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Research article

Some new Grüss inequalities associated with generalized fractional derivative

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Abstract: In this paper, we prove several new integral inequalities for the k-Hilfer fractional derivative operator, which is a fractional calculus operator. As a result, we have a whole new set of fractional integral inequalities. For the generalized fractional derivative, we also use Young's inequality to find new forms of inequalities. Such conclusions for this novel and generalized fractional derivative are extremely useful and valuable in the domains of differential equations and fractional differential calculus, both of which have a strong connections to real-world situations. These findings may stimulate additional research in a variety of fields of pure and applied sciences.

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1. Introduction

The study of fractional order integral and derivative operators is known as fractional calculus. This subject is as important as calculus itself, and it has played a considerable role in recent decades (see, for example, [1–4, 9–12, 14, 15]). Fractional calculus is used in a variety of fields, including engineering, science, finance, applied mathematics and bioengineering. Mathematical inequalities play a significant role in the study of mathematics and many related subjects, and their applications are diverse. In the case of fractional partial differential equations, fractional integral inequalities are useful in determining the uniqueness of solutions. They also give upper and lower boundaries for fractional boundary value problem solutions. These recommendations have led various researchers in the field of integral inequalities to inquire into certain extensions by involving fractional calculus operators. More information related to the subject can be found in the following research articles: [8, 13, 16].

The Grüss inequality is in fact a connection between the integral of the product of two functions and the product of their integrals. The continuous and discrete cases of Grüss-type variants play a considerable role in examining the qualitative conduct of differential and integral equations. Our main purpose is to show some new and modified versions of the Grüss inequality by using a generalized k-fractional derivative. Such new versions of the inequalities are supposed to be vital and the exploration has continued to develop investigations for such kinds of variants.

The Grüss inequality is one of the most fascinating inequalities amongst the field of inequalities and is stated in the next theorem.

Theorem 1.1. *[6] Let $\mathfrak{R}$ be a set of real numbers, $m, M, n, N \in \mathfrak{R}$, and $\Omega, \Upsilon : [\tau_1, \tau_2] \to \mathfrak{R}$ be two positive functions such that $m \le \Omega(\xi) \le M$, $n \le \Upsilon(\xi) \le N$, for $\xi \in [\tau_1, \tau_2]$. Then,*

$$\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Omega(\xi)\Upsilon(\xi)d\xi - \frac{1}{(\tau_2 - \tau_1)^2} \int_{\tau_1}^{\tau_2} \Omega(\xi)d\xi \int_{\tau_1}^{\tau_2} \Upsilon(\xi)d\xi \right| \le \frac{1}{4}(M - m)(N - n), \tag{1.1}$$

where the constant $\frac{1}{4}$ cannot be improved.

Diaz et al. in [5] originated the definition of the gamma k-function, which is defined as:

Definition 1.1. *The Γ_k function is the generalization of the classical Γ function and is defined as follows.*

$$\Gamma_k(t) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{t}{k}-1}}{(t)_{n,k}},\ k > 0,\ \Re(t) > 0,$$

where $(t)_{n,k} = t(t+k)(t+2k)\ldots,(t+(n-1)k)$, $n \ge 1$, is called the Pochhammer k symbol. The integral representation is given by

$$\Gamma_k(t) = \int_0^\infty x^{t-1}e^{\frac{-x^k}{k}}dx,\ \Re(t) > 0. \tag{1.2}$$

Clearly, for $k = 1$, $\Gamma_1(t) = \Gamma(t)$.

We denote by $L_p[a, b]$ the set of those Lebesgue complex-valued measurable functions on a finite or infinite interval of real number set $\mathbb{R}$, denote L^p-space with the power weight consist of those complex-valued Lebesgue measurable function f on (a, b) for the absolute value of f is finite, and denote by $AC^n[a, b]$ the space of complex-valued functions f which have continuous derivatives up to order $(n-1)$ on $[a, b]$ such that $f^{n-1}(x)$ belongs to $AC[a, b]$.

The next definition is presented in [7].

Definition 1.2. *Let $f \in L_1[a,b]$, $k>0$, $f * K_{(1-\eta)(1-\xi)} \in AC^1[a,b]$. The k- fractional derivative operator ${}^kD^{\xi,\eta}_{a+}f$ of order $0<\xi<1$ and type $0<\eta\leq 1$ with respect to $x\in[a,b]$ is defined by*

$$\left({}^kD^{\xi,\eta}_{a+}f\right)(x) := I^{\eta(1-\xi)}_{a+,k}\frac{d}{dx}\left(I^{(1-\eta)(1-\xi)}_{a+,k}f(x)\right), \tag{1.3}$$

whenever the right hand side exists. The derivative (1.3) is usually called the Hilfer k-fractional derivative. The more general integral representation of Eq (1.3) is defines as follows.
Let $f \in L^1[a,b], f * K_{(1-\eta)(n-\xi)} \in AC^n[a,b], n-1<\xi<n, 0<\eta\leq 1, n\in\mathbb{N}$; then, the following equation holds true:

$$\left({}^kD^{\xi,\eta}_{a+}f\right)(x) = I^{\eta(n-\xi)}_{a+,k}\frac{d^n}{dx^n}\left(I^{(1-\eta)(n-\xi)}_{a+,k}f(x)\right). \tag{1.4}$$

