Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 76–101.
DOI: 10.3934/math.2023004
Received: 11 August 2022
Revised: 11 September 2022
Accepted: 15 September 2022
Published: 27 September 2022

Research article

Existence of S-asymptotically ω-periodic solutions for non-instantaneous impulsive semilinear differential equations and inclusions of fractional order $1 < \alpha < 2$

Zainab Alsheekhhussain[1,*], Ahmed Gamal Ibrahim[2] and Rabie A. Ramadan[1,3]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Hail Saudi Arabia
[2] Department of Mathematics, College of Sciences, King Faisal University, Saudi Arabia
[3] Cairo University, Department of Computer Engineering, Faculty of Engineering, Cairo, 12613, Egypt

* **Correspondence:** Email: za.hussain@uoh.edu.sa.

Abstract: It is known that there is no non-constant periodic solutions on a closed bounded interval for differential equations with fractional order. Therefore, many researchers investigate the existence of asymptotically periodic solution for differential equations with fractional order. In this paper, we demonstrate the existence and uniqueness of the S-asymptotically ω-periodic mild solution to non-instantaneous impulsive semilinear differential equations of order $1 < \alpha < 2$, and its linear part is an infinitesimal generator of a strongly continuous cosine family of bounded linear operators. In addition, we consider the case of differential inclusion. Examples are given to illustrate the applicability of our results.

Keywords: non-instantaneous impulses; cosine family; asymptotically periodic solutions; differential inclusions
Mathematics Subject Classification: 26A33, 34A08, 34A60

1. Introduction

It is known that the action of instantaneous impulses seems not describe some certain dynamics of evolution processes in Pharmacotherapy. For example, in the case of a decompensation, (high or low levels of glucose) one can prescribe some intravenous drugs (insulin). The introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. Thus, we do not expect to use the instantaneous impulses to describe such a process. In fact, the above situation is fallen in a new case of impulsive action, which starts at any arbitrary fixed point and stays active on a finite time interval. To this end, Hernándaz and O'Regan [1] introduced the non-instantaneous impulsive differential equations. For recent contributions on non-instantaneous impulsive differential equations and inclusions, we refer the reader to [2–7].

There are some papers where the nonexistence of non-constant periodic solutions on closed bounded interval for differential equations with fractional order are considered such as [8–12]. Many authors investigated the existence of S-asymptotically ω-periodic solutions for many types of differential equations of fractional order. For example, Maghsoodi et al. [13] considered an evolution equation of order $\alpha \in (0,1)$ generated by an evolution system $U(\theta,s)$. Ren et al. [12] studied semilinear differential equation of order $\alpha \in (0,1)$ and generated by exponentially stable C_0-semigroup. Ren et al. [14] considered semilinear differential equations of order $\alpha \in (1,2)$ generated by a sectorial operator. Mu et al. [15] investigated an evolution equation with the Weyl-Liouville fractional derivative of order $\alpha \in (0,1)$ and generated by C_0-semigroup. Zhao at al. [16] demonstrated the existence of an asymptotically almost automorphic mild solution to a semilinear fractional differential equation, and Wang et al. [17] studied delay fractional differential equations with an almost sectorial operator of order $\alpha \in (0,1)$. Moreover, Muslim et al. [18] investigated the existence, uniqueness and stability of solutions to second order nonlinear differential equations with non- instantaneous impulses. Very recently, Alsheekhhussain et al. [19] proved the existence of S-asymptotically w-periodic solutions for non-instantaneous impulsive differential equations and inclusions generated by sectorial operators. For more information regarding this subject, we refer the reader to [20–25].

It is worth noting that the problems considered in all the cited works above, except [19], do not contain impulseses effects and the right-hand side is a single-valued function. Moreover, to the best of the authors' knowledge, the literature concerning S-asymptotically w-periodic solutions for differential inclusions subject to non-instantaneous impulses and generated by an infinitesimal generator of a cosine family $\{C(\theta) : \theta \geq 0\}$ is very new, and this fact is the main aim in the present paper.

When the considered problem contains non-instantaneous impulses, there are two approaches in the literature to prove the existence of the solution. The first one is by keeping the lower limit of the fractional derivative at zero. The second one is by switching it at the impulsive points, which will be considered in the present paper.

Let $\alpha \in (1,2)$, E be a Banach space, $\mathbb{N}$ be the set of natural numbers, $m \in \mathbb{N}$, $\omega > 0$, $J = [0,\infty)$,

$$0 = s_0 < \theta_1 < s_1 < \cdots < \theta_m < s_m = \omega < \theta_{m+1} = \omega + \theta_1 < s_{m+1} = s_1 + \omega < ...,$$

with $\lim_{i\to\infty} \theta_i = \infty$, $s_{m+i} = s_i + \omega$; $i \in \{0\} \cup \mathbb{N}$, $\theta_{m+i} = \theta_i + \omega$; $i \in \mathbb{N}$, and A is the infinitesimal generator of cosine family $\{C(\theta) : \theta \geq 0\}$. Moreover, let $\Pi : J \times E \to E$, $g_i : [\theta_i, s_i] \times E \longrightarrow E$; $i \in \mathbb{N}$, $x_0 \in D(A)$ (the domain of A), and $x_1 \in E$ a fixed point.

Motivated by the above cited works, we demonstrate the existence and uniqueness of an S-asymptotically ω-periodic solution to the following non-instantaneous impulsive semilinear differential equation:

$$\begin{cases} {}^{c}D^{\alpha}_{0,\theta}x(\theta) = Ax(\theta) + \Pi(\theta, x(\theta)), \ a.e.\ \theta \in (s_i,\ \theta_{i+1}], i \in \mathbb{N} \cup \{0\}, \\ x(\theta) = g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i\ s_i], i \in \mathbb{N}, \\ x(0) = x_0, x'(0) = x_1, \end{cases} \tag{1.1}$$

where, ${}^{c}D^{\alpha}_{0,\theta}x(\theta)$ is the Caputo derivative of the function x at the point θ with lower limit at 0 [26].

After that, we prove the existence of S-asymptotically ω-periodic solutions for the following non-instantaneous impulsive semilinear differential inclusion:

$$\begin{cases} {}^{c}D^{\alpha}_{0,\theta}x(\theta) \in Ax(\theta) + F(\theta, x(\theta)), \ a.e.\ \theta \in (s_i,\ \theta_{i+1}], i \in \mathbb{N} \cup \{0\}, \\ x(\theta) = g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i\ s_i], i \in \mathbb{N}, \\ x(0) = x_0, x\acute{}(0) = x_1, \end{cases} \tag{1.2}$$

where $F : J \times E \to 2^E - \{\phi\}$ is a multi-valued function.

Unlike the differential equations of integer order, the existence of non-constant periodic solutions for fractional differential equations is not guaranteed. For this reason, the concept of an asymptotically periodic solution is introduced for fractional differential equations. Many researchers uses this approach to investigate the existence of the solution for fractional differential equations. However, up to now, there are no work studying the problem mentioned above. In this paper, we construct sufficient conditions that assure the existence of asymptotically periodic mild solutions for Problems (1.1) and (1.2). Moreover, our results generalize the obtained ones in [12], and our method can be used to study the existence of asymptotically periodic mild solutions for the problems considered in [13, 15–17, 20–25], when these problems contain impulseses effects and the right hand side is a multi-valued function.

Since a multivalued function is a function values are sets, so, our technique to find an asymptotically periodic solution for Problem (2) can be used to extend many recent publications on the same subject in which the right hand side is a single-function see, for example, [27–29].

In Section 3, we prove the existence and uniqueness of S-asymptotically ω-periodic solution for Problem (1.1). Section 4 is devoted to prove the existence of S-asymptotically ω-periodic solutions to Problem (1.2). Finally, examples are given to show that the obtained results are applicable.

2. Preliminaries and notations

Let $J_0 = [0, \theta_1]$, $J_i = (\theta_i, \theta_{i+1}]$, and $i \in \mathbb{N}$. Because Problem (1.1) contains non-instantaneous impulses effect, we consider the two Banach spaces:

$$PC(J, E) := \{x : J \to E,\ x|_{J_i} \in C(J_i, E), x(\theta_i^+) \text{ and } x(\theta_i^-) \text{ exist}, i \in \mathbb{N}\ \},$$

and

$$PC_b(J, E) := \{x\ \in PC(J, E) : x \text{ is bounded},\ x|_{J_i} \in C(J_i, E)\},$$

where

$$\|x\|_{PC(J,E)} := \max_{\theta \in J} \|x(\theta)\|_E,$$

$$\|x\|_{PC_b(J,E))} := \max_{\theta \in J} \|x(\theta)\|_E,$$

and $x(\theta_i^+)$ and $x(\theta_i^-)$ are the right and left limits of x at θ_i.

Definition 2.1. *Let ω be a positive real number. A function $x \in PC_b(J, E)$ is said to be S-asymptotically ω-periodic if it satisfies the relation:*

$$\lim_{\theta \to \infty} \|x(\theta + \omega) - x(\theta)\| = 0.$$

Definition 2.2. *[19] By $SAP_\omega PC_b(J,E)$, we mean the Banach space of all S-asymptotically ω-periodic functions $x \in PC_b(J,E)$, where the norm is given by*

$$\|x\|_{PC_b(J,E))} := \max_{\theta\in J} \|x(\theta)\|_E.$$

Definition 2.3. *[30] A family $\{C(\theta) : \theta \in \mathbb{R}\}$, where $C(\theta) : D(C(\theta)) = E \to E$ is a bounded linear operator, is called a strongly cosine family if:*

(i) $C(0) = I$,
(ii) $C(\theta+\tau) + C(\tau-\theta) = 2C(\tau)C(\theta)$ for all $\tau, \theta \in \mathbb{R}$,
(iii) the map $\theta \longmapsto C(\theta)x$ is continuous for each $x \in E$.

If $\{C(\theta) : \theta \in \mathbb{R}\}$ is a strongly cosine family, then the strongly continuous sine family associated with it is defined by:

$$S(\theta)x = \int_0^\theta C(s)xds; \theta \in \mathbb{R}, x \in E.$$

Definition 2.4. *The infinitesimal generator of a cosine family $\{C(\theta) : \theta \in \mathbb{R}\}$ is an operator $A : D(A) \longmapsto E$ defined by*

$$Ax = \frac{d^2}{d\theta^2} C(\theta)x|_{\theta=0},$$

where $D(A) = \{x \in E : C(t)x$ is twice continuously differentiable of $t\}$.

