ \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)A_{n-1}(y,t)dy + \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)A_{n-1}(y,t)dy \quad \forall n \geq 2,\end{aligned} \tag{4.4}$$

where the Adomain polynomial, A_n for $n = 0, 1, 2, \ldots$ is evaluated by using the equation as follows:

$$A_n(u_0(x), u_1(x), \ldots, u_n(x), y; l) = \frac{1}{n!}\left[\frac{d^n}{d\rho^n}\left(\int_a^b H_2(y,t)\sum_{i=0}^{\infty}\rho^i u_i(t)dt\right)^l\right]_{|\rho=0}. \tag{4.5}$$

Theorem 4.1. *The approximate solution determined by Eqs (4.2)–(4.4) for the NVFIE (3.17) converges to the exact solution $u(x)$ while satisfying the conditions of Theorem 3.3.*

Proof. Let $\{S_k(x)\}$ be a sequence of partial sums where

$$S_k(x) = \sum_{i=0}^{k} u_i(x). \tag{4.6}$$

Let $n, m \in \mathbb{Z}^+$ such that $n > m \geq 1$. Then

$$\begin{aligned}
\|S_n(x) - S_m(x)\|_\infty &= |\sum_{i=m+1}^{n} u_i(x)| \\
&\leq \frac{1}{|\omega|}\int_a^b |W(x,t)\sum_{i=m}^{n-1} u_i(t)|dt + \frac{|\lambda_1|}{|\omega|}\int_a^b\int_a^x |R(x,y;1)H_2(y,t)\sum_{i=m}^{n-1} u_i(t)|dydt \qquad (4.7)\\
&+\frac{|\lambda_2|}{|\omega|}\int_a^b\int_a^b |R(x,y;1)H_2(y,t)\sum_{i=m}^{n-1} u_i(t)|dydt \\
&+\frac{|\lambda_1|}{|\omega|}\int_a^x \sum_{l=2}^{p} |R(x,y;l)\sum_{i=m}^{n-1} A_{n-1}(y,t)|dydt \\
&+\frac{|\lambda_2|}{|\omega|}\int_a^b \sum_{l=2}^{p} |R(x,y;l)\sum_{i=m}^{n-1} A_{n-1}(y,t)|dydt \\
&\leq \frac{\alpha}{\omega}\|S_{n-1} - S_{m-1}\|_\infty + \frac{|\lambda_1|}{|\omega|}(b-a)^3\int_a^b |R(x,y;1)\sum_{i=m}^{n-1} u_i(t)|dt \\
&+\frac{|\lambda_2|}{|\omega|}(b-a)^3\int_a^b |R(x,y;1)\sum_{i=m}^{n-1} u_i(t)|dt + \frac{|\lambda_1| + |\lambda_2|}{|\omega|}(b-a)^3 e(l) \\
&\leq \frac{1}{|\omega|}(\alpha + (|\lambda_1| + |\lambda_2|)\Lambda)\|S_{n-1} - S_{m-1}\|_\infty. \qquad (4.8)
\end{aligned}$$

Let $\theta = \dfrac{(\alpha + (|\lambda_1| + |\lambda_2|)\Lambda)}{|\omega|}$ and $\theta < 1$. Then

$$\|S_n(x) - S_m(x)\|_\infty \leq \theta\|S_{n-1}(x) - S_{m-1}(x)\|_\infty. \tag{4.9}$$

Take $n = m + 1$; then,

$$\begin{aligned}
\|S_{m+1} - S_m\|_\infty &\leq \theta\|S_m(x) - S_{m-1}(x)\|_\infty \\
&\leq \theta^2\|S_{m-1}(x) - S_{m-2}(x)\|_\infty \\
&\leq \cdots \leq \theta^m\|S_1(x) - S_0(x)\|_\infty \\
&= \theta^m\|u_1\|_\infty.
\end{aligned} \tag{4.10}$$

Substituting the inequality of Eq (4.10) into the inequality of Eq (4.9), after applying the triangle inequality and setting $n > m > N \in \mathbb{N}$, we get

$$\|S_n - S_m\|_\infty \leq \frac{\theta^n}{1-\theta}\|u_1\|_\infty = \epsilon,$$

where

$$\lim_{n\longrightarrow\infty} \theta^n = 0.$$

Therefore,

$$\|S_n - S_m\|_\infty < \epsilon, \quad \forall n, m \in \mathbb{N}.$$

So, the sequence $S_n(x)$ is a Cauchy sequence in the Banach space $C(I, \mathbb{R})$.