Applying the properties of the Riemann-Liouville fractional integral, the relation (1.4) can be rewritten in the form

$$\begin{aligned}\left({}^kD^{\xi,\eta}_{a+}f\right)(x) &= I^{\eta(n-\xi)}_{a+,k}\left(\left(D^{n-(1-\eta)(n-\xi)}_{a+,k}f\right)(x)\right)\\ &= \frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^x (x-y)^{\frac{\eta(n-\xi)}{k}-1}\left(\left(D^{\xi+\eta(n-\xi)}_{a+,k}f\right)(y)\right)dy.\end{aligned} \tag{1.5}$$

From the derivative (1.4), we obtain different classical fractional derivatives,

(i) By setting $k=1$, we get the Hilfer fractional derivative presented in [7].
(ii) By Setting $k=1, \eta=0$, $D^{\xi,0}_{a+}f = D^{\xi}_{a+}f$, we arrive at the Riemann- Liouville fractional derivative of order ξ given in [17].
(iii) By Setting $k=1, \eta=1, n=1$, it is a Caputo fractional derivative $D^{\xi,1}_{a+}f = ^C D^{\xi}_{a+}f$ of order ξ provided in [11].

2. Main results

The first main result is given in the next theorem.

Theorem 2.1. *Let $k>0$ and $(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)$ be a positive function on $[0,\infty)$, and let $\left({}^kD^{\xi,\eta}_{a+}f\right)$ denote the Hilfer k-fractional derivative of order ξ, $0<\xi<1$, and type $0<\eta\leq 1$. Suppose that there exist $(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_1), (D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_2)$ such that*

$$(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_1)(\xi) \leq (D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\xi) \leq (D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_2)(\xi), \tag{2.1}$$

for all $\xi\in[0,\infty)$. Then,

$$\begin{aligned}&\left({}^kD^{\xi,\eta}_{a+}\Psi_1\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi) + \left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)\\ \geq\ &\left({}^kD^{\xi,\eta}_{a+}\Psi_1\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi) + \left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi).\end{aligned} \tag{2.2}$$

Proof. Using (2.1) for all $\gamma\geq 0$, $\delta\geq 0$, we have

$$[(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_2)(\gamma) - (D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\gamma)]$$

$$\times \quad [(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)] \geq 0,$$

and then

$$\begin{aligned} & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) \\ \geq \ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta). \end{aligned} \tag{2.3}$$

If we multiply by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides of (2.3) and integrate the resulting identity for the variable γ over the interval (a, ξ), we get

$$\begin{aligned} & (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)d\gamma \\ + \ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)d\gamma \\ \geq \ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)d\gamma \\ + \ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)d\gamma, \end{aligned}$$

which can be written as follows by applying (1.5):

$$\begin{aligned} & (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi) \\ \geq \ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)\left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi). \end{aligned} \tag{2.4}$$

Now, multiplying by $\frac{(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides of (2.4) and integrating the resulting identity for the variable δ on the interval (a, ξ), we get

$$\begin{aligned} & \left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi) \\ \geq \ & \left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi). \end{aligned}$$

This completes the proof. □

Corollary 2.1. *Let $m, M \in \Re$, with $m < M$, and $k, \xi > 0$. Let $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)$ be a positive function such that $m \leq (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi) \leq M$. Then,*

$$mZ_k(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi) + MZ_k(\xi){}^kD_{a+}^{\xi,\eta}\Omega(\xi) \geq mMZ_k(\xi)Z_k(\xi) + ({}^kD_{a+}^{\xi,\eta}\Omega(\xi))({}^kD_{a+}^{\xi,\eta}\Omega)(\xi),$$

where

$$Z_k(\xi) = \frac{(\xi-a)^{\frac{\eta(n-\xi)}{k}}}{\Gamma_k(\eta(n-\xi)+k)}. \tag{2.5}$$

Remark 2.1. *Take* $k=1, \eta=0$, $D_{a+}^{\xi,0}\Omega = D_{a+}^{\xi}\Omega$, $D_{a+}^{\xi,0}\Psi_1 = D_{a+}^{\xi}\Psi_1$, $D_{a+}^{\xi,0}\Psi_2 = D_{a+}^{\xi}\Psi_2$, *and Theorem 2.1 and Corollary 2.1 we get Theorem 2 and Corollary 3 of [18], respectively.*