Lemma 2.1. *([30], Propositions 2.2 and 2.3]) Let $\{C(t) : t \in \mathbb{R}\}$ be a strongly cosine family in E with infinitesimal generator A and*

$$Z = \{z \in E : C(\theta)x \text{ is once continuously differentiable of } \theta\ \}.$$

Then, the following statements hold:

1- $D(A)$ is dense in E, and A is a closed operator.
2- If $z \in E$, then $S(\theta)z \in Z$.
3- If $z \in Z$, then
(i) $S(\theta)z \in D(A)$ and $\frac{d^2}{d\theta^2} S(\theta)z = AS(\theta)z$,
(ii) $S(\theta)z \in D(A)$ and $\frac{d}{d\theta} C(\theta)z = AS(\theta)z$.
4- If $z \in D(A)$, then
(i) $C(\theta)z \in D(A)$ and $\frac{d^2}{d\theta^2} C(\theta)z = AC(\theta)x = C(\theta)Az$;
(ii) $S(\theta)z \in D(A)$ and $AS(\theta)z = S(\theta)Az$.

Definition 2.5. *([31]) By a mild solution for Problem (1.1), we mean a function $x \in PC(J,E)$ such that*

$$x(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ + \int_0^\theta (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau, \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta-s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta-s_i)g_i'(s_i, x(\theta_i^-)) \\ - \int_0^{s_i} (s_i-\tau)^{q-1} P_q(s_i-\tau)\Pi(\tau, x(\tau))d\tau \\ + \int_0^\theta (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $q = \frac{\alpha}{2}$, and, for $\vartheta \geq 0$,

$$C_q(\vartheta) = \int_0^\infty \xi_q(\theta)C(\vartheta^q\theta)d\theta, K_q(\vartheta) = \int_0^\vartheta C_q(\tau)d\tau,$$

$$P_q(\vartheta) = q\int_0^\infty \theta\xi_q(\theta)S(\vartheta^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}w_q(\theta^{-\frac{1}{q}}),\ \theta \in (0,\infty),$$

and

$$w_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in (0,\infty).$$

Remark 2.1. *The solution function given by (2.1) satisfies the following properties:*

1- $x(0) = C_q(0)x_0 = x_0$.
2 $x\'(0) = x_1$.
3- x is continuous on J_i; $i \in \{0\} \cup \mathbb{N}$.

We will need the following lemma which gives some properties for the operators $C_q(\theta), K_q(\theta)$ and $P_q(\theta)$.

Lemma 2.2. *([31], Lemma 8). Assume that*

(HA) $A : D(A) \to E$ is the infinitesimal generator of strongly continuous cosine family of linear operators $\{C(\theta) : \theta \geq 0\}$ which is uniformly bounded by $M > 0$. Then,

(i) For any fixed $\theta \geq 0, C_q(\theta),\ K_q(\theta)$ and $P_q(\theta)$ are linear bounded operators.
(ii) For $\gamma \in [0,1]$, $\int_0^\infty \theta^\gamma\xi_\alpha(\theta)d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}$.
(iii) If $\|C_q(\theta)\| \leq M$, $\theta \geq 0$, then for any $x \in E$, $\|C_q(\theta)x\| \leq M\|x\|$, $\|K_q(\theta)x\| \leq \theta M\|x\|$ and $\|P_q(\theta)x\| \leq \frac{M}{\Gamma(2q)}\|x\|\theta^q$.
(iv) $\{C_q(\theta), \theta \geq 0\}, \{K_q(\theta), \theta \geq 0\}$ and $\{\theta^{q-1}P_q(\theta), \theta \geq 0\}$ are strongly continuous.

3. Existence and uniqueness of an S-asymptotically ω-periodic mild solution for Problem (1.1).

We make the following assumptions:

$(HA)^*A : D(A) \to E$ satisfies (HA), and the family $\{C(\theta) : \theta \geq 0\}$ is exponentially stable. That is, there exist positive numbers a, M such that $\|C(\theta)\| \leq e^{-a\theta}M,\ \theta \geq 0$.

$(H\Pi)$ $\Pi : J \times E \to E$ is a strongly measurable function, and there are $h_1, h_2 \in C(J, \mathbb{R}^+)$ such that $h_1$is bounded,

$$\|\Pi(\theta,x) - \Pi(\theta,y)\|_E \leq h_1(\theta)\|x-y\|_E, \forall\theta \in J,\ x,y \in E, \tag{3.1}$$

and

$$\|\Pi(\theta+\omega,x) - \Pi(\theta,x)\|_E \leq h_2(\theta)(\|x\|_E + 1), \forall\theta \in J,\ x \in E. \tag{3.2}$$

(Hg) For any $i \in \mathbb{N}$, $g_i : [\theta_i, s_i] \times E \longrightarrow E$ $(i \in \mathbb{N})$ such that, for any $x \in E$, the function $\theta \mapsto g_i(\theta,x)$ is differentiable at s_i , and that:

(i)

$$\lim_{\substack{\theta\to\infty\\ i\to\infty}} \|g_{i+m}(\theta+\omega,z)-g_i(\theta,z)\|_E = 0, \forall z\in E, \tag{3.3}$$

and

$$\lim_{i\to\infty} \|g'_{i+m}(s_i+\omega,z)-g'_i(s_i,z)\|_E = 0, \forall z\in E. \tag{3.4}$$

(ii) There are $N > 0$ such that

$$\|g_i(\theta,z_1)-g_i(\theta,z_2)\|_E \leq N\|z_1-z_2\|_E, \forall \theta\in[\theta_i,s_i],\ \forall z_1,z_2\in E. \tag{3.5}$$

(iii) There is $\mathcal{N} > 0$ such that

$$\|g'_i(s_i,z_1)-g'_i(s_i,z_2)\|_E \leq \mathcal{N}\|z_1-z_2\|_E, \forall\ z_1,z_2\in E. \tag{3.6}$$

(iv) There is $\kappa_1 > 0$ such that

$$\sup_{i\in\mathbb{N}}\sup_{\theta\in J}\|g_i(\theta,z)\|_E \leq \kappa_1(\|z\|_E+1), \forall z\in E. \tag{3.7}$$

(v) There is $\kappa_2 > 0$ with

$$\sup_{i\in\mathbb{N}}\|g'_i(s_i,z)\|_E \leq \kappa_2(\|z\|_E+1), \forall z\in E. \tag{3.8}$$

The following lemma provides additional properties for the operators $C_q(\theta)$ and $P_q(\theta)$ when $\{C(\theta):\theta\geq 0\}$ is exponentially stable.

Lemma 3.1. *([32], Proposition 2.1). If $(HA)^*$ is verified, then there is $L > 0$ such that*

$$\|C_q(\theta)\| \leq \frac{L}{(1+\theta)^q}, \|P_q(\theta)\| \leq \frac{L}{(1+\theta)^{2q}}, \forall\theta\in J. \tag{3.9}$$

Lemma 3.2. *([33], Lemma 2.11]) Let $\gamma\in[0,1]$, $0 < a < b$. Then, $|b^\gamma - a^\gamma| \leq (b-a)^\gamma$.*

Remark 3.1. *In what follows, we mean by $\|\ \ \|$ the norm in the Banach space E.*

Theorem 3.1. *Under conditions $(HA)^*, (H\Pi), (Hg_i)$ and (H), Problem (1.1) has a unique S-asymptotically ω-periodic mild solution provided that the following assumptions are verified:*

$$\varsigma = \sup_{\theta\in J}\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} h_1(\tau)d\tau < \infty, \tag{3.10}$$

$$MN + M\omega\mathcal{N} + 2L\varsigma < 1, \tag{3.11}$$

$$\xi = \sup_{\tau\in[0,\omega]}\|\Pi(\tau,0)\|_E < \infty, \tag{3.12}$$

and

$$\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} h_2(\tau)d\tau = 0, \tag{3.13}$$

where h_1 and h_2 are specified in $(H\Pi)$.

Proof. First, we clarify that if $x \in SAP_\omega PC_b(J, E)$, then the function $\Phi(x)$ defined by

$$\Phi(x)(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ + \int_0^\theta (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau, \theta \in [0, \theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta - s_i)g_i'(s_i, x(\theta_i^-)) \\ - \int_0^{s_i} (s_i-\tau)^{q-1} P_q(s_i-\tau)\Pi(\tau, x(\tau))d\tau \\ + \int_0^\theta (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}, \end{cases} \tag{3.14}$$

belongs to $SAP_\omega PC_b(J, E)$. The proof will be given in the following steps.
Step 1. we will show that $\lim_{\theta\to\infty} \|\Phi(x)(\theta+\omega) - \Phi(x)(\theta)\| = 0$.

Let $\epsilon > 0$. Because $x \in SAP_\omega PC(J, E)$, $\lim_{\theta\to\infty} \|x(\theta+\omega) - x(\theta)\|_E = 0$, and hence there is $\theta_\epsilon > \theta_1$ such that

$$\sup_{\theta>\theta_\epsilon} \|x(\theta+\omega) - x(\theta)\|_E < \frac{\epsilon}{L\varsigma}. \tag{3.15}$$

Let $\theta > \theta_\epsilon$. If $\theta \in (\theta_i, s_i]$, $i \in \mathbb{N}$, then $\theta + \omega \in (\theta_i + \omega, s_i + \omega] = (\theta_{i+m}, s_{i+m}]$. So, relations (3.3), (3.5) and (3.14) imply that

$$\begin{aligned} & \lim_{\theta\to\infty} \|\Phi(x)(\theta+\omega) - \Phi(x)(\theta)\|_E \\ = \; & \lim_{\theta\to\infty} \|g_{i+m}(\theta+\omega, x(\theta_{i+m}^-)) - g_i(\theta, x(\theta_i^-))\| \\ \le \; & \lim_{\substack{\theta\to\infty \\ i\to\infty}} \|g_{i+m}(\theta+\omega, x(\theta_i^- + \omega)) - g_{i+m}(\theta, x(\theta_i^- + \omega))\| \\ & + \lim_{\substack{\theta\to\infty \\ i\to\infty}} \|g_{i+m}(\theta, x(\theta_i^- + \omega)) - g_i(\theta, x(\theta_i^-))\| \\ \le \; & N \lim_{\theta_i\to\infty} \|x(\theta_i^- + \omega) - x(\theta_i^-)\|_E = 0. \end{aligned} \tag{3.16}$$