Therefore,

$$\lim_{n\longrightarrow\infty} S_n(x) = u(x).$$

5. HAM for the NVFIE

This section is devoted to applying the HAM to the NVFIE (3.17) while satisfying the conditions of Theorem 3.3 from Eq (1.2) as follows:

$$\begin{aligned} u(x) &+ \frac{1}{\omega}\Big(\int_a^b [W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy \\ &- \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy]u(t)dt\Big) \\ &- \frac{1}{\omega}F(x) - \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy \\ &- \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)u(t)dt\right)^l dy = 0. \end{aligned} \tag{5.1}$$

We define the nonlinear operator N by

$$\begin{aligned}N[u(x)] = u(x) + \frac{1}{\omega}(\int_a^b [W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy \\ - \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy]u(t)dt) \\ - \frac{1}{\omega}F(x) - \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)(\int_a^b H_2(y,t)u(t)dt)^l dy \\ - \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)(\int_a^b H_2(y,t)u(t)dt)^l dy.\end{aligned} \tag{5.2}$$

From Eqs (5.1) and (5.2) we have

$$N[u(x)] = 0, \forall x \in I. \tag{5.3}$$

We define the homotopy of the unknown function $u(x)$ as below

$$\chi^*[\phi(x;\hbar,\iota)] = (1-\iota)\mathcal{L}(\phi(x;\hbar,\iota) - u_0(x)) - \iota\hbar N[\phi(x;\hbar,\iota)], \tag{5.4}$$

where

(1) the function $u_0(x)$ is the initial approximation of the unknown solution $u(x)$;

(2) the parameter $\hbar \in \mathbb{R} - 0$ is used as a control tool to manage the convergence of the proposed technique;

(3) the parameter $\iota \in [0,1]$ is an embedding in Eq (5.4) and called the homotopy parameter;

(4) the operator $\mathcal{L}$ is an auxiliary linear operator satisfying the property $\mathcal{L}[\iota(x)] = 0$ when $\iota(x) = 0$;

(5) the operator N denotes Eq (5.2), that is

$$
\begin{aligned}
N[\phi(x;\hbar,\iota)] &= \phi(x;\hbar,\iota) + \frac{1}{\omega}\Big(\int_a^b [W(x,t) - \lambda_1 \int_a^x R(x,y;1)H_2(y,t)dy \\
&- \lambda_2 \int_a^b R(x,y;1)H_2(y,t)dy]\phi(t;\hbar,\iota)dt\Big) - \frac{1}{\omega}F(x) \\
&- \frac{\lambda_1}{\omega}\int_a^x \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)\phi(t;\hbar,\iota)dt\right)^l dy \\
&- \frac{\lambda_2}{\omega}\int_a^b \sum_{l=2}^{p} R(x,y;l)\left(\int_a^b H_2(y,t)\phi(t;\hbar,\iota)dt\right)^l dy.
\end{aligned} \tag{5.5}
$$

$$\chi^*[\phi(x;\hbar,\iota)] = 0. \tag{5.6}$$

Solving Eq (5.6) yields

$$(1-\iota)\mathcal{L}[\phi(x;\hbar,\iota) - u_0(x)] = \iota\hbar N[\phi(x;\hbar,\iota)].$$

$$u(x) = u_0(x) + \sum_{k=1}^{\infty} u_k(x) = \sum_{k=0}^{\infty} u_k(x) \tag{5.7}$$

where

$$u_k(x) = \frac{1}{k!}\frac{\partial^k \phi(x;\hbar,\iota)}{\partial \iota^k}|_{\iota=0}, \tag{5.8}$$

$$u_1(x) = \hbar \mathbf{R_1}[u_0(x)], \tag{5.9}$$

$$u_n(x) = u_{(n-1)}(x) + \hbar \mathbf{R_n}[u_{(n-1)}(x)], \quad \forall n \geq 2, \tag{5.10}$$

$$u_{(n-1)}(x) = (u_0(x), u_1(x), \ldots, u_{n-1}(x)), \tag{5.11}$$

$$\mathbf{R_n}[u_{n-1}(x)] = \frac{1}{(n-1)!}\left[\frac{\partial^{n-1}}{\partial \iota^{n-1}} N\left(\sum_{i=0}^{\infty} u_i(x)\iota^i\right)|_{\iota=0}\right]. \tag{5.12}$$