Theorem 2.2. *Let* $k>0$, *and let* $\left({}^kD_{a+}^{\xi,\eta}f\right)$ *denote the Hilfer k-fractional derivative of order* ξ, $0<\xi<1$, *and type* $0<\eta\leq 1$, $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)$. *Let* $(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)$ *be two positive functions on* $[0,\xi)$. *Suppose that (2.1) holds and there exist integrable functions* $(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)$ *and* $(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)$ *on* $[0,\xi)$ *such that*

$$(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\xi) \leq (D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\xi) \leq (D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)(\xi). \tag{2.6}$$

Then, the following inequalities hold:

$$\begin{aligned}(a)\quad &\left({}^kD_{a+}^{\xi,\eta}\varphi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi)\\ &\geq \left({}^kD_{a+}^{\xi,\eta}\varphi_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi),\end{aligned}$$

$$\begin{aligned}(b)\quad &\left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\varphi_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\\ &\geq \left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\varphi_2\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi),\end{aligned}$$

$$\begin{aligned}(c)\quad &\left({}^kD_{a+}^{\xi,\eta}\Psi_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\varphi_2\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi)\\ &\geq \left({}^kD_{a+}^{\xi,\eta}\Theta_2\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\varphi_2\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi),\end{aligned}$$

$$\begin{aligned}(d)\quad &\left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\varphi_1\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi)\\ &\geq \left({}^kD_{a+}^{\xi,\eta}\Psi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\varphi_1\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi).\end{aligned}$$

Proof. For all $\xi \in [0,\infty)$, it follows from (2.1) and (2.6) we get

$$\begin{aligned}&[(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))]\\ \times\ &[(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\delta))] \geq 0.\end{aligned}$$

Then,

$$\begin{aligned}&(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta) + (D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)\\ \geq\ &(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta).\end{aligned}$$

Multiplying by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides and integrating the resulting identity for the variable γ over the interval $[0,\infty)$, we have that

$$\begin{aligned}&(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^\xi (\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)d\gamma\\ +\ &(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^\xi (\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)d\gamma\end{aligned}$$

$$\geq (D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_1)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_2)(\gamma)d\gamma$$
$$+ (D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\delta)\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\xi}(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\gamma)d\gamma,$$

which can be written as follows by applying (1.5):

$$\begin{aligned}&(D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\delta)\left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi)+(D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_1)(\delta)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)\\ \geq\ &(D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_1)(\delta)\left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi)+(D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\delta)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi).\end{aligned}$$

Again multiplying by $\frac{(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ on both sides and integrating the resulting identity for the variable δ over the interval $[0,\infty)$, we have that

$$\begin{aligned}&\left({}^kD^{\xi,\eta}_{a+}\varphi_1\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)+\left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Upsilon\right)(\xi)\\ \geq\ &\left({}^kD^{\xi,\eta}_{a+}\varphi_1\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Psi_2\right)(\xi)+\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Upsilon\right)(\xi).\end{aligned}$$

This completes part (a).
To prove parts (b)–(d), the following inequalities shall be used.

$$\begin{aligned}(b)\quad &((D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_2)(\gamma)-(D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\gamma))\\ &\times((D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\delta)-(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_1)(\delta)))\geq 0.\end{aligned}$$

$$\begin{aligned}(c)\quad &((D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_2)(\gamma)-(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\gamma))\\ &\times((D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\delta)-(D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_2)(\delta)))\leq 0.\end{aligned}$$

$$\begin{aligned}(d)\quad &(D^{\xi+\eta(n-\xi)}_{a+,k}\Psi_1(\gamma)-(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\gamma))\\ &\times((D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\delta)-(D^{\xi+\eta(n-\xi)}_{a+,k}\varphi_1)(\delta)))\leq 0.\end{aligned}$$

□

Corollary 2.2. *Let* $k>0$, *and let* $\left({}^kD^{\xi,\eta}_{a+}\right)$ *denotes the Hilfer k-fractional derivative of order* ξ, $0<\xi<1$, *and type* $0<\eta\leq 1$. *Let* $(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)$ *and* $(D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)$ *be two positive functions on* $[0,\xi)$. *Suppose that there exist real constants* m, M, n, N *such that* $m\leq(D^{\xi+\eta(n-\xi)}_{a+,k}\Omega)(\xi)\leq M$, $n\leq(D^{\xi+\eta(n-\xi)}_{a+,k}\Upsilon)(\xi)\leq N$ *for all* $\xi\in[0,\infty)$. *Then,*

$$\begin{aligned}(a)\quad &nZ_k(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)+MZ_k(\xi)\left({}^kD^{\xi,\eta}_{a+}\Upsilon\right)(\xi)\\ &\geq nMZ_k(\xi)Z_k(\xi)+\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)\left({}^kD^{\xi,\eta}_{a+}\Upsilon\right)(\xi),\end{aligned}$$

$$(b)\quad mZ_k(\xi)\left({}^kD^{\xi,\eta}_{a+}\Upsilon\right)(\xi)+NZ_k(\xi)\left({}^kD^{\xi,\eta}_{a+}\Omega\right)(\xi)$$