Let $\theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}$. Then, $\theta + \omega \in [s_i + \omega, \theta_{i+1} + \omega] = [s_{i+m}, \theta_{i+m+1}]$. By arguing as in (3.16), one obtains

$$\begin{aligned} & \lim_{\theta\to\infty} \|C_q(\theta + \omega - (s_i + \omega))g_{i+m}(s_i + \omega, x(\theta_{i+\omega}^-)) - C_q(\theta - s_i)g_i(s_i, x(\theta_i^-))\| \\ = \; & M \lim_{\theta\to\infty} \|g_{i+m}(s_i + \omega, x(\theta_i^- + \omega)) - g_i(s_i, x(\theta_i^-))\| = 0. \end{aligned} \tag{3.17}$$

Similarly, by (3.4) and (3.6), we get

$$\begin{aligned} & \lim_{\theta\to\infty} \|K_q(\theta + \omega - (s_i + \omega))g_{i+m}'(s_i + \omega, x(\theta_{i+\omega}^-)) - K_q(\theta - s_i)g_i'(s_i, x(\theta_i^-))\| \\ = \; & \lim_{\theta\to\infty} \|K_q(\theta - s_i)\| \, \|g_{i+m}'(s_i + \omega, x(\theta_i^- + \omega)) - g_i'(s_i, x(\theta_i^-))\| \\ \le \; & \lim_{\theta\to\infty} M\,(\theta - s_i)\|g_{i+m}'(s_i + \omega, x(\theta_i^- + \omega)) - g_i'(s_i, x(\theta_i^-))\| \\ \le \; & M\,(\theta_{i+1} - s_i)[\lim_{\substack{\theta\to\infty \\ i\to\infty}} \|g_{i+m}'(s_i, x(\theta_i^- + \omega)) - g'(s_i, x(\theta_i^-))\| \\ & + \mathcal{N} \lim_{\theta\to\infty} \|x(\theta_i^- + \omega) - x(\theta_i^-)\| = 0. \end{aligned} \tag{3.18}$$

Next, notice that

$$\int_0^{\theta+\omega} (\theta + \omega - \tau)^{q-1} P_q(\theta + \omega - \tau)\Pi(\tau, x(\tau))d\tau$$

$$= \int_{-\omega}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau+\omega, x(\tau+\omega))d\tau.$$

Then,

$$\begin{aligned}
& \| \int_{0}^{\theta+\omega} (\theta+\omega-\tau)^{q-1} P_q(\theta+\omega-\tau)\Pi(\tau, x(\tau))d\tau \\
& - \int_{0}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau\| \\
= & \| \int_{-\omega}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau+\omega, x(\tau+\omega))d\tau \\
& - \int_{0}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau\| \\
\leq & \| \int_{-\omega}^{0} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau+\omega, x(\tau+\omega))d\tau\| \\
& +\| \int_{0}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)(\Pi(\tau+\omega, x(\tau+\omega)) \\
& -\Pi(\tau, x(\tau+\omega)))d\tau\| \\
& +\| \int_{0}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)(\Pi(\tau, x(\tau+\omega)) \\
& -\Pi(\tau, x(\tau)))d\tau\|. \\
= & Q_1 + Q_2 + Q_3.
\end{aligned} \tag{3.19}$$

Note that, from Lemma 3.1, $(\theta+\omega)^q - \theta^q \leq \omega^q$. Hence, by taking into account $\tau \in [-\omega, 0] \Longrightarrow \tau+\omega \in [0, \omega]$, it yields from (3.9)

$$\begin{aligned}
Q_1 = & \| \int_{-\omega}^{0} (\theta-\tau)^{q-1} P_q(\theta-\tau)\Pi(\tau+\omega, x(\tau+\omega))d\tau\| \\
\leq & L \sup_{s\in[0,\omega], \|v\|\leq\|x\|_{SAP_\omega PC(J,E)}} \|\Pi(s, v)\| \int_{-\omega}^{0} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} d\tau \\
\leq & \frac{L\varkappa.}{(1+\theta)^{2q}} \int_{-\omega}^{0} (\theta-\tau)^{q-1} d\tau = \frac{L\sigma_x,}{q(1+\theta)^{2q}}((\theta+\omega)^q - \theta^q) \\
\leq & \frac{L\varkappa.\omega^q}{q(1+\theta)^{2q}},
\end{aligned} \tag{3.20}$$

where, $\varkappa = \sup_{s\in[0,\omega], \|v\|\leq\|x\|_{SAP_\omega PC_b(J,E)}} \|\Pi(s, v)\|$.

Next, by (3.1), (3.2), (3.9), (3.10) and (3.15), we get

$$\begin{aligned}
Q_2 = & \| \int_{0}^{\theta} (\theta-\tau)^{q-1} P_q(\theta-\tau)(\Pi(\tau+\omega, x(\tau+\omega)) \\
& -\Pi(\tau, x(\tau+\omega)))d\tau\| \\
\leq & L \int_{0}^{\theta} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} (1+\|x(\tau+\omega))\| h_2(\tau)d\tau
\end{aligned}$$

$$\leq \quad L(1+\|x\|_{SAP_\omega PC_b(J,E)})\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}h_2(\tau)d\tau, \tag{3.21}$$

and

$$\begin{aligned}
Q_3 &= \int_0^\theta (\theta-\tau)^{q-1}\|P_q(\theta-\tau)\|\;\|\Pi(\tau,x(\tau+\omega))-\Pi(\tau,x(\tau))\|d\tau \\
&\leq L\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}\|x(\tau+\omega))-x(\tau)\|h_1(\tau)d\tau \\
&\leq L\int_0^{\theta_\epsilon} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}\|x(\tau+\omega))-x(\tau)\|h_1(\tau)d\tau \\
&\quad +L\int_{\theta_\epsilon}^{\theta} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}\|x(\tau+\omega))-x(\tau)\|h_1(\tau)d\tau \\
&< c_1c_2L\int_0^{\theta_\epsilon} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau+\epsilon \\
&< c_1c_2L\int_0^{\theta_\epsilon}(\theta-\tau)^{-q-1}d\tau+\epsilon \\
&< c_1c_2L\frac{\theta^{-q}-(\theta-\theta_\epsilon)^{-q}}{q}+\epsilon,
\end{aligned} \tag{3.22}$$

where $c_1=\sup_{\tau\in[0,\theta_\epsilon]}\|x(\tau+\omega))-x(\tau)\|$ and $c_2=\sup_{\tau\in[0,\theta_\epsilon]}h_1(\tau)$. Combining (3.19–3.22), one obtains,

$$\begin{aligned}
&\lim_{\theta\to\infty}\|\int_0^{\theta+\omega}(\theta+\omega-\tau)^{q-1}P_q(\theta+\omega-\tau)\Pi(\tau,x(\tau))d\tau \\
&-\int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)\Pi(\tau,x(\tau))d\tau\| \\
<\;&\lim_{\theta\to\infty}\frac{L\varkappa.\omega^q}{q(1+\theta)^{2q}}+L(1+\|x\|)\lim_{\theta\to\infty}\int_0^\theta\frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}h_2(\tau)d\tau \\
&+c_1c_2L\lim_{\theta\to\infty}\frac{(\theta-\tau)^{-q}-\theta^{-q}}{q}+\epsilon.
\end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned}
&\|\int_0^{s_i+\omega}(s_i+\omega-\tau)^{q-1}P_q(s_i+\omega-\tau)\Pi(\tau,x(\tau))d\tau \\
&-\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)\Pi(\tau,x(\tau))d\tau\| \\
<\;&\frac{L\varkappa.\omega^q}{q(1+s_i)^{2q}}+L(1+\|x\|)\int_0^{s_i}\frac{(s_i-\tau)^{q-1}}{(1+s_i-\tau)^{2q}}h_2(\tau)d\tau \\
&+c_1c_2L\frac{\theta^{-q}-(\theta-\theta_\epsilon)^{-q}}{q}+\epsilon.
\end{aligned} \tag{3.24}$$

Note that $s_i\to\infty$ when $\theta\to\infty$. Therefore, using (3.16)–(3.18), (3.13) and (3.24), we derive $\lim_{\theta\to\infty}\|\Phi(x)(\theta+\omega)-\Phi(x)(\theta)\|=0$.

Step 2. We show that, for any $x \in SAP_{\omega}PC_b(J, E)$, $\Phi(x)$ is bounded.

Let $\theta \in J$.

(i) Let $\theta \in [0, \theta_1]$. Then, applying Lemma (1.2) (iii), (3.9) and (3.14), one gets

$$\begin{aligned} \|\Phi(x)(\theta)\| &\leq M\|x_0\| + M\omega\|x_1\| \\ &\quad + L\int_0^{\theta} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}\|\Pi(\tau, x(\tau))\|d\tau. \end{aligned} \tag{3.25}$$

On the hand, from (3.1), we get

$$\begin{aligned} \|\Pi(\tau, x(\tau))\| &\leq \leq \|\Pi(\tau, 0\| + h_1(\tau)\|x(\tau)\| \\ &\leq \xi + h_1(\tau)\|x\|_{SAP_{\omega}PC_b(J,E)}. \end{aligned} \tag{3.26}$$

By (3.25) and (3.26), we get

$$\begin{aligned} \|\Phi(x)(\theta)\| &\leq M\|x_0\| + M\omega\|x_1\| \\ &\quad + L\xi\int_0^{\theta} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau + \varsigma L\|x\|_{SAP_{\omega}PC_b(J,E)} \\ &\leq M\|x_0\| + M\omega\|x_1\| \\ &\quad + L\xi\int_0^{\theta} \frac{\delta^{q-1}}{(1+\delta)^{2q}}d\delta + \varsigma L\|x\|_{SAP_{\omega}PC_b(J,E)} \\ &\leq M\|x_0\| + M\omega\|x_1\| \\ &\quad + L\xi\int_0^{\infty} \frac{\delta^{q-1}}{(1+\delta)^{2q}}d\delta + \varsigma L\|x\|_{SAP_{\omega}PC_b(J,E)} \\ &= M\|x_0\| + M\omega\|x_1\| \\ &\quad + L\xi B(q,q) + \varsigma L\|x\|_{SAP_{\omega}PC_b(J,E)}, \end{aligned} \tag{3.27}$$

where B is the beta function. Hence, y is bounded on $[0, \theta_1]$.

