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Research article

A new family of hybrid three-term conjugate gradient method for unconstrained optimization with application to image restoration and portfolio selection

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Abstract: The conjugate gradient (CG) method is an optimization method, which, in its application, has a fast convergence. Until now, many CG methods have been developed to improve computational performance and have been applied to real-world problems. In this paper, a new hybrid three-term CG method is proposed for solving unconstrained optimization problems. The search direction is a three-term hybrid form of the Hestenes-Stiefel (HS) and the Polak-Ribiére-Polyak (PRP) CG coefficients, and it satisfies the sufficient descent condition. In addition, the global convergence properties of the proposed method will also be proved under the weak Wolfe line search. By using several test functions, numerical results show that the proposed method is most efficient compared to some of the existing methods. In addition, the proposed method is used in practical application problems for image restoration and portfolio selection.

Keywords: conjugate gradient method; unconstrained optimization; sufficient descent condition; global convergence; image restoration; portfolio selection
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1. Introduction

We consider the unconstrained optimization problems with the form

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and bounded below. There are many methods for solving (1.1) such as the Newton methods, Quasi-Newton methods, Steepest Descent method, and Levenberg-Marquardt methods [36]. However, these methods are efficient to use for low-dimensional problems, while high-dimensional problems require many iterations and a long time. Therefore, to overcome the drawbacks of the previous methods, a nonlinear conjugate gradient (CG) method is proposed. Generally, the CG method is used to solve large-scale optimization problems because it has simple iterations, fast convergence properties, and low memory requirements [5, 22, 36].

The CG method was first introduced by Hestenes and Stiefel in 1952 and was used to solve a system of linear equations. Subsequently, in 1964, Fletcher and Reeves extended the form of the CG method to solve large-scale nonlinear systems of equations, and they used it to solve the general form of the optimization problem without constraints. The results of the expansion carried out by Fletcher and Reeves triggered many further studies [36]. For instance, in [24], Ibrahim et al. proposed a new hybrid method by combining the Liu-Storey [29] and Kamandi-Amini [28] conjugate gradient parameters. Likewise, Jian et al. [26] proposed a simple spectral CG method for solving large-scale unconstrained optimization problems. The method was based on the Fletcher-Reeves [17] and the Dai-Yuan methods [11]. Under the weak Wolfe line search structure, the convergence analysis was presented. In [44], Salleh et al. proposed a modified Liu and Storey [29] method by formulating the new parameter. Depending on the strong Wolfe line search, the search direction satisfies the descent condition and fulfills the convergence properties. Besides this, Zheng and Shi [53] proposed another formula for the CG parameter. The parameter is formulated by replacing the denominator of the PRP formula. The direction satisfies a trust region property and by using the Armijo line search, the global convergence properties were proved. Furthermore, motivated by the idea of Zhang et al. in [52], Tian et al. [49] proposed a new descent hybrid three-term CG algorithm. The new method satisfies the sufficient descent condition and is independent of the line search structure. For uniformly convex objective functions, global convergence is established under mild conditions. The modified secant condition is also used to establish global convergence for general functions, without the assumption of convexity. According to the numerical results, the proposed algorithm by Tian et al. [49] is effective and reliable.

The CG method is an iteration method where each step produces approximation points from the following formula:

$$x_{k+1} = x_k + \alpha_k d_k, \;\; k = 0, 1, 2, ...,$$

where d_k is the search direction, x_{k+1} and x_k are the current and previous points, respectively. The $\alpha_k > 0$ is the stepsize, obtained by exact or inexact line searches. However, considering the level of efficiency, inexact line search is more often used than exact line search. The two most frequently used inexact line searches are weak Wolfe and strong Wolfe line searches. The weak Wolfe line search is calculated such that α_k satisfy