$$\geq mNZ_k(\xi)mZ_k(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi),$$

$$\begin{aligned}(c) \qquad & NMZ_k(\xi)Z_k(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) \\ & \geq MZ_k(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) + NZ_k(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi),\end{aligned}$$

$$\begin{aligned}(d) \qquad & nmZ_k(\xi)Z_k(\xi)\left({}^kD_{a+}^{\xi,\eta}\varphi_1\right)(\xi) + \left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) \\ & \geq mZ_k(\xi)\left({}^kD_{a+}^{\xi,\eta}\Upsilon\right)(\xi) + nmZ_k(\xi)\left({}^kD_{a+}^{\xi,\eta}\Omega\right)(\xi),\end{aligned}$$

where $Z_k(\xi)$ *is defined by (2.5).*

Remark 2.2. *Take* $k = 1, \eta = 0$, $D_{a+}^{\xi,0}\Omega = D_{a+}^{\xi}\Omega$, $D_{a+}^{\xi,0}\Omega = D_{a+}^{\xi}\Upsilon$ *and in Theorem 2.2 and Corollary 2.2 we get Theorem 5 and Corollary 6 of [18], respectively.*

Lemma 2.1. *Let* $k > 0$, *and let* $\left({}^kD_{a+}^{\xi,\eta}f\right)$ *denote the Hilfer k-fractional derivative of order* ξ, $0 < \xi < 1$, *and type* $0 < \eta \leq 1$. *Let* $(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)$, *and* $(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)$ *be two integrable functions on* $[0, \infty)$. *Then,*

$$\begin{aligned} & Z_k(\xi)(({}^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)) - (({}^kD_{a+}^{\xi,\eta}\Omega)(\xi))^2 \\ = \; & (({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - ({}^kD_{a+}^{\xi,\eta}\Omega)(\xi))({}^kD_{a+}^{\xi,\eta}\Omega(\xi) - {}^kD_{a+}^{\xi,\eta}\Psi_1(\xi)) \\ - \; & Z_k(\xi)(({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - ({}^kD_{a+}^{\xi,\eta}\Omega)(\xi))(({}^kD_{a+}^{\xi,\eta}\Omega)(\xi) - ({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)) \\ + \; & Z_k(\xi){}^kD_{a+}^{\xi,\eta}(\Psi_1(\xi)\Omega(\xi)) - (({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)) \\ + \; & Z_k(\xi){}^kD_{a+}^{\xi,\eta}(\Psi_2(\xi)\Omega(\xi)) - ({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi) \\ - \; & Z_k(\xi){}^kD_{a+}^{\xi,\eta}(\Psi_1(\xi)\Psi_2(\xi)) + ({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi), \end{aligned}$$

where $Z_k(\xi)$ *is defined by (2.5).*

Proof. Since $\gamma, \delta > 0$, we have

$$\begin{aligned} & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)) \\ + \; & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)) \\ - \; & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)) \\ - \; & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)) \\ = \; & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))^2 + ((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))^2 - 2(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\ + \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\ + \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\ + \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) \end{aligned}$$

$$\begin{aligned} - \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\gamma) \\ - \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\ + \; & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta). \end{aligned}$$

Multiplying both sides by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ and integrating for the variable γ over the interval $[0,\xi]$, we get

$$\begin{aligned}
& ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))((^kD_{a+}^{\xi,\eta}\Omega)(\xi) - (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)) \\
+ & ((^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - (^kD_{a+}^{\xi,\eta}\Omega)(\xi))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta))) \\
- & (^kD_{a+}^{\xi,\eta}(((\Psi_2)(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\delta))) \\
- & ((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)Z_k(\xi) \\
= & (^kD_{a+}^{\xi,\eta}\Omega^2(\gamma)) + Z_k(\xi)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^2)(\delta) - 2(^kD_{a+}^{\xi,\eta}\Omega)(\xi)D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\xi) \\
+ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) + (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\
- & (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta) + (^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta) \\
+ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) - (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) \\
- & (^kD_{a+}^{\xi,\eta})(\Psi_2(\delta)\Omega(\gamma)) + (^kD_{a+}^{\xi,\eta}\Psi_1)(\gamma)(^kD_{a+}^{\xi,\eta}\Psi_1(\gamma))Z_k(\xi) \\
+ & (^kD_{a+}^{\xi,\eta}\Psi_1)(\gamma)(^kD_{a+}^{\xi,\eta}\Omega)(\gamma) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)Z_k(\xi) \\
+ & (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\delta)Z_k(\xi) + (D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)Z_k(\xi).
\end{aligned}$$