(ii) If $\theta \in (\theta_i, s_i]$, $i \in \mathbb{N}$, then, by (3.7), it yields

$$\|\Phi(x)(\theta)\| = \|g_i(s_i, x(\theta_i^-))\| \leq \kappa_1(\|x\| + 1), \forall z \in E. \tag{3.28}$$

(iii) If $\theta \in (s_i, \theta_{i+1}]$, then it follows from (3.8) and Lemma (1.2) that

$$\begin{aligned} &\|C_q(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta - s_i)g_i'(s_i, x(\theta_i^-))\| \\ \leq\ & M\kappa_1(1 + \|x(\theta_i^-)\|) + M\ \omega\kappa_1(1 + \|x(\theta_i^-)\|). \end{aligned} \tag{3.29}$$

Moreover, as in (3.27),

$$\begin{aligned} &\left\|\int_0^{\theta}(\theta-\tau)^{q-1}P_q(\theta-\tau)\Pi(\tau, x(\tau))d\tau\right\| \\ \leq\ & L\int_0^{\theta}\frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}\Pi(\tau, x(\tau))d\tau\| \\ \leq\ & L\xi B(q,q) + \varsigma L\|x\|_{SAP_{\omega}PC_b(J,E)}. \end{aligned} \tag{3.30}$$

Similarly, we can derive

$$\begin{aligned}&\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)\Pi(\tau,x(\tau))d\tau\\ \leq\ & L\xi B(q,q)+\varsigma L\|x\|_{SAP_\omega PC_b(J,E)}.\end{aligned}\tag{3.31}$$

As a result of (3.27)–(3.31), we conclude that y is bounded on J.

Now, $\Phi(x)$ is continuous on J_i $i\in\{0\}\cup\mathbb{N}$, and, hence, from Steps 1 and 2, we confirm that $\Phi(x)\in SAP_\omega PC_b(J,E)$. Thus, Φ is a function from $SAP_\omega PC_b(J,E)$ to itself.

Step 3. We show in this step that Φ is a contraction mapping from $SAP_\omega PC_b(J,E)$ to $SAP_\omega PC_b(J,E)$.

To show this, let $x,y\in SAP_\omega PC_b(J,E)$.We have three cases.

Case 1. $\theta\in[0,\theta_1]$

Using (3.14), it yields

$$\begin{aligned}&\|\Phi(x)(\theta)-\Phi(y)(\theta)\|\\ \leq\ & \|\int_0^{\theta}(\theta-\tau)^{q-1}P_q(\theta-\tau)\Pi(\tau,x(\tau))d\tau\\ &-\int_0^{\theta}(\theta-\tau)^{q-1}P_q(\theta-\tau)\Pi(\tau,y(\tau))d\tau\|.\end{aligned}\tag{3.32}$$

Using Lemma 1.2, (3.2), (3.4), (3.8) and (3.9), relation (3.32) becomes

$$\begin{aligned}&\|\Phi(x)(\theta)-\Phi(y)(\theta)\|\\ \leq\ & L\|x-y\|_{SAP_\omega PC_b(J,E)}\int_0^{\theta}\frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}h_1(\tau)d\tau\\ \leq\ & L\varsigma\|x-y\|_{SAP_\omega PC_b(J,E)}.\end{aligned}\tag{3.33}$$

Case 2. $\theta\in(\theta_i,s_i]$, $i\in\mathbb{N}$. Relations (3.5) and (3.14) lead to

$$\begin{aligned}&\|\Phi(x)(\theta)-\Phi(y)(\theta)\|\\ =\ & \|g_i(\theta,x(\theta_i^-))-g_i(\theta,y(\theta_i^-))\|\\ \leq\ & N\|x(\theta_i^-)-y(\theta_i^-)\|\leq N\|x-y\|_{SAP_\omega PC_b(J,E)},\end{aligned}\tag{3.34}$$

where $N=\max_{1\leq i\leq m}\{N_i\}$.

Case 3. $\theta\in[s_i,\theta_{i+1}]$, $i\in\mathbb{N}$. It yields from (3.5), (3.6) and (3.14)

$$\begin{aligned}&\|C_q(\theta-s_i)g_i(s_i,x(\theta_i^-))-C_q(\theta-s_i)g_i(s_i,y(\theta_i^-))\|\\ \leq\ & MN\|x-y\|_{SAP_\omega PC_b(J,E)},\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}&\|K_q(\theta-s_i)g_i'(s_i,x(\theta_i^-))-K_q(\theta-s_i)g_i'(s_i,y(\theta_i^-))\|\\ \leq\ & M\omega\mathcal{N}\|x(\theta_i^-)-y(\theta_i^-)\|\leq M\omega\mathcal{N}\|x-y\|_{SAP_\omega PC_b(J,E)},\end{aligned}\tag{3.36}$$

where $\mathcal{N}=\max_{1\leq i\leq m}\{\mathcal{N}_i\}$.

Moreover, similar to (3.33),

$$\begin{aligned}
&\| \int_0^{\theta} (\theta - s)^{q-1} P_q(\theta - \tau)\Pi(\tau, x(\tau))d\tau \\
&- \int_0^{\theta} (\theta - \tau)^{q-1} P_q(\theta - \tau)\Pi(\tau, y(\tau))d\tau\| \\
\leq\ & L\varsigma\|x - y\|_{SAP_{\omega}PC_b(J,E)}, \qquad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
&\| \int_0^{s_i} (s_i - \tau)^{q-1} P_q(s_i - \tau)\Pi(\tau, x(\tau))d\tau| \\
&- \int_0^{s_i} (s_i - \tau)^{q-1} P_q(s_i - \tau)\Pi(\tau, y(\tau))d\tau\| \\
\leq\ & L\varsigma\|x - y\|_{SAP_{\omega}PC_b(J,E)}. \qquad (3.38)
\end{aligned}$$

Due to (3.33)–(3.38), we conclude that

$$\begin{aligned}
&\|\Phi(x) - \Phi(y)\| \\
\leq\ & \|x - y\|_{SAP_{\omega}PC_b(J,E)}(MN + M\omega\mathcal{N} + 2L\varsigma). \qquad (3.39)
\end{aligned}$$

It yields from (3.11) and (3.39) that Φ is contraction. Applying the Banach fixed point theorem, we have that Φ has a unique fixed-point which is an S-asymptotically ω-periodic solution to Problem (1.1).

Remark 3.2. *If $h_1$is bounded on J, then relation (3.10) is verified. In fact, suppose that $h_1(\tau) \leq \kappa, \forall \tau \in J$. We have*

$$\begin{aligned}
\varsigma &= \int_0^{\theta} \frac{(\theta - \tau)^{q-1}}{(1 + \theta - \tau)^{2q}} h_1(\tau)d\tau \\
&\leq \kappa \int_0^{\theta} \frac{(\theta - \tau)^{q-1}}{(1 + \theta - \tau)^{2q}} d\tau \\
&\leq \kappa \int_0^{\theta} \frac{\delta^{q-1}}{(1 + \delta)^{2q}} d\delta. \\
&\leq \kappa \int_0^{\infty} \frac{\delta^{q-1}}{(1 + \delta)^{2q}} d\delta \\
&= \kappa B(q, q) < \infty, \qquad (3.40)
\end{aligned}$$

where B is the beta function. Thus, (3.10) is verified.

Remark 3.3. *If $\lim_{\theta\to\infty} \int_0^{\theta} h_2(\tau)d\tau = 0$, then relation (3.13) is verified. In fact, we have*

$$\begin{aligned}
&\lim_{\theta\to\infty} \int_0^{\theta} \frac{(\theta - \tau)^{q-1}}{(1 + \theta - \tau)^{2q}} h_2(\tau)d\tau \\
\leq\ & \lim_{\theta\to\infty} \int_0^{\theta} h_2(\tau)d\tau = 0. \qquad (3.41)
\end{aligned}$$

Corollary 3.1. *Suppose that conditions* (HA) *and* (Hg_i)) *are satisfied. If* $(H\Pi)$ *is verified with* $h_1(\tau) \leq \kappa, \forall \tau \in J$, *and* $\lim_{\theta \to \infty} \int_0^\theta h_2(\tau)d\tau = 0$ *then, by (3.41) and Theorem (1.1), Problem (1.1) has a unique* S*-asymptotically* ω*-periodic provided that*

$$MN + M\omega\mathcal{N} + 2L\kappa B(q,q) < 1. \tag{3.42}$$

Remark 3.4. *If there is no impulses effect, then* $N = \mathcal{N} = 0$. *Hence, relations (3.42) becomes*

$2L\kappa B(q,q) < 1$.

4. S-asymptotically ω-periodic mild solutions for Problem (1.2)

In this section, we demonstrate the existence of S-asymptotically ω-periodic mild solutions for 1.2. We denote by $P_{ck}(E)$ the family of non-empty, convex and compact subsets of E.

Consider the following assumptions:

(HF) $F : J \times E \to P_{ck}(E)$ is a multi-valued function such that:

(i) For any $z \in E$, the multi-valued function $\theta \to F(.,z)$ is strongly measurable.

(ii) For any $x \in PC(J,E)$, the set

$$S^1_{F(..x(.))} := \{\varphi : J \to E, \varphi \text{ is locally integrable, and } \varphi(\tau) \in F(\tau, x(\tau)), a.e. \theta \in J\}$$

is not empty.

(iii) There is a measurable bounded, almost everywhere, function $L_1 : J \to J$ such that

$$h(F(\theta,z_1), F(\theta,z_2)) \leq L_1(\theta)\|z_1 - z_2\|, \forall \theta \in J,\ u, z_2 \in E, \tag{4.1}$$

where h is the Hausdorff distance.

(iv) There is $L_2 \in C(J, \mathbb{R}^+)$ such that

$$h(F(\theta+\omega, z), F(\theta, z)) \leq L_2(\theta)\|1 + z\|, \forall \theta \in J,\ z \in E. \tag{4.2}$$

(v) The function

$$t \longmapsto \sigma(\tau) := \|F(\tau,0)| = \sup_{z \in F(\tau,0)} \|z\| \tag{4.3}$$

is bounded almost everywhere on J.

We need the following Lemma, which is due to Covitz and Nadler [34].

Lemma 4.1. *Let* (X,d) *be a metric space and G be a contraction multi-valued function from X to the family of non-empty closed subsets of X. Then, G has a fixed point.*

Theorem 4.1. *Under conditions* $(HA)^*, (HF), (Hg_i)$ *and* (H), *Problem (1.2) has an* S*-asymptotically* ω*-periodic mild solution provided that*

$$\lim_{\theta \to \infty} \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} \sigma(\tau)d\tau = 0, \tag{4.4}$$

$$\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}L_1(\tau)d\tau = 0, \tag{4.5}$$

$$\lim_{\theta\to\infty}\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}L_2(\tau)d\tau = 0, \tag{4.6}$$

and

$$MN + M\omega\mathcal{N} + 2L\omega_1 B(q,q) < 1, \tag{4.7}$$

where $|L_1(t)| \le \lambda_1, a.e.\, t \in J$.