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \tag{1.2}$$
$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k, \tag{1.3}$$

and the strong Wolfe line search is calculated such that α_k satisfy

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \quad |g(x_k + \alpha_k d_k)^T d_k| \leq -\sigma g_k^T d_k,$$

where $g_k = g(x_k) = \nabla f(x_k)$ is a gradient f at point x_k, and the parameters δ and σ are required to satisfy $0 < \delta < \sigma < 1$. One condition that must be met by the CG method is the descent condition. This condition guarantees that the approximation point leads to the minimum point, and this condition is defined as follows:

$$g_k^T d_k < 0.$$

Over time, an Omani scientist Al-Baali [4] proposed another version of the descent condition, which plays an important role in the convergence of CG methods called the sufficient descent condition. The definition of sufficient descent condition is given below.

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable function and the search direction d_k satisfies*

$$g_k^T d_k \leq -C\|g_k\|^2, \ \ \forall k, \tag{1.4}$$

where $C > 0$ is a constant, then d_k is said to fulfill the sufficient descent condition.

For standard CG method, the search direction d_k is defined by

$$d_k := \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k > 0, \end{cases}$$

where β_k is a parameter corresponding to a distinct CG method. Some of the well-known standard CG methods are the Hestenes-Stiefel (HS) method [19], the Fletcher-Reeves (FR) method [17], the Polak-Ribiére-Polyak (PRP) method [40, 41], the Conjugate-Descent (CD) method [16], the Dai-Yuan (DY) method [11], the Liu-Storey (LS) method [29], and the Rivaie-Mustafa-Ismail-Leong (RMIL) method [43] and their β_k parameters are

$$\beta_k^{HS} = \frac{g_k^T r_{k-1}}{d_{k-1}^T r_{k-1}}, \quad \beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_k^{PRP} = \frac{g_k^T r_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_k^{CD} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}},$$

$$\beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T r_{k-1}}, \quad \beta_k^{LS} = \frac{g_k^T r_{k-1}}{-g_{k-1}^T d_{k-1}}, \quad \beta_k^{RMIL} = \frac{g_k^T r_{k-1}}{\|d_{k-1}\|^2},$$

respectively, where $r_{k-1} := g_k - g_{k-1}$ and $\|.\|$ is a symbol for Euclidean norm on $\mathbb{R}^n$. We know that the HS, PRP, and LS methods are efficient but fail to meet the convergence property, even when using the exact line searches for non-convex functions. So, to improve the convergence of the PRP, HS, and LS methods, Powell [42] suggested modifying the parameter values to be non-negative. On other hand, while this makes the FR, PRP, DY, and RMIL methods robust and able to converge, the numerical performance remains not efficient. When compared to the standard CG method, the hybrid [30, 33] and three-term [8, 23, 25, 38, 51] CG methods always show good theoretical properties and numerical performance, such as the sufficient descent property regardless of the line search structure.

Recently, Abubakar et al. [1] proposed a hybrid three-term CG method in which the search direction is generated from the limited memory Broyden-Fletcher-Goldferb-Shanno (LBFGS) Quasi-Newton method. The method satisfies the sufficient descent condition and fulfills the trust region. Under a condition, the global convergence properties were established, and compared with some of the existing methods, the method is efficient. Likewise, Deepho et al. in [12] proposed a modification of the hybrid three-term CG method. The modification was done by combining the conjugate gradient parameters that already exist. Numerical experiments on several test problems for the method showed good results compared to other existing methods. In addition, the methods have also been applied to solve risk optimization problems in portfolio selection.

Motivated by the above contributions, in this paper, we propose a hybrid three-term CG method based on the structure of the LBFGS method of Nocedal [39] and Shanno [45], which can give a better numerical performance. The following are some of this paper's contributions:

(1) Based on the LBFGS method, a new hybrid three-term CG method for solving unconstrained optimization is proposed.

(2) The search direction of the proposed method satisfies the sufficient descent property without requiring any line search.

(3) The global convergence of the proposed method is demonstrated using the weak Wolfe line search.

(4) The computational performance of the new method is presented on several standard test problems.