6. Numerical examples

In this section, several examples demonstrate the accuracy and efficiency of the proposed methods under the conditions of Theorems 3.2 and 3.3. All of them were performed on a computer using programs written in Matlab. It also contains a numerical comparison of the MADM and HAM. We report in tables the values of the exact solutions, approximate solutions and the ∞−norm of the error that was calculated at certain considered points; some figures might be included with each example for clarification.

Example 6.1. *Consider the following boundary value problem:*

$$4\psi''(x) + 2x\psi'(x) + \psi(x) = \frac{-1}{100}\left(\frac{(1+x)(1+x^{10})}{10}\right) + 48x^2 + 9x^4 \\ + 0.01\int_0^x (t+tx)\psi^2(t)dt + 0.01\int_0^1 (t+tx)\psi^2(t)dt, \tag{6.1}$$

where $x \in [0,1], \psi(0) = 0$, *and* $\psi(1) = 1$ *and the exact solution is* $\psi(x) = x^4$.

Applying $u(x) = \psi''(x)$, *we get an NVFIE with the form of Eq (3.5). Since the kernel* $k(x,t) = t + tx$ *is a real valued continuous function in* $x\,\forall t \in [0,x]$ *and* $\left(\int_0^1 (t+tx)^2 dt\right)^{\frac{1}{2}} = \frac{1+x}{\sqrt{3}} \leq \frac{2}{\sqrt{3}}\ \forall x \in [0,1]$, *it is clear that* $(\alpha + (|\lambda_1| + |\lambda_2|)C^*(1)) = 3.266$. *So Eq (3.5) has a solution from Theorem 3.2.*

Table 1 presents the $\infty-$*norm of the absolute errors between the exact solution and the approximate solutions obtained by using the MADM and HAM with* $\hbar = -0.089609$, -0.089608 *and* -0.089607 *where the initial value of* u_0 *is* $\dfrac{x^4 - x^2}{100}$.

Table 1. Exact solution u_e of Example (6.1) along with the approximate solution $S_4(x)$ obtained by using the MADM; u_M, and HAM; u_H, where $S_n(x) = \sum_{i=0}^{n} u_i(x)$ as in Theorem 3.3. Here u_M corresponds to an approximate solution $u(x)$ obtained by using the MADM and u_H corresponds to an approximate solution $u(x)$ obtained by using the HAM with varying values of $\hbar$.

x	u_e	u_M	u_H $\hbar = -0.089609$	u_H $\hbar = -0.089608$	u_H $\hbar = -0.089607$	$\|\|u_e - u_M\|\|$	$\|\|u_e - u_H\|\|$ $\hbar = -0.089607$
0	0	-0.000080410	-0.000223803	-0.000223800	-0.000223798	0.000080410	0.000223798
0.2	0.48	0.483369160	0.307489406	0.307485869	0.307482332	0.003369160	0.172517667
0.4	1.92	1.925978977	1.517787774	1.517770649	1.517753524	0.005978977	0.402246475
0.6	4.32	4.325333773	3.739156228	3.739114352	3.739072477	0.005333773	0.580927522
0.8	7.68	7.671674552	7.150736302	7.150656697	7.150577092	0.008325447	0.529422907
1	12	11.939887850	12.000329477	12.000196681	12.000063885	0.060112149	0.000063885

Figure 1 illustrates the absolute errors of the MADM and HAM, where $\hbar = -0.089607$, *corresponding to the exact solution at any considered point in Table 1.*

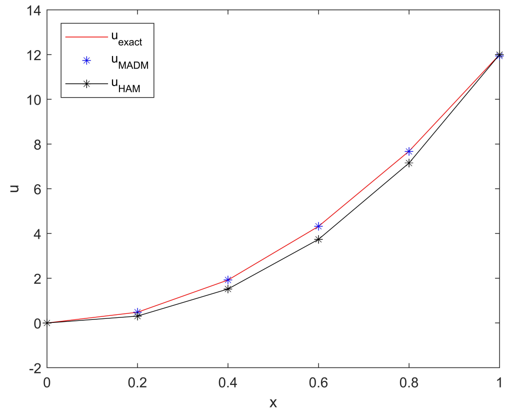

Figure 1. Comparison of exact and approximate solutions of Example (6.1).