Proof. Due to $(HF)(ii)$, for any $x \in SAP_\omega PC_b(J,E)$, the set $S^1_{F(.,x(.))}$ is not empty. Therefore, for any $x \in SAP_\omega PC_b(J,E)$, we can define a multi-valued function $R(x)$ as follows: an element $y \in R(x)$ if and only if

$$y(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ + \int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta - s_i)g_i'(s_i, x(\theta_i^-)) \\ - \int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f(\tau)d\tau \\ + \int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $f \in S^1_{F(.,x(.))}$. Since the proof is similar to what was shown in Theorem 1.1, we will illustrate only the differences.

Step 1. We show that if $x \in SAP_\omega PC_b(J,E)$ and $y \in R(x)$, then $\lim_{\theta\to\infty}\|y(\theta+\omega) - y(\theta)\| = 0$.

Let $\epsilon > 0$. Because $x \in SAP_\omega PC_b(J,E)$, then $\lim_{\theta\to\infty}\|x(\theta+\omega) - x(\theta)\| = 0$ and, hence, there is $\theta_\epsilon > \theta_1$ such that (3.15) is verified.

Let $y \in R(x)$ and $\theta \in [s_i, \theta_{i+1}]$. According to (4.8), we have

$$\begin{aligned} & \|\int_0^{\theta+\omega}(\theta+\omega-\tau)^{q-1}P_q(\theta+\omega-\tau)f(\tau)d\tau \\ & - \int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau\| \\ = \; & \|\int_{-\omega}^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau+\omega)d\tau \\ & - \int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau\| \\ \le \; & \|\int_{-\omega}^0(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau+\omega)d\tau\| \\ & +\|\int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau+\omega)d\tau \\ & - \int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau\| \\ = \; & I_1 + I_2. \end{aligned} \tag{4.9}$$

Let $\tau \in [-\omega, 0]$ be fixed. Since $F(\tau+\omega, 0)$ is compact, there is $v_{\tau+\omega} \in F(\tau+\omega, 0)$ such that

$$\begin{aligned}\|f(\tau+\omega) - v_{\tau+\omega}\| &= d(f(\tau+\omega), F(\tau+\omega, 0)\\ &\leq h(F(\tau+\omega, x(\tau+\omega)), F(\tau+\omega, 0)).\end{aligned} \tag{4.10}$$

From (4.1), (4.3) and (4.10), we get

$$\begin{aligned}\|f(\tau+\omega)\| &\leq h(F(\tau+\omega, x(\tau+\omega)), F(\tau+\omega, 0)) + \|v_{\tau+\omega}\|\\ &\leq L_1(\tau+\omega)\|x(\tau+\omega)\| + \sigma(\tau+\omega)\\ &\leq \|x\|L_1(\tau+\omega) + \sigma(\tau+\omega), \forall \tau \in [-\omega, 0].\end{aligned} \tag{4.11}$$

Then, by (3.9) and (4.11), it follows that

$$\begin{aligned}\lim_{\theta\to\infty} I_1 &= \lim_{\theta\to\infty} \|\int_{-\omega}^{0} (\theta-\tau)^{q-1} P_q(\theta-\tau) f(\tau+\omega) d\tau\|\\ &\leq \lim_{\theta\to\infty} (\omega_1\|x\| + \omega_2) L \int_{-\omega}^{0} \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} d\tau\\ &\leq (\lambda_1\|x\| + \lambda_2) L \lim_{\theta\to\infty} \frac{1}{(1+\theta)^{2q}} \int_{-\omega}^{0} (\theta-\tau)^{q-1} d\tau\\ &= (\lambda_1\|x\| + \lambda_2) L \lim_{\theta\to\infty} \frac{(\theta+\omega)^q - \theta^q}{q(1+\theta)^{2q}}\\ &\leq (\lambda_1\|x\| + \lambda_2) L \lim_{\theta\to\infty} \frac{\omega^q}{q(1+\theta)^{2q}} = 0,\end{aligned} \tag{4.12}$$

where λ_2 is a positive number such that $\sigma(\theta) \leq \lambda_2, a.e., \theta \in J$.

Next, let $\tau \in [0, \theta]$ be fixed. From the fact that $F(\tau+\omega, x(\tau))$ is compact, there are $z_{\tau+\omega}, z_\tau \in F(\tau, x(\tau+\omega))$ such that $d(f(\tau+\omega), z_{\tau+\omega}) = d(f(\tau+\omega), F(\tau, x(\tau+\omega)))$ and $d(f(\tau), z_\tau) = d(f(\tau), F(\tau, x(\tau+\omega)))$. Then, by (4.1) and (4.2), it yields

$$\begin{aligned}&\|f(\tau+\omega) - f(\tau)\|\\ \leq\ &\|f(\tau+\omega) - z_{\tau+\omega}\| + \|z_{\tau+\omega} - z_\tau\| + \|z_\tau - f(\tau)\|\\ \leq\ &d(f(\tau+\omega), F(\tau + x(\tau+\omega))) + \|z_{\tau+\omega} - z_\tau\|\\ &+d(f(\tau, F(\tau, x(\tau+\omega)))\\ \leq\ &h(F((\tau+\omega), x(\tau+\omega)), F(\tau + x(\tau+\omega)))\\ &+2\|F(\tau, x(\tau+\omega))\|\\ &+h(F(\tau, x(\tau+\omega))), F(\tau, x(\tau))\\ \leq\ &L_1(\tau)\|x((\tau+\omega) - x(\tau)\|\\ &+2\|F(\tau, x(\tau+\omega))\| + L_2(\tau)\|1 + x(\tau)\|\\ \leq\ &2\|x\|L_1(\tau) + 2\|F(\tau, x(\tau+\omega))\| + L_2(\tau)\|1 + x(\tau)\|.\end{aligned} \tag{4.13}$$

Moreover, according to (4.1) and (4.3), we get

$$\|F(\tau, x(\tau+\omega))\| \leq \|F(\tau, 0)\| + L_1(\tau)\|x(\tau+\omega)\|$$

$$= \quad \sigma(\tau) + L_1(\tau)\|x\|. \tag{4.14}$$

Then, by (4.13) and (4.14), one obtains

$$\begin{aligned} I_2 &\leq \| \int_0^\theta (\theta-\tau)^{q-1} \|P_q(\theta-\tau)\| \, f(\tau+\omega) - f(\tau) d\tau \| \\ &\leq 4\|x\| L \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} L_1(\tau) d\tau \\ &\quad +2L \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} \sigma(\tau) d\tau \\ &\quad +L(1+\|x\|) \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} L_2(\tau) d\tau. \end{aligned} \tag{4.15}$$

Using (4.4)–(4.6) and (4.15), it yields

$$\begin{aligned} \lim_{\theta\to\infty} I_2 &= \lim_{\theta\to\infty} \| \int_0^{\theta+\omega} (\theta+\theta-\tau)^{q-1} P_q(\theta+\omega-\tau) f(\tau+\omega) d\tau \\ &\quad - \int_0^\theta (\theta-\tau)^{q-1} P_q(\theta-\tau) f(\tau) ds \| = 0. \end{aligned} \tag{4.16}$$

Note that $\tau_i \to \infty$ when $\theta \to \infty$. Hence, as above, we derive

$$\begin{aligned} &\lim_{\theta\to\infty} \| \int_0^{s_i+\omega} (s_i+\omega-\tau)^{q-1} P_q(s_i+\omega-\tau) f(\tau) d\tau \\ &- \int_0^{s_i} (s_i-\tau)^{q-1} P_q(s_i-\tau) f(\tau) d\tau \| \\ = \quad & 0. \end{aligned} \tag{4.17}$$

Then, due to (3.16)–(3.18), (4.9), (4.12), (4.16) and (4.17), we conclude that

$$\lim_{\theta\to\infty} \|y(\theta+\omega) - y(\theta)\| = 0$$

Step 2. In this step, we show that if $x \in SAP_\omega PC_b(J,E)$ and $y \in R(x)$, then y is bounded.

Let $\theta \in [0, \theta_1]$. Then, using Lemma 1.2, (3.9) and (4.8), one has

$$\begin{aligned} \|y(\theta)\| &\leq M\|x_0\| + M\omega\|x_1\| \\ &\quad +L \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}} \|f(\tau)\| d\tau. \end{aligned} \tag{4.18}$$

On the hand, from (4.1), we get

$$\begin{aligned} \|f(\tau)\| &\leq \|F(\tau, x(\tau))\| \leq \|F(\tau, 0)\| + L_1(\tau)\|x(\tau)\| \\ &\leq \sigma(\tau) + L_1(\tau)\|x\|, \forall \tau \in J. \end{aligned} \tag{4.19}$$

By (4.18) and (4.19), it yields

$$\|y(\theta)\| \quad \leq \quad M\|x_0\| + M\omega\|x_1\|$$

$$\begin{aligned}
&\quad + L(\lambda_2 + \|x\|\lambda_1)\int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau \\
&= M\|x_0\| + M\omega\|x_1\| \\
&\quad + L(\omega_2 + \|x\|\omega_1)\int_0^\theta \frac{\delta^{q-1}}{(1+\delta)^{2q}}d\tau \\
&= M\|x_0\| + M\omega\|x_1\| \\
&\quad + L(\omega_2 + \|x\|\omega_1)B(q,q),
\end{aligned} \tag{4.20}$$

where B is the beta function. Therefore, y is bounded on $[0, \theta_1]$. Similarly, one can show that if $\theta \in [s_i, \theta_{i+1}]$, then

$$\begin{aligned}
&\|\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau))d\tau\| \\
\leq\ & L(\omega_2 + \|x\|\omega_1)B(q,q),
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
&\int_0^{s_i} (s_i-\tau)^{q-1}P_q(s_i-\tau)\Pi(\tau, x(\tau))d\tau \\
\leq\ & L(\omega_2 + \|x\|\omega_1)B(q,q).
\end{aligned} \tag{4.22}$$

Then, by (4.20)–(4.22) and by arguing as in (3.28) and (3.29), we deduce that y is bounded on J, and our claim in this step is proved.

As a result of Eqs 1.1 and 1.2 , R is a multivalued function from $SAP_\omega PC_b(J, E)$ to the non-empty subsets of $SAP_\omega PC_b(J, E)$.

Next, in order to apply Lemma 3.2 and show that R has a fixed point, we have to show that R is a contraction where its set of values is closed. We do this in two steps.