Multiplying both sides by $\frac{(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ and integrating for the variable δ over the interval $[0,\xi]$, we get

$$\begin{aligned}
& ((^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - (^kD_{a+}^{\xi,\eta}\Omega)(\xi))((^kD_{a+}^{\xi,\eta}\Omega)(\xi) - (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)) \\
+ & ((^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - (^kD_{a+}^{\xi,\eta}\Omega)(\xi))((^kD_{a+}^{\xi,\eta}\Omega)(\xi) - (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi))) \\
- & (^kD_{a+}^{\xi,\eta})(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))Z_k(\xi) \\
- & ((^kD_{a+}^{\xi,\eta})(\Psi_2(\xi) - \Omega(\xi)))(\Omega(\xi) - \Psi_1(\xi))Z_k(\xi) \\
= & ((^kD_{a+}^{\xi,\eta}\Omega^2)(\xi))Z_k(\xi) + Z_k(\xi)(^kD_{a+}^{\xi,\eta}\Omega^2)(\xi) - 2(^kD_{a+}^{\xi,\eta}\Omega)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) \\
+ & (^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) + (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) \\
+ & (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) + (^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) \\
+ & (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi) - (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) \\
- & Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\Psi_2(\xi)\Omega(\xi)) + (^kD_{a+}^{\xi,\eta})(\Psi_1(\xi)\Psi_1(\xi))Z_k(\xi) \\
- & Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\Psi_1(\gamma)(\Omega)(\xi)) - (^kD_{a+}^{\xi,\eta})(\Psi_2(\xi)\Omega(\xi))Z_k(\xi) \\
+ & Z_k(\xi)(^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi) - (^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi)Z_k(\xi).
\end{aligned}$$

This completes the proof of lemma. □

Corollary 2.3. *Let $m < M$, $k > 0$, and let $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)$ be a positive function on $[0,\xi)$ such that $m \leq (D_{a+,k}^{\xi+\eta(n-\xi)})\Omega(\xi) \leq M$. Then,*

$$\begin{aligned}
& Z_k(\xi)(^kD_{a+}^{\xi,\eta}\Omega^2(\xi)) - (^kD_{a+}^{\xi,\eta}\Omega(\xi))^2 \\
= & \left(MZ_k(\xi) - {}^kD_{a+}^{\xi,\eta}\Omega(\xi)\right)\left({}^kD_{a+}^{\xi,\eta}\Omega(\xi) - mZ_k(\xi)\right) - {}^kD_{a+}^{\xi,\eta}((M - \Omega(\xi))(\Omega(\xi) - m)).
\end{aligned}$$

$Z_k(\xi)$ is defined by (2.5).

Remark 2.3. *Take* $k=1,\eta=0$, $D_{a+}^{\xi,0}\Omega=D_{a+}^{\xi}\Omega$, $D_{a+}^{\xi,0}\Psi_1=D_{a+}^{\xi}\Psi_1$, $D_{a+}^{\xi,0}\Psi_2=D_{a+}^{\xi}\Psi_2$, *and in Lemma 2.1 and Corollary 2.3 we get Lemma 7 and Corollary of [18], respectively.*

Theorem 2.3. *Let* $f\in L_1[a,b]$, $k>0$, *and let* $\left({}^kD_{a+}^{\xi,\eta}f\right)$ *denote the Hilfer k-fractional derivative of order* ξ, $0<\xi<1$, *and type* $0<\eta\leq 1$. *Let* $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega,)$ $(D_{a+,k}^{\xi+\eta(n-\xi)})\Upsilon$, $(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)$, $(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)$, $(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)$ *and* $(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)$ *be integrable functions on* $[0,\xi)$. *If conditions (2.1) and (2.6) are satisfied, then*

$$\begin{aligned}&\left|Z_k(\xi)({}^kD_{a+}^{\xi,\eta}(\Omega(\xi))\Upsilon(\xi))-({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)\right|\\ &\leq\sqrt{T(\Omega,\Psi_1,\Psi_2)T(\Upsilon,\varphi_1,\varphi_2)},\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}T(\Omega,\Psi_1,\Psi_2)&=(({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)-({}^kD_{a+}^{\xi,\eta}\Omega)(\xi))(({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)-({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi))\\&+Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\Psi_1(\xi)\Omega(\xi))-({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)\\&+Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\Psi_2(\xi)\Omega(\xi))-({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)\\&+({}^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)({}^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)-Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\Psi_1(\xi)\Psi_2(\xi)),\end{aligned}$$

and

$$\begin{aligned}T(\Upsilon,\varphi_1,\varphi_2)&=(({}^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)-({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi))(({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)-({}^kD_{a+}^{\xi,\eta}\varphi_1)(\xi))\\&+Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\varphi_1(\xi)\Upsilon(\xi))-({}^kD_{a+}^{\xi,\eta}\varphi_1)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)\\&+Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\varphi_2(\xi)\Upsilon(\xi))-({}^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)\\&+({}^kD_{a+}^{\xi,\eta}\varphi_1)((\xi)^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)-Z_k(\xi)({}^kD_{a+}^{\xi,\eta})(\varphi_1(\xi)\varphi_2(\xi)),\end{aligned}$$

where $Z_k(\xi)$ *is defined by (2.5).*

Proof. Let $\xi>0$, $\gamma,\delta\in[0,\xi]$. Let $({}^kD_{a+}^{\xi,\eta}\Omega)$, $({}^kD_{a+}^{\xi,\eta}\Upsilon)$ be two positive functions on $[0,\infty]$ such that conditions (2.1) and (2.6) are satisfied, and $T(\gamma,\delta)$ is defined by

$$T(\gamma,\delta)=((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)-D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\delta))(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon(\gamma)-D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon(\delta)).\tag{2.8}$$