The values of $\hbar$ that ensure the convergence of the approximate solution are represented in Figure 2.

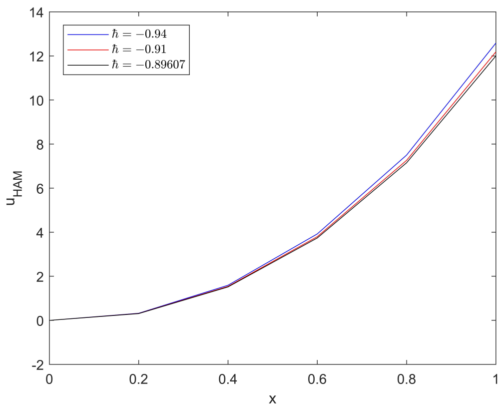

Figure 2. Illustration of the optimal value of the control parameter $\hbar_n$ corresponding to $S_4(x)$.

where

Table 2. Values of $\hbar$ that ensure the convergence of the approximate solution

x	u_H $\hbar = -0.94$	u_H $\hbar = -0.91$	u_H $\hbar = -0.089607$
0	-0.000234746223448	-0.000227270037349	-0.000223798068881
0.2	0.323041090007669	0.312411224185282	0.307482332540245
0.4	1.593021768935646	1.541612692662778	1.517753524195289
0.6	3.923057382146835	3.797407545034087	3.739072477211836
0.8	7.500234778270934	7.261462205176178	7.150577092704081
1	12.583195260825082	12.185024506254976	12.000063885259873

Example 6.2. *Consider the following boundary value problem:*

$$8\psi''(x) = 8(\cos(x) - sin(x)) - \frac{1}{50}\left(1 - 3\cos(x) - 2\sin(x) + \frac{\sin(2x)}{3}\right) + 48x^2 + 9x^4$$
$$+ 0.02\int_0^x \sin(x-t)\psi^2(t)dt + 0.02\int_0^\pi \sin(x-t)\psi^2(t)dt, \tag{6.2}$$

where $x \in [0, \pi]$, $\psi(0) = -1$, *and* $\psi(\pi) = 1$ *and the exact solution is* $\psi(x) = \sin(x) - \cos(x)$.

Applying $u(x) = \psi''(x)$, *we get an NVFIE with the form of Eq (3.5). Since the kernel* $k(x,t) = \sin(x-t)$ *is a real valued continuous function in* $x\,\forall t \in [0,x]$ *and* $\left(\int_0^\pi \sin(x-t)^2 dt\right)^{\frac{1}{2}} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} = \gamma\ \forall x \in [0,\pi]$, *the value of* $(\alpha + (|\lambda_1| + |\lambda_2|)C^*(1)) = 1.012664$. *So Eq (3.5) has a solution from Theorem 3.2.*

Table 3 presents the $\infty-$*norm of the absolute errors between the exact solution and the approximate solutions obtained by using the MADM and HAM with* $\hbar = -0.16579$, -0.16578 *and* -0.16577 *where the initial value of* u_0 *is* $\dfrac{\cos(x)}{200}$.

Table 3. Exact solution u_e of Example (6.2) along with the approximate solution $S_3(x)$ using MADM and HAM where $S_n(x) = \sum_{i=0}^{n} u_i(x)$ as in Theorem 3.3. See the caption of Table 1 for the description of u_M and u_H.