Step 3. The set of values of R is closed.

Let $x \in SAP_\omega PC_b(J, E)$ and $(y_n)_{n\geq1}$be a sequences in $R(x)$ with $y_n \to y$ in $SAP_\omega PC_b(J, E)$. Then, there is $f_n \in S^1_{F(.,x(.))}$ such that

$$y_n(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ + \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_n(\tau)d\tau, \theta \in [0, \theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta - s_i)g_i'(s_i, x(\theta_i^-)) \\ - \int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f_n(\tau)d\tau \\ + \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_n(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{4.23}$$

We have to show that $y \in R(x)$. By arguing as in (4.19) , one obtains

$$\|f_n(\tau)\| \leq \sigma(\tau) + L_1(\tau)\|x\|, \forall \tau \in J. \tag{4.24}$$

Now, let θ be a fixed point in J, and $J_\theta = [0, \theta]$. From the fact that σ and L_1 are bounded almost everywhere, we can deduce, from (4.24), that the family $\{f_n : n \geq 1\}$ is bounded in $L^2(J_\theta, E)$ and, hence, it is weakly compact in $L^2(J_\theta, E)$. Thus, it has a subsequence, denoted again by $(f_n)_{n\geq1}$, such that $f_n \to f$ weakly in $L^2(J_\theta, E)$. According to Mazur's lemma, we can find a sequence $(z_n)_{n\geq 1}$ of convex combinations of f_n with $z_n \to f$ strongly in $L^2(J_\theta, E)$. Then, we can assume, without loss of generality, that $z_n(\tau) \to f(\tau), a.e.\, \tau \in J_\theta$. Moreover, from (4.24) and Lemma 1.2, we get

$$\begin{aligned} & (\theta-\tau)^{q-1}\|P_q(\theta-\tau)f_n(\tau)\| \\ \leq\ & \frac{M}{\Gamma(2q)}(\theta-\tau)^{2q-1}(\lambda_2+\lambda_1\|x\|), a.e.\tau \in [0,\theta]. \end{aligned}$$

Note that the function $\tau \to (\theta-\tau)^{2q-1}$ belongs to $L^1([0,\theta], E)$. Therefore, by the continuity of $P_q(.)$ and applying the Lebesgue dominated convergence theorem, it yields

$$\begin{aligned} & \lim_{n\to\infty}\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_n(\tau)d\tau \\ =\ & \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau. \end{aligned} \tag{4.25}$$

Thus, from (4.25) and the continuity $P_q(.)$, it follows, by taking the limit as $n \to \infty$ in (4.23), that

$$\lim_{n\to\infty} y_n(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta-s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta-s_i)g_i'(s_i, x(\theta_i^-)) \\ -\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f(\tau)d\tau \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{4.26}$$

Note that $(HF)(iv)$ leads to $f(s) \in F(s, x(s)), a.e.\ s \in J$ and, hence, (4.26) leads to

$$y(\theta) = \begin{cases} C_q(\theta)(x_0 - g(x)) + K_q(\theta)(x_1 - p(x)) \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [0,\theta_1], \\ g_i(\theta, x(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta-s_i)g_i(s_i, x(\theta_i^-)) + K_q(\theta-s_i)g_i'(s_i, x(\theta_i^-)) \\ -\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f(\tau)d\tau \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases}$$

Then, $y \in R(x)$.

Step 4. We show that R is a contraction.

Let $u_1, u_2 \in SAP_\omega PC(J, E)$ and $y_1 \in R(u_1)$. Then, there is $f \in S^1_{F(.,u(.))}$ such that

$$y_1(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_1(\tau)d\tau, \theta \in [0,\theta_1], \\ g_i(\theta, u_1(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta-s_i)g_i(s_i, u_1(\theta_i^-)) + K_q(\theta-s_i)g_i'(s_i, u_1(\theta_i^-)) \\ -\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f_1(\tau)d\tau \\ +\int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_1(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{4.27}$$

Consider the multivalued function $\Theta : J \to 2^E$ defined by:

$$\Theta(\theta) = \{z \in E : \|z - f_1(\theta)\| \le L_1(\theta)\|u_1(\theta) - u_2(\theta)\|, a.e.\theta \in J\}.$$

We show that the set of values of Θ is non-empty. Let $\theta \in J$. From (4.1), we get

$$h(F(\theta, u_1(\theta)), F(\theta, u_2(\theta))) \le L_1(\theta)\,\|u_1(\theta) - u_2(\theta)\|\,.$$

Thus, from the compactness of $F(\theta, u_2(\theta))$, there is $z_\theta \in F(\theta, u_2(\theta))$ such that

$$||f_1(\theta) - z_\theta||\ \le h(F(\theta, u_1(\theta)), F(\theta, u_2(\theta))) \le L_1(\theta)\,\|u_1(\theta) - u_2(\theta)\|\,,$$

which leads to $\Theta(\theta) \neq \phi$, $\theta \in J$. Moreover, the set $\Lambda(\theta) = \Theta(\theta) \cap F(\theta, u_2(\theta))$, $\theta \in J$ is not empty. Because the functions f_1, L_1, u_1, u_2 are measurable, Proposition 3.4 in [35] (Corollary 1.3.1(a) in [36]) guarantees that the multivalued map $\theta \to \Lambda(\theta)$ is measurable. Note that $\Theta(\theta)$, $\theta \in J$ is closed. Consequently, the set of values of Λ is non-empty and compact and, hence, by Theorem 3.1.1 in [37], there exists a measurable selection f_2 for Λ with

$$\|f_1(\theta) - f_2(\theta)\| \le L_1(\theta)\,\|u_1(\theta) - u_2(\theta)\|\,, a.e.\theta \in\ J. \tag{4.28}$$

Set

$$y_2(\theta) = \begin{cases} C_q(\theta)x_0 + K_q(\theta)x_1 \\ +\int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f_2(\tau)d\tau, \theta \in [0, \theta_1], \\ g_i(\theta, u_2(\theta_i^-)), \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\ C_q(\theta - s_i)g_i(s_i, u_2(\theta_i^-)) + K_q(\theta - s_i)g_i'(s_i, u_2(\theta_i^-)) \\ -\int_0^{s_i}(s_i-\tau)^{q-1}P_q(s_i-\tau)f_2(\tau)d\tau \\ +\int_0^\theta(\theta-\tau)^{q-1}P_q(\theta-\tau)f_2(\tau)d\tau, \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}. \end{cases} \tag{4.29}$$

Obviously, $y_2 \in R(u_1)$. Now, we estimate the value of $\|y_1 - y_2\|$.

Let $\theta \in [0, \theta_1]$. Using Lemma 1.2, (3.8), (3.9) and (4.27)–(4.29), we get

$$\begin{aligned} &\|y_1(\theta) - y_2(\theta)\| \\ \le\ &\| \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)\|f_1(\tau) - f_2(\tau)\|d\tau \\ \le\ &L\omega_1\|u_1 - u_2\| \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau \\ \le\ &L\omega_1\|u_1 - u_2\| \int_0^\infty \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau \\ \le\ &\|u_1 - u_2\|L\omega_1 B(q, q)). \end{aligned} \tag{4.30}$$

Let $\theta \in [s_i, \theta_{i+1}]$, $i \in \mathbb{N}$. As in (4.30), one can show that

$$\begin{aligned} &\| \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_1(\tau)d\tau \\ &- \int_0^\theta (\theta-\tau)^{q-1}P_q(\theta-\tau)f_2(\tau)d\tau\| \end{aligned}$$

$$
\begin{aligned}
&\leq \;\; L\omega_1\|u_1-u_2\| \int_0^\theta \frac{(\theta-\tau)^{q-1}}{(1+\theta-\tau)^{2q}}d\tau \\
&\leq \;\; L\omega_1 B(q,q))\|u_1-u_2\|, \qquad (4.31)
\end{aligned}
$$

and

$$
\begin{aligned}
&\| \int_0^{s_i} (s_i-\tau)^{q-1}P_q(s_i-\tau)f_1(\tau)d\tau| \\
&- \int_0^{s_i} (s_i-\tau)^{q-1}P_q(s_i-\tau)f_2(\tau)d\tau\| \\
\leq &\;\; L\omega_1\|u_1-u_2\| \int_0^{s_i} \frac{(s_i-\tau)^{q-1}}{(1+s_i-\tau)^{2q}}d\tau \\
\leq &\;\; L\omega_1 B(q,q)\|u_1-u_2\|. \qquad (4.32)
\end{aligned}
$$

Combining relations (3.34)–(3.36) and (4.30)–(4.32), it yields

$$
\|y_1(\theta)-y_2(\theta)\| \leq \|u_1-u_2\|.(MN+M\omega\mathcal{N}+2L\omega_1 B(q,q)). \tag{4.33}
$$

Due to (4.7), relation (4.33) becomes

$$
\|y_1(\theta)-y_2(\theta)\| < \vartheta\|u_1-u_2\|, \tag{4.34}
$$

where $\vartheta = MN + M\omega\mathcal{N} + 2L\lambda_1 B(q,q) < 1$. By interchanging the role of y_1 and y_2 in the above discussion and using (4.7) and (4.34), we conclude that R is a contraction.

As a result of Steps 1.1–3.1 and by applying Lemma (3.2), R has a fixed-point which is S-asymptotically ω-periodic solution to Problem(1.2).

Remark 4.1. *As in Remark (2.1), if* $\lim_{\tau\to\infty}\sigma(\tau) = \lim_{\tau\to\infty} L_1(\tau) = \lim_{\tau\to\infty} L_2(\tau) = 0$*, then relations (4.4)–(4.6) are verified .*

Remark 4.2. *If there is no impulses effect, then* $N = \mathcal{N} = 0$ *and, hence, relation (4.7) becomes* $2L\lambda_1 B(q,q) < 1$.

5. Examples

In this section, we give two examples as applications of our results.

Example 5.1. *Let* $\alpha = \frac{3}{2}$, $q = \frac{3}{4}$, $E = L^2[0,\pi]$, $m = 4$, $\omega = 2\pi$, $J = [0,\infty)$, $s_i = i\frac{\pi}{2}$, $i \in \{0\}\cup\mathbb{N}$, *and* $\theta_i = (2i-1)\frac{\pi}{2}; i \in \mathbb{N}$. *Observe that* $s_4 = \omega$ *and for* $i \in \mathbb{N}$, $s_{i+m} = s_{i+4} = (i+4)\frac{\pi}{2} = i\frac{\pi}{2} + 2\pi = s_i + \omega$, *and* $\theta_{i+m} = \theta_{i+4} = (2i+7)\frac{\pi}{4} = (2i-1)\frac{\pi}{4} + 2\pi = \theta_i + 2\pi = \theta_i + \omega$.