Multiplying both sides of (2.8) by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{2(k\Gamma_k(\eta(n-\xi)))^2}$ and integrating for the variables γ and δ over the interval $[a,\xi]$, we get

$$\begin{aligned}&\int_a^{\xi}\int_a^{\xi}\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{2(k\Gamma_k(\eta(n-\xi)))^2}T(\gamma,\delta)d\gamma d\delta\\ =&\ \frac{\xi^{\frac{\eta(n-\xi)}{k}}}{\Gamma_k(\eta(n-\xi)+k)}({}^kD_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi))-({}^kD_{a+}^{\xi,\eta}\Omega)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi).\end{aligned}\tag{2.9}$$

Applying the Cauchy Schwarz inequality, we get

$$\left(\int_a^{\xi}\int_a^{\xi}\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{2(k\Gamma_k(\eta(n-\xi)))^2}\right.$$

$$
\begin{aligned}
&((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta))((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta))d\gamma d\delta\Big)^2\\
\le\ & \int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{2(k\Gamma_k(\eta(n-\xi)))^2}((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\delta)))^2 d\gamma d\delta\\
\times\ & \int_a^\xi\int_a^\xi \frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{2(k\Gamma_k(\eta(n-\xi)))^2}((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta))^2 d\gamma d\delta.
\end{aligned}
\tag{2.10}
$$

From (2.9) and (2.10), we get

$$
\begin{aligned}
&(Z_k(\xi)((^kD_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi))-(^kD_{a+}^{\xi,\eta}\Omega)(\xi)(^kD_{a+}^{\xi,\eta}\Upsilon)(\xi))^2\\
\le\ &(Z_k(\xi)(^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)-((^kD_{a+}^{\xi,\eta}\Omega)(\xi))^2)(Z_k(\xi)(^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi)-((^kD_{a+}^{\xi,\eta}\Upsilon)(\xi))^2).
\end{aligned}
$$

Since

$$((D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\xi)))\ge 0,$$

and

$$((D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\xi))((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\xi)))\ge 0,$$

we have

$$Z_k(\xi)(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_2)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi))((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Psi_1)(\xi)))\ge 0,$$

and

$$Z_k(\xi)((D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\xi))((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\xi)-(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_1)(\xi))\ge 0.$$

Thus, from Lemma 2.1, we have

$$
\begin{aligned}
&Z_k(\xi)((^kD_{a+}^{\xi,\eta}\Omega^2)(\xi))-((^kD_{a+}^{\xi,\eta}\Omega)(\xi))^2\\
\le\ &((^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)-(^kD_{a+}^{\xi,\eta}\Omega)(\xi))((^kD_{a+}^{\xi,\eta}\Omega)(\xi)-(^kD_{a+}^{\xi,\eta}\Psi_1)(\xi))\\
+\ &Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\Psi_1(\xi)\Omega(\xi))-(^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi)\\
+\ &Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\Psi_2(\xi)\Omega(\xi))-(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)(^kD_{a+}^{\xi,\eta}\Omega)(\xi)\\
+\ &(^kD_{a+}^{\xi,\eta}\Psi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Psi_2)(\xi)-Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\Psi_1(\xi)\Psi_2(\xi))\\
=\ &T(\Omega,\Psi_1,\Psi_2),
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&Z_k(\xi)((^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi))-((^kD_{a+}^{\xi,\eta}\Upsilon)(\xi))^2\\
\le\ &((^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)-(^kD_{a+}^{\xi,\eta}\Upsilon)(\xi))((^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)-(^kD_{a+}^{\xi,\eta}\varphi_1)(\xi))\\
+\ &Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\varphi_1(\xi)\Upsilon(\xi))-(^kD_{a+}^{\xi,\eta}\varphi_1)(\xi)(^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)\\
+\ &Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\varphi_2(\xi)\Upsilon(\xi))-(^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)(^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)\\
+\ &(^kD_{a+}^{\xi,\eta}\varphi_1)(\xi)(^kD_{a+}^{\xi,\eta}\varphi_2)(\xi)-Z_k(\xi)(^kD_{a+}^{\xi,\eta})(\varphi_1(\xi)\varphi_2(\xi))\\
=\ &T(\Upsilon,\varphi_1,\varphi_2).
\end{aligned}
\tag{2.12}
$$