x	u_e	u_M	u_H $\hbar = -0.16579$	u_H $\hbar = -0.16578$	u_H $\hbar = -0.16577$	$\|u_e - u_M\|$	$\|u_e - u_H\|$ $\hbar = -0.16577$
0	1.0000000000	1.0131227836	1.0131227836	1.0131227836	1.0004121726	0.0131227836	0.0004121726
0.6283185307	0.2212317420	0.2337366092	0.2337366092	0.2337366092	0.2288946915	0.0125048671	0.0076629494
1.2566370614	-0.6420395219	-0.6363285798	-0.6363285798	-0.6363285798	-0.6329909938	0.0057109420	0.0090485280
1.8849555921	-1.2600735106	-1.2601671258	-1.2601671258	-1.2601671258	-1.2543201179	0.0000936151	0.0057533927
2.5132741228	-1.3968022466	-1.3943052273	-1.3943052273	-1.3943052273	-1.3947135856	0.0024970193	0.0020886610
3.1415926535	-1.0000000000	-0.9862687835	-0.9862687835	-0.9862687835	-0.9984737858	0.0137312164	0.0015262141

Figure 3 illustrates the absolute errors of the MADM and HAM corresponding to the exact solution at any considered point in Table 3.

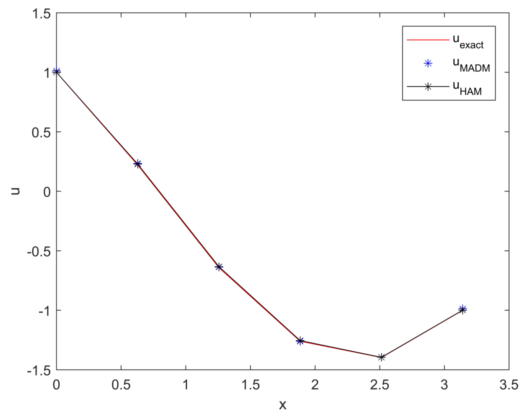

Figure 3. Comparison of the exact and approximate solutions of Example (6.2).

The values of ℏ that ensure the convergence of the approximate solution are represented in Figure 4.

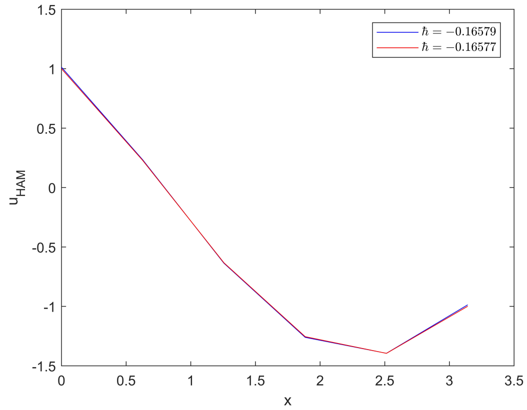

Figure 4. Illustration of the optimal value of the control parameter $\hbar_n$ corresponding to $S_3(x)$.

Example 6.3. *Consider the following boundary value problem:*

$$1000\psi''(x) + \psi(x) = 1001e^{-x} - \left(\frac{-3e^{-3x} + 2e^{x}}{8} + \frac{2e^{-x}}{4}\right) - \left(\frac{-e^{x-4}}{8} - \frac{e^{-x-2}}{4}\right) + \int_0^x \cosh(x-t)\psi^3(t)dt + \int_0^1 \cosh(x-t)\psi^3(t)dt, \tag{6.3}$$

where $x \in [0, 1]$, $\psi(0) = 1$ *and* $\psi(1) = e^{-1}$ *and the exact solution is* $\psi(x) = e^{-x}$.

The value of $(\alpha + (|\lambda_1| + |\lambda_2|)C^*(1)) \approx 7.5$. *So Eq (3.5) has a solution from Theorem 3.2. And since the value of* $\frac{(\alpha+(|\lambda_1|+|\lambda_2|)C^*(1))}{\omega}$ *is close to zero, the approximate solution approaches the* u_e *solution rapidly.*

Table 4 presents the $\infty-$*norm of the absolute errors between the exact solution and the approximate solutions obtained by using the MADM and HAM with* $\hbar = -0.16655$, -0.16654 *and* -0.16653 *where the initial value of* u_0 *is* $\dfrac{x}{1000}$.