Consider an operator $A : D(A) \subset E \to E$ defined as follows: $Av = v''$and

$$
D(A) := \{v \in L^2[0,\pi] : \; v_{yy} \in L^2[0,1], v(0) = v(\pi) = 0\}.
$$

Note that the operator A has the representation

$$
Ax = \sum_{n=1}^{\infty} -n^2 < x, \; x_n > x_n, x \in D(A), \tag{5.1}
$$

where $x_n(y) = \sqrt{2}\,\sin ny, n = 1, 2, ...,$ is the orthonormal set of eigenfunctions of A. Moreover, A is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t\in\mathbb{R}}$ which is given by

$$C(t)(x) = \sum_{n=1}^{\infty} \cos nt < x\,,\ x_n > x_n, x \in E,$$

and the associated sine family $S(t)_{t\in\mathbb{R}}$ on E is defined by

$$S(t)(x) := \sum_{n=1}^{\infty} \frac{\sin nt}{n} < x\,,\ x_n > x_n, x \in E.$$

It is known that $\|C(t)\| \leq e^{-\pi^2 t}$ and $\|S(t)\| \leq e^{-\pi^2 t}$ for $t \geq 0$ (see [38], P.1307). Therefore, the family $\{C(\theta) : \theta \geq 0\}$ is exponentially stable and the operator A satisfies $(HA)^*$ with $M = 1$.

Consider a function $\Pi : J \times E \to E$ defined by

$$\Pi(\theta, u)(s) := \kappa \sin u(s) + \cos\theta;\ \theta \in J,\ u \in E,\ s \in [0, \pi], \tag{5.2}$$

where $\kappa > 0$. We demonstrate that Π satisfies the conditions of Corollary (1.1). Let $u, v \in E = L^2[0, \pi]$. One has

$$\begin{aligned}
&\|\Pi(\theta, u) - \Pi(\theta, v)\|_{L^2[0,\pi]} \\
=\ & (\int_0^{\pi} |\Pi(\theta, u)(s) - \Pi(\theta, v)(s)|^2 ds)^{\frac{1}{2}} \\
=\ & \kappa(\int_0^{\pi} |\sin u(s) - \sin v(s)|^2 ds)^{\frac{1}{2}} \\
\leq\ & \kappa(\int_0^{\pi} |u(s) - v(s)|^2 ds)^{\frac{1}{2}} = \kappa\|u - v\|_{L^2[0,\pi]}.
\end{aligned} \tag{5.3}$$

Moreover,

$$\begin{aligned}
&\|\Pi(\theta + 2\pi, u) - \Pi(\theta, u)\|_{L^2[0,\pi]} \\
=\ & (\int_0^{\pi} |\Pi(\theta + 2\pi, u)(s) - \Pi(\theta, u)(s)|^2 ds)^{\frac{1}{2}} = 0.
\end{aligned} \tag{5.4}$$

Relations (5.3) and (5.4) leads to $(H\Pi)$, where $h_1(\theta) = \kappa$ and $h_2(\theta) = 0$, $\theta \in J$.

Next, for any $i \in \mathbb{N}$, let $g_i : [t_i, s_i] \times E \to E$, be defined as:

$$g_i(\theta, u)(s) := \frac{\lambda(\sin i\theta)}{i^2} u(s)\ ;(\theta, u) \in [t_i, s_i] \times E, s \in [0, \pi], \tag{5.5}$$

where λ is a positive real number. Then,

$$g_i'(s_i, u)(s) := \frac{\lambda(\cos is_i)}{i} u(s); u \in E, s \in [0, \pi], i \in \mathbb{N}.$$

Obviously, g_i is bounded on bounded subsets. Note that, for any $i \in \mathbb{N}$, any $\theta \in J$, and any $u, v \in E$, we have

$$\lim_{\substack{\theta\to\infty \\ i\to\infty}} (\|g_{i+m}(\theta + 2\pi, u) - g_i(\theta, u)\|_{L^2[0,\pi]})^2$$

$$
\begin{aligned}
&= \lim_{\substack{\theta\to\infty\\ i\to\infty}} \int_0^{\pi} |g_{i+m}(\theta+2\pi,u)(s) - g_i(\theta,u)(s)|^2 ds \\
&= \lim_{\substack{\theta\to\infty\\ i\to\infty}} \lambda^2 \int_0^{\pi} |(\frac{\sin(i+m)(\theta+2\pi))u(s)}{(i+m)^2} - \frac{(\sin i\theta)u(s)}{i^2}|^2 ds \\
&= \lambda^2 \lim_{\substack{\theta\to\infty\\ i\to\infty}} \int_0^{\pi} |(\frac{\sin(i+m)\theta)u(s)}{(i+m)^2} - \frac{(\sin i\theta)u(s)}{i^2}|^2 ds \\
&\leq 4\lambda^2 \lim_{\substack{\theta\to\infty\\ i\to\infty}} \int_0^{\pi} |\frac{u(s)}{(i+m)^2} + \frac{u(s)}{i^2}|^2 ds \\
&\leq \lim_{\substack{\theta\to\infty\\ i\to\infty}} \frac{4\lambda^2}{i^4} \int_0^{\pi} |u(s)|^2 ds = \frac{4\lambda^2}{i^4} ||u||^2_{L^2[0,\pi]} = 0, \qquad (5.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{i\to\infty} (||g'_{i+m}(s_i+2\pi,u) - g'_i(s_i,u)||_{L^2[0,\pi]})^2 \\
&= \lim_{i\to\infty} \int_0^{\pi} |g'_{i+m}(s_i+2\pi,u)(s) - g'_i(s_i,u)(s)|^2 ds \\
&= \lambda^2 \lim_{i\to\infty} \int_0^{\pi} |(\frac{\cos(i+m)(s_i+2\pi)u(s)}{i+m} - \frac{(\cos is_i)u(s)}{i}|^2 ds \\
&= \lambda^2 \lim_{i\to\infty} \int_0^{\pi} |(\frac{\cos(i+m)s_i)u(s)}{i+m} - \frac{(\cos is_i)u(s)}{i}|^2 ds \\
&\leq 4\lambda^2 \lim_{i\to\infty} \int_0^{\pi} |\frac{u(s)}{i+m} + \frac{u(s)}{i}|^2 ds \\
&\leq \lim_{i\to\infty} \frac{4\lambda}{i} \int_0^{\pi} |u(s)|^2 ds \lim_{i\to\infty} \frac{4\lambda}{i} ||u||^2_{L^2[0,\pi]} = 0. \qquad (5.7)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&||g_i(\theta,u) - g_i(\theta,v)||_{L^2[0,\pi]} \\
&= \lambda(\int_0^{\pi} |\frac{(\sin i\theta)u(s)}{i^2} - \frac{(\sin i\theta)v(s)}{i^2}|^2 ds)^{\frac{1}{2}} \\
&\leq \lambda||u-v||, \qquad (5.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&||g'_i(s_i,u) - g'_i(s_i,v)||_{L^2[0,\pi]} \\
&= \lambda(\int_0^{\pi} |\frac{(\sin i\theta)u(s)}{i} - \frac{(\sin i\theta)v(s)}{i}|^2 ds)^{\frac{1}{2}} \\
&\leq \lambda||u-v||. \qquad (5.9)
\end{aligned}
$$

Furthermore,

$$
||g_i(\theta,u)||_{L^2[0,\pi]} = \lambda(\int_0^{\pi} |\frac{(\sin i\theta)u(s)}{i^2}|^2 ds)^{\frac{1}{2}} \leq \lambda||u||, \qquad (5.10)
$$

and

$$\|g_i'(s_i,u)\|_{L^2[0,\pi]} = \lambda(\int_0^\pi |\frac{(\cos i\theta)u(s)}{i}|^2 ds)^{\frac{1}{2}} \leq \lambda\|u\|. \tag{5.11}$$

As a result of relations (5.6)–(5.11), (Hg_i) is satisfied where $N = \mathcal{N} = \lambda$ and $\kappa_1 = \kappa_2 = \lambda$. By applying Corollary (1.1), we conclude that Problem (1.1) has a unique $S-$asymptotically $2\pi-$periodic mild solution provided that

$$\lambda(1+\omega) + 2\kappa LB(q,q) < 1, \tag{5.12}$$

where A, Π, g_i are given by (5.1), (5.2) and (5.5), respectively, and L appears in (3.9). By choosing λ and κ sufficiently small, we can derive (5.12).

Example 5.2. *Assume that* $A, \alpha, q,\ E,\ m,\ \omega,\ J,\ s_i,\ \theta_i,\ i \in \mathbb{N}$ *are as in Example (1.1). Let* Z *be a non-empty convex compact subset of* E, $L_1 : J \to J$ *be a measurable bounded almost everywhere function such that* $Lim_{\theta\to\infty} L_1(\theta) = 0$ *and* $F : J \times E \to P_{ck}(E)$ *be a multi-valued function defined by*

$$F(\theta,u) = \frac{L_1(\theta)\|u\|\sin\theta}{\sigma\,(1+\|u\|)} Z; (\theta,u) \in J \times E, \tag{5.13}$$

where σ is a constant such that $Sup\{\ \|z\|\ :\ z \in Z\} \leq \sigma$. Clearly, for every $u \in E$, $\theta \to F(\theta,u)$ is strongly measurable and, for any $x \in PC(J,E)$, the function $f(\theta) = \frac{L_1(\theta)\ \|x(\theta)\|\ \sin\theta}{\sigma\ (1+\|x(\theta)\|)} z_0,\ z_0 \in Z$ is locally integrable, and $f(\theta) \in F(\theta, x(\theta))$, $\theta \in J$. Moreover, using (5.13), for any $u, v \in E$ and any $\theta \in J$, we have

$$\begin{aligned} H(F(\theta,u),F(\theta,v)) &\leq L_1(\theta)|\sin\theta|\,|\frac{\|u\|}{(1+\|u\|)} - \frac{\|v\|}{(1+\|v\|)}| \\ &\leq L_1(\theta)\|u-v\|, \end{aligned} \tag{5.14}$$

$$H(F(\theta+2\pi,u),F(\theta,u)) = 0, \tag{5.15}$$

and

$$\sup_{\theta\in J} \|F(\theta,0)\| = \{0\}. \tag{5.16}$$

Then, from (5.14)–(5.16), it follows that assumption (HF) is verified where $L_2(\theta) = \sigma(\theta) = 0,\ \theta \in J$. Thus, applying Theorem 1.2, Problem (1.2), where A, F, g_i are given by (5.1), (5.13) and (5.5), respectively, and L appears in (3.9), has an $S-$asymptotically $2\pi-$periodic mild solution provided that

$$\lambda + 2\pi\lambda + 2L\lambda_1 B(q,q) < 1,$$

where λ_1 is a positive number such that $|L_1(\theta)| \leq \lambda_1, a.e.$ for $\theta \in\ J$.