Therefore, the inequality (2.7) follows from (2.11) and (2.12). This completes the proof. □

Corollary 2.4. *Let $m, M, n, N \in \Re$, $T(\Omega, \Psi_1, \Psi_2) = T(\Omega, m, M)$ and $T(\Upsilon, \varphi_1, \varphi_2) = T(\Upsilon, n, N)$. Then, the inequality (2.7) reduces to*

$$\left| Z_k(\xi)({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi)) - ({}^k D_{a+}^{\xi,\eta}\Omega)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon)(\xi) \right| \leq (Z_k(\xi))^2 (M-m)(N-n).$$

Remark 2.4. *Take $k = 1, \eta = 0$, $D_{a+}^{\xi,0}\Omega = D_{a+}^{\xi}\Omega$ in Theorem 2.3 and Corollary 2.4, and we get Theorem 9 and Remark 10 of [18], respectively.*

Theorem 2.4. *Let $k > 0$ and $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)$ and $(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)$ be two positive functions defined on $[0, \infty)$. Then, the following inequalities hold:*

$$\begin{aligned}
(1) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) \\
& \geq pq\frac{1}{Z_k(\xi)}({}^k D_{a+}^{\xi,\eta}\Upsilon)(\rho)({}^k D_{a+}^{\xi,\eta}\Omega)(\xi),
\end{aligned}$$

$$\begin{aligned}
(2) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^p)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Omega^q)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) \\
& \geq pq(({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi)))^2,
\end{aligned}$$

$$\begin{aligned}
(3) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Omega^q)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^p)(\xi) \\
& \geq pq(({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon^{p-1}(\xi))(({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon^{q-1}(\xi)),
\end{aligned}$$

$$\begin{aligned}
(4) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) \\
& \geq pq({}^k D_{a+}^{\xi,\eta})(\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi))({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi)),
\end{aligned}$$

$$\begin{aligned}
(5) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^2)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Omega^2)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) \\
& \geq pq({}^k D_{a+}^{\xi,\eta})(\Omega(\xi)\Upsilon(\xi))({}^k D_{a+}^{\xi,\eta})(\Omega^{\frac{2}{q}}(\xi)\Upsilon^{\frac{2}{p}}(\xi)),
\end{aligned}$$

$$\begin{aligned}
(6) \quad & q\ ({}^k D_{a+}^{\xi,\eta}\Omega^2)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^q)(\xi) + p\ ({}^k D_{a+}^{\xi,\eta}\Omega^p)(\xi)({}^k D_{a+}^{\xi,\eta}\Upsilon^2)(\xi) \\
& \geq pq({}^k D_{a+}^{\xi,\eta})(\Omega^{\frac{2}{p}}(\xi)\Upsilon^{\frac{2}{q}}(\xi))({}^k D_{a+}^{\xi,\eta})(\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi)),
\end{aligned}$$

$$\begin{aligned}
(7) \quad & q\ ({}^k D_{a+}^{\xi,\eta})(\Omega^2(\xi)\Upsilon^q(\xi)) + p\ ({}^k D_{a+}^{\xi,\eta})(\Omega^2(\xi)\Upsilon^p)(\xi) \\
& \geq pq\frac{1}{Z_k(\xi)}({}^k D_{a+}^{\xi,\eta})(\Omega^{\frac{2}{p}}(\xi)\Upsilon^{q-1}(\xi))({}^k D_{a+}^{\xi,\eta})(\Omega^{\frac{2}{q}}(\xi)\Upsilon^{p-1}(\xi)),
\end{aligned}$$

where $Z_k(\xi)$ is defined by (2.5).

Proof. By Young's inequality, we have

$$\frac{a^p}{p} + \frac{a^q}{q} \geq ab, (a, b \geq 0, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1).$$

Let us choose $a = (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\gamma))$ and $b = (D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)$ and we have

$$\frac{((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))^p}{p} + \frac{((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta))^q}{q} \geq (D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta),$$

for all $(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma),(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\geq 0$.
Multiplying by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ and integrating for the variable γ over the interval $[0,\xi]$, we get

$$\int_a^{\xi}\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}\frac{((D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma))^p}{p}d\gamma$$
$$+\int_a^{\xi}\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}\frac{((D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta))^q}{q}d\gamma$$
$$\geq\int_a^{\xi}\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\gamma)D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon(\delta)d\gamma,$$

and it becomes

$$\frac{1}{p}{}^kD_{a+}^{\xi,\eta}\Omega^p(\xi)+\frac{1}{q}\Upsilon^q(\delta)Z_k(\xi)\geq(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)({}^kD_{a+}^{\xi,\eta})\Omega(\xi).$$