Table 4. Exact solution u_e of Example (6.3) along with the approximate solution $S_3(x)$ using MADM and HAM. See the caption of Table 1 for the description of u_M and u_H.

x	u_e	u_M	u_H $\hbar = -0.16655$	u_H $\hbar = -0.16654$	u_H $\hbar = -0.16653$	$\|\|u_e - u_M\|\|$	$\|\|u_e - u_H\|\|$ $\hbar = -0.16655$
0	1.0000000000	1.0006689598	1.0001208391	1.0000607907	1.0000007423	-0.0006689598	-0.0001208391
0.2	0.8187307530	0.8189818777	0.8185775678	0.8185284312	0.8184792946	-0.0002511246	0.0001531852
0.4	0.6703200460	0.6702463215	0.6700094163	0.6699692118	0.6699290073	0.0000737244	0.0003106296
0.6	0.5488116360	0.5484759072	0.5484108756	0.5483779840	0.5483450925	0.0003357288	0.0004007604
0.8	0.4493289641	0.4487727932	0.4488794418	0.4488525383	0.4488256349	0.0005561708	0.0004495223
1	0.3678794411	0.3671295391	0.3674095283	0.3673875286	0.3673655288	0.0007499019	0.0004699128

Figure 5 illustrates the absolute errors for the MADM and HAM corresponding to the exact solution at any considered point in Table 4.

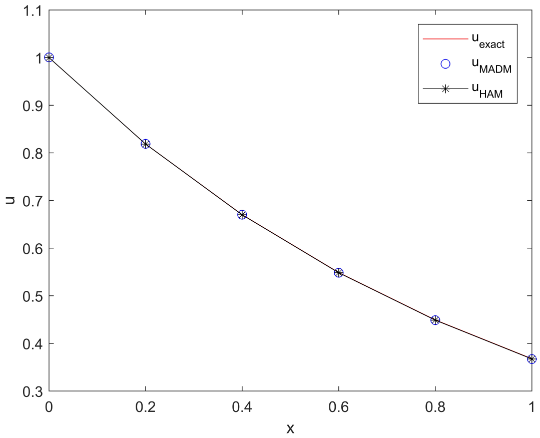

Figure 5. Comparison of the exact and approximate solutions of Example (6.3).

The values of $\hbar$ (see Table 5), that ensure the convergence of the approximate solution are represented in Figure 6.

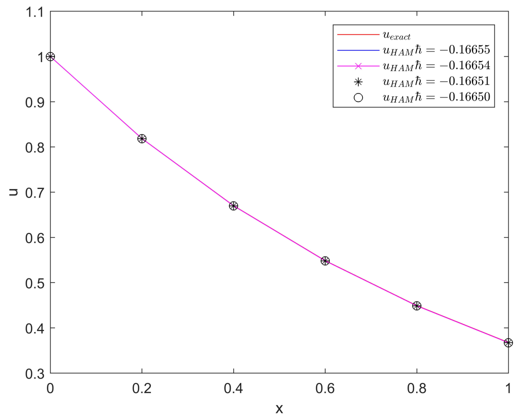

Figure 6. Illustration of the optimal value of the control parameter $\hbar_n$ corresponding to $S_3(x)$.

Table 5. Results for the approximate solution u_H, as obtained via the HAM, for Example (6.3) corresponding to the value of $\hbar$.

x	u_e	u_H $\hbar = -0.16655$	u_H $\hbar = -0.16654$	u_H $\hbar = -0.16651$	u_H $\hbar = -0.16650$
0	1.0000000000	1.0001208391	1.0000607907	0.9998806455	0.9998205971
0.2	0.8187307530	0.8185775678	0.8185284312	0.8183810213	0.8183318847
0.4	0.6703200460	0.6700094163	0.6699692118	0.6698485984	0.6698083939
0.6	0.5488116360	0.5484108756	0.5483779840	0.5482793095	0.5482464180
0.8	0.4493289641	0.4488794418	0.4488525383	0.4487718280	0.4487449246
1	0.3678794411	0.3674095283	0.3673875286	0.3673215294	0.3672995296

7. Conclusions

If the value of $\dfrac{(\alpha + (|\lambda_1| + |\lambda_2|)C^*(1))}{|\omega|}$ is near zero, then the approximate solution approaches the exact solution rapidly. As illustrated through the examples, the MADM gives a better approximate solution in case of a separate kernel of polynomial functions (Example 6.1), while for the case of different kernels of trigonometric functions the HAM gives the better approximate solution (Examples 6.2 and 6.3). In terms of the running time, in both cases, the HAM is significantly faster than the MADM.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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