6. Conclusions

Because, in some works, it was demonstrated that there are no non-stationary periodic solutions of fractional differential equations, studying the existence of S-asymptotically ω-periodic solutions for fractional differential equations is necessary and important. Sufficient conditions that assure the existence of S-asymptotically ω-periodic solutions for non-instantaneous impulsive semilinear differential equations of order $1 < \alpha < 2$ and generated by the infinitesimal generator of a strongly continuous cosine family of bounded linear operators have been obtained. Also, the case when the single-valued function in the right-hand side is replaced by a multi-valued function is investigated. Examples are given to demonstrate the possibility of applicability of our results. Moreover, our results generalize the obtained one in [12] into the case where the order is $1 < \alpha < 2$, there are non-instantaneous impulse effects, and the right-hand side is a multi-valued function instead of a single-valued-function. Furthermore, our technique can be used to extend many problems that are considered in the literatures such as [13, 15–17, 20–25, 27–29] to the case where there are non-instantaneous impulse effects and the right-hand side is a multi-valued function instead of a single-valued-function.

Acknowledgments

This research has been funded by the Scientific Research Deanship at University of Ha'il-Saudi Arabia through project number RG-21 101.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. E. Hernandez, D. O'Regan, On a new class of abstract impulsive differential equation, *Proc. Amer. Math. Soc.*, **141** (2013), 1641–1649. https://doi.org/10.1090/S0002-9939-2012-11613-2
2. A. G. Ibrahim, A. A. Elmandouh, Existence and stability of solutions of ψ-Hilfer fractional functional differential inclusions with non-instantaneous impulses, *AIMS Math.*, **6** (2021), 10802–10832. https://doi.org/10.3934/math.2021628
3. J. R. Wang, M. Li, D. O'Regan, M. Fečkan, Robustness for nonlinear evolution equation with non-instantaneous effects, *B. Sci. Math.*, **159** (2020), 102827. https://doi.org/10.1016/j.bulsci.2019.102827
4. J. R. Wang, A. G. Ibrahim, D. O'Regan, Global attracting solutions to Hilfer fractional non-instantaneous impulsive semilinear differential inclusions of Sobolev type and with nonlocal conditions, *Nonlinear Anal. Model.*, **24** (2019), 775–803. https://doi.org/10.15388/NA.2019.5.6
5. J. R. Wang, A. G. Ibrahim, D. O'Regan, Hilfer type fractional differential switched inclusions with non-instantaneous impulsive and nonlocal conditions, *Nonlinear Anal. Model.*, **23** (2018), 921–941. https://doi.org/10.15388/NA.2018.6.7
6. J. R. Wang, A. G. Ibrahim, D. O'Regan, Y. Zhou, A general class of non-instantaneous impulsive semilinear differential inclusions in Banach spaces, *Adv. Differ. Equ.*, **2017** (2017), 287. https://doi.org/10.1186/s13662-017-1342-8
7. J. R. Wang, A. G. Ibrahim, D. O'Regan, Noeemptness and compactness of the solution set for fractional differential inclusions with non-instantaneous impulses, *Electron. J. Differ. Eq.*, **2019** (2019), 37.

8. M. S. Tavazoei, M. Haeri, A proof for non existence of periodic solutions in time invariant fractional order systems, *Automatica*, **45** (2009), 1886–1890. https://doi.org/10.1016/j.automatica.2009.04.001
9. I. Area, J. Losada, J. J. Nieto, On fractional derivatives and primitives of periodic of periodic functions, *Abstr. Appl. Anal.*, **2014** (2014), 392598. https://doi.org/10.1155/2014/392598
10. E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, *Nonlinear Anal. Real*, **13** (2012), 1489–1497. https://doi.org/10.1016/j.nonrwa.2011.11.013
11. M. D. Ortigueira, J. D. Machado, J. J. Trujillo, Fractional derivatives and periodic functions, *Int. J. Dynam. Control*, **5** (2017), 72–78. https://doi.org/10.1007/s40435-015-0215-9
12. L. Ren, J. Wang, M. Fečkan, Asymptotically periodic behavior solutions for Caputo type fractional evolution equations, *Fract. Calc. Appl. Anal.*, **21** (2019), 1294–1312. https://doi.org/10.1515/fca-2018-0068
13. S. Maghsoodi, A. Neamaty, Existence and uniqueness of asymptotically w-periodic solution for fractional semilinear problem, *J. Appl. Comput. Math.*, **8** (2019), 1–5.
14. L. Ren, J. R. Wang, D. O'Regan, Asymptotically periodic behavior of solutions of fractional evolution equations of order $1 < \alpha < 2$, *Math. Slovaca*, **69** (2019), 599–610. https://doi.org/10.1515/ms-2017-0250
15. J. Mu, Y. Zhou, L. Peng, Periodic solutions and S-asymptotically periodic solutions to fractional evolution equations, *Discrete Dyn. Nat. Soc.*, **2017** (2017), 1364532. https://doi.org/10.1155/2017/1364532
16. J. Q. Zhao, Y. K. Chang, G. M. N. Guérékata, Asymptotic behavior of mild solutions to semilinear fractional differential equations, *J. Optim. Theory Appl.*, **156** (2013), 106–114. https://doi.org/10.1007/s10957-012-0202-7
17. H. Wang, F. Li, S-asymptotically T-periodic solutions for delay fractional differential equations with almost sectorial operator, *Adv. Differ. Equ.*, **2016** (2016), 315. https://doi.org/10.1186/s13662-016-1043-8
18. M. Muslim, A. Kumar, M. Fečkan, Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses, *J. King Saud Uni. Sci.*, **30** (2018), 204–213. https://doi.org/10.1016/j.jksus.2016.11.005
19. Z. Alsheekhhussain, J. Wang, A. G. Ibrahim, Asymptotically periodic behavior of solutions to fractional non-instantaneous impulsive semilinear differential inclusions with sectorial operators, *Adv. Differ. Equ.*, **2021** (2021), 330. https://doi.org/10.1186/s13662-021-03475-w
20. F. Li, J. Liang, H. Wang, S-Asymptotically ω-periodic solution for fractional differential equations of order $q \in (0,1)$ with finite delay, *Adv. Differ. Equ.*, **2017** (2017), 83. https://doi.org/10.1186/s13662-017-1137-y
21. A. M. Abou-El-Elai, A. L. Sadek, A. M. Mahmoud, E. Farghalyi, Asymptotic stability of solutions for a certain non-autonomous second-order stochastic delay differential equation, *Turk. J. Math.*, **41** (2017), 576–584. https://doi.org/10.3906/mat-1508-62
22. T. Zhang, L. Xiong, Periodic motion for impulsive fractional functional equations with piecewise Caputo derivative, *Appl. Math. Lett.*, **101** (2020), 106072. https://doi.org/10.1016/j.aml.2019.106072

23. J. Andra, Coexistence of periodic solutions with various periods of impulsive differential equations and inclusions on tori via Poincare operators, *Topol. Appl.*, **255** (2019), 128–140. https://doi.org/10.1016/j.topol.2019.01.008
24. M. Fecčkan, R. J. Wang, Periodic impulsive fractional differential equations, *Adv. Nonlinear Anal.*, **8** (2019), 482–496. https://doi.org/10.1515/anona-2017-0015
25. H. R. Henrique, Periodic solutions of abstract neutral functional differential equations with infinite delay, *Acta Math. Hung.*, **121** (2008), 203–227. https://doi.org/10.1007/s10474-008-7009-x
26. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
27. T. Zhang, Y. Li, S-Asymptotically periodic fractional functional differential equations with off-diagonal matrix Mittag-Leffler functional kernels, *Math. Comput. Simul.*, **193** (2022), 313–347. https://doi.org/10.1016/j.matcom.2021.10.006
28. T. Zhang, Y. Li, Exponential Euler scheme of multi-delay Caput-Fabrizio fractional-order differential quations, *Appl. Math. Lett.*, **124** (2022), 107709. https://doi.org/10.1016/j.aml.2021.107709
29. T. Zhang, J. Zhou, Y. Liao, Exponentially stable periodic oscillation and Mittag-Leffler stabilization for fractional-order impulsive control neutral networks with piecewise Caputo derivatives, *IEEE T. Cybernetics*, **52** (2022), 9670–9683. https://doi.org/10.1109/TCYB.2021.3054946
30. C. C. Travis, G. F. Webb, Cosine families abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. H.*, **32** (1978), 75–96. https://doi.org/10.1007/BF01902205
31. J. W. He, Y. Liang, B. Ahmed, Y. Zhou, Nonlocal fractional evolution inclusions of order $\alpha \in (1,2)$, *Mathematics*, (2019) 2019, 7. https://doi.org/10.3390/math7020209
32. T. Ke, N. Lu, V. Obukhovskii, Decay solutions for a class of reactional differential varational inequalities, *Fract. Calc. Appl. Anal.*, **18** (2015), 531–553. https://doi.org/10.1515/fca-2015-0033
33. J. R. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. Real*, **12** (2011), 3642–3653. https://doi.org/10.1016/j.nonrwa.2011.06.021
34. H. Covitz, S. B. Nadler, Multivalued contraction mapping in generalized metric space, *Israel J. Math.*, **8** (1970), 5–11. https://doi.org/10.1007/BF02771543
35. C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Springer-Verlag, 1977.
36. F. Hiai, H. Umegaki, Integrals conditional expectation and martingales of multivalued functions, *J. Multivariate Anal.*, **7** (1977), 149–182. https://doi.org/10.1016/0047-259X(77)90037-9
37. M. Kamenskii, V. Obukhowskii, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, New York: Walter de Gruyter, 2001. https://doi.org/10.1515/9783110870893
38. G. Arthi, Ju H. Park, H. Y. Jung, Exponential stability for second-order neutral stochastic differential equations with impulses, *Int. J, Control*, **88** (2015), 1300–1309. https://doi.org/10.1080/00207179.2015.1006683

AIMS Press