Again multiplying by $\frac{(\xi-\delta)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ and integrating for the variable δ over the interval $[0,\xi]$, we get

$$\frac{1}{p}({}^kD_{a+}^{\xi,\eta}\Omega^p)(\xi)Z_k(\xi)+\frac{1}{q}{}^kD_{a+}^{\xi,\eta}\Upsilon^q(\xi)Z_k(\xi)\geq{}^kD_{a+}^{\xi,\eta}\Upsilon(\xi){}^kD_{a+}^{\xi,\eta}\Omega(\xi),$$

which implies that

$$\frac{1}{p}({}^kD_{a+}^{\xi,\eta}\Omega^p)(\xi)+\frac{1}{q}({}^kD_{a+}^{\xi,\eta}\Upsilon^q)(\delta)\geq\frac{1}{Z_k(\xi)}({}^kD_{a+}^{\xi,\eta}\Upsilon)(\xi)({}^kD_{a+}^{\xi,\eta}\Omega)(\xi).$$

This completes the proof of part (a).
The remaining inequalities can be proved using Young's inequality in a similar manner by taking

(2) $a=(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)$,
and $b=(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)$.

(3) $a=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)}$, and $b=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)}$,
$(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\neq 0$.

(4) $a=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)}$, and $b=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)}$,
$(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\neq 0$.

(5) $a=(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon^{\frac{2}{p}})(\delta)$,

and $b=(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^{\frac{2}{q}})(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)$.

(6) $a=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^{\frac{2}{p}})(\gamma)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\delta))}$, and $b=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon^{\frac{2}{q}})(\gamma)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)}$,
$(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\delta)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\neq 0$.

(7) $a=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^{\frac{2}{p}})(\gamma)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)}$, and $b=\frac{(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^{\frac{2}{q}})(\delta)}{(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)}$,
$(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\delta)\neq 0$.

□

Example 2.1. *Let* $k>0$ *and* $(D_{a+,k}^{\xi+\eta(n-\xi)}\varphi_2)(\gamma)$ *be a positive function defined on* $[0,\infty)$*, and let* $m=\min_{0\le\gamma\le\xi}\frac{D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\gamma)}{D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon(\gamma)}$ *and* $M=\max_{0\le\gamma\le\xi}\frac{D_{a+,k}^{\xi+\eta(n-\xi)}\Omega(\gamma)}{D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon(\gamma)}$*. Then, one can have*

$$0\le({}^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi)\le\frac{(m+M)^2}{4mM}({}^kD_{a+}^{\xi,\eta})(\Omega\Upsilon(\xi))^2. \tag{2.13}$$

Proof. It follows from (2.13) that

$$\left(\frac{({}^kD_{a+}^{\xi,\eta}\Omega)(\gamma)}{({}^kD_{a+}^{\xi,\eta}\Upsilon)(\gamma)}-m\right)\left(M-\frac{({}^kD_{a+}^{\xi,\eta}\Omega)(\gamma)}{({}^kD_{a+}^{\xi,\eta}\Upsilon)(\gamma)}\right)({}^kD_{a+}^{\xi,\eta}\Upsilon^2)(\gamma)\ge 0,$$

and

$$(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^2)(\gamma)+mM(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon^2)(\gamma)\le(m+M)(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma).$$

Multiplying by $\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}$ and integrating for the variable γ over the interval $[0,\xi]$, we get

$$\int_a^\xi\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega^2)(\xi)d\gamma+mM\int_a^\xi\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon^2)(\xi)d\gamma$$
$$\le(m+M)\int_a^\xi\frac{(\xi-\gamma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}(D_{a+,k}^{\xi+\eta(n-\xi)}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma)d\gamma.$$

This implies that

$$({}^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)+mM({}^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi)\le(m+M)({}^kD_{a+}^{\xi,\eta}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma). \tag{2.14}$$

Alternatively, it follows from

$$\left(\sqrt{{}^kD_{a+}^{\xi,\eta}\Omega^2(\xi)}-\sqrt{mM{}^kD_{a+}^{\xi,\eta}\Upsilon^2(\xi)}\right)^2\ge 0$$

that

$$2\sqrt{({}^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)}\sqrt{mM({}^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi)} \leq (m+M)({}^kD_{a+}^{\xi,\eta}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma). \tag{2.15}$$

Therefore,

$$4mM({}^kD_{a+}^{\xi,\eta}\Omega^2)(\xi)({}^kD_{a+}^{\xi,\eta}\Upsilon^2)(\xi) \leq (m+M)^2(({}^kD_{a+}^{\xi,\eta}\Omega)(\gamma)(D_{a+,k}^{\xi+\eta(n-\xi)}\Upsilon)(\gamma))^2$$

follows from (2.14) and (2.15), and the proof is complete. □

3. Conclusions

It is always interesting for researchers to give more generalized results. In the present paper, we have used the generalized Hilfer-k fractional derivative and Young's inequality to produce a variety of new fractional inequalities. Such results have wide applications in applied mathematics and mathematical physics. The technique provided in this paper to establish the new fractional estimates may motivate researchers to do further work in this direction.

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Conflict of interest

The authors declare no conflict of interest.

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