(5) Finally, the proposed method is effectively applied to image restoration and minimizing risk in portfolio selection problems.

The paper is organized as follows. In Section 2, we present a modified hybrid three-term CG method and give the algorithm for our proposed method. In Section 3, we establish the sufficient descent condition and prove the global convergence property of our proposed method under a certain line search. Numerical experiments are outlined in Section 4 to see the computational performance by using several test functions and comparing them with other existing methods. In Section 5, we provide problem-solving to image restoration and portfolio selection problems by using our proposed method. Finally, the conclusions are presented in Section 6.

2. Formulation and algorithm

We start this section by describing our formulation and then present an algorithm of our proposed method.

In [1], Abubakar et al. proposed a hybrid three-term HTT CG method in which the search direction is formulated as follows:

$$d_0 := -g_0,\ d_k := -g_k + \beta_k^{HTT} d_{k-1} + \gamma_k g_k,\ k \geq 1,$$

where

$$\beta_k^{HTT} := \frac{\|g_k\|^2}{z_k} - \frac{\|g_k\|^2 g_k^T d_{k-1}}{z_k^2},\ \gamma_k := -v_k \frac{g_k^T d_{k-1}}{z_k},$$

and

$$z_k := \max\{\lambda\|d_{k-1}\|\|g_k\|, d_{k-1}^T r_{k-1}, \|g_{k-1}\|^2\},\ \lambda > 0,\ 0 \le v_k \le \bar{v} < 1.$$

Similarly, Deepho, et al. [12] proposed a hybrid three-term TTCDDY CG method in which the search direction owns the form

$$d_0 := -g_0,\ d_k := -g_k + \beta_k^{TTCDDY} d_{k-1} + \varrho_k g_k,\ k \ge 1,$$

where

$$\begin{aligned} \beta_k^{TTCDDY} &:= \frac{\|g_k\|^2}{h_k} - \frac{\|g_k\|^2 g_k^T d_{k-1}}{h_k^2}, \\ \varrho_k &:= -e_k \frac{g_k^T d_{k-1}}{h_k}, \end{aligned}$$

and

$$h_k := \max\{\varpi\|d_{k-1}\|\|g_k\|, -d_{k-1}^T g_{k-1}, d_{k-1}^T r_{k-1}\},\ \varpi > 0,\ 0 \le e_k \le \bar{e} < 1.$$

Under some assumptions, HTT and TTCDDY methods satisfy the sufficient descent condition, and the global convergence is proved. The numerical experiments showed that the HTT and TTCDDY methods perform better than the other existing methods.

Motivated by the HTT and TTCDDY methods, we propose a new hybrid three-term CG method based on the LBFGS Quasi-Newton method. In the same way, we first recall the search direction of the LBFGS method as

$$\begin{aligned} d_k^{LBFGS} &:= -M_k g_k, \\ M_k &= -\left(I - \frac{s_{k-1} r_{k-1}^T}{s_{k-1}^T r_{k-1}} - \frac{r_{k-1} s_{k-1}^T}{s_{k-1}^T r_{k-1}} + \frac{s_{k-1} r_{k-1}^T r_{k-1} s_{k-1}^T}{s_{k-1}^T r_{k-1}} + \frac{s_{k-1} s_{k-1}^T}{s_{k-1}^T r_{k-1}}\right), \end{aligned}$$

where $s_{k-1} = x_k - x_{k-1} = \alpha_{k-1} d_{k-1}$ and I is the identity matrix. By simplifying the form of d_k^{LBFGS}, we can define the search direction as follows:

$$d_k^{LBFGS} := -g_k + \left(\beta_k^{HS} - \frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{(d_{k-1}^T r_{k-1})^2}\right) d_{k-1} + \frac{g_k^T d_{k-1}}{d_{k-1}^T r_{k-1}} (r_{k-1} - s_{k-1}). \tag{2.1}$$

Next, replacing the term $(r_{k-1} - s_{k-1})$ in (2.1) with $c_k r_{k-1}$, where c_k is a parameter, we get the following three-term search direction:

$$d_k^{TTHS} := -g_k + \left(\beta_k^{HS} - \frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{(d_{k-1}^T r_{k-1})^2}\right) d_{k-1} + c_k \frac{g_k^T d_{k-1}}{d_{k-1}^T r_{k-1}} r_{k-1}. \tag{2.2}$$

Further, replacing β_k^{HS} with β_k^{PRP}, $\frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{(d_{k-1}^T r_{k-1})^2}$ with $\frac{\|g_k\|^2 g_k^T d_{k-1}}{\|g_{k-1}\|^4}$ and $(r_{k-1} - s_{k-1})$ with $c_k r_{k-1}$ in (2.1), we can write a three-term search direction as

$$d_k^{TTPRP} := -g_k + \left(\beta_k^{PRP} - \frac{\|g_k\|^2 g_k^T d_{k-1}}{\|g_{k-1}\|^4}\right) d_{k-1} + c_k \frac{g_k^T d_{k-1}}{\|g_{k-1}\|^2} r_{k-1}. \tag{2.3}$$

In the following, we will rewrite how to find the parameter c_k as in [1, 12]. The c_k parameter is obtained by solving the univariate problem as follows:

$$\min_{c\in\mathbb{R}} \|(r_{k-1} - s_{k-1}) - c g_k\|_F^2,$$

where $\|\cdot\|_F$ is the Frobenious norm.

Let $A_k = (r_{k-1} - s_{k-1}) - c g_k$, then

$$\begin{aligned} A_k A_k^T &= [(r_{k-1} - s_{k-1}) - c g_k][(r_{k-1} - s_{k-1}) - c g_k]^T \\ &= [(r_{k-1} - s_{k-1}) - c g_k][(r_{k-1} - s_{k-1})^T - c g_k^T] \\ &= c^2 g_k g_k^T - c[(r_{k-1} - s_{k-1}) g_k^T + g_k (r_{k-1} - s_{k-1})^T] + (r_{k-1} - s_{k-1})(r_{k-1} - s_{k-1})^T. \end{aligned}$$

Letting $B_k = r_{k-1} - s_{k-1}$, then

$$A_k A_k^T = c^2 g_k g_k^T - c[B_k g_k^T + g_k B_k^T] + B_k B_k^T,$$

and

$$\begin{aligned} tr(A_k A_k^T) &= c^2 \|g_k\|^2 - c[tr(B_k g_k^T) + tr(g_k B_k^T)] + \|B_k\|^2 \\ &= c^2 \|g_k\|^2 - 2c g_k^T B_k + \|B_k\|^2. \end{aligned}$$

Differentiating the above with respect to c and equating to zero, we have

$$2c\|g_k\|^2 - 2 g_k^T B_k = 0,$$

which implies

$$c = \frac{g_k^T (r_{k-1} - s_{k-1})}{\|g_k\|^2}.$$

Hence, we select c_k as

$$c_k := \min\{\bar{c}, \max\{0, c\}\}, \tag{2.4}$$

which implies $0 \le c_k \le \bar{c} < 1$.

Based on the search direction of the three-term CG defined in (2.2) and (2.3), we construct a new search direction of the hybrid three-term CG method as follows:

$$d_k := \begin{cases} -g_0, & k = 0, \\ -g_k + \beta_k^{HTHP} d_{k-1} + \kappa_k^{HTHP} r_{k-1}, & k \ge 1, \end{cases} \tag{2.5}$$

where

$$\beta_k^{HTHP} := \frac{g_k^T r_{k-1}}{n_k} - \frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{n_k^2}, \tag{2.6}$$

$$\kappa_k^{HTHP} := c_k \frac{g_k^T d_{k-1}}{n_k}, \tag{2.7}$$

and

$$n_k := \max\{\mu\|d_{k-1}\| \, \|r_{k-1}\|, d_{k-1}^T r_{k-1}, \|g_{k-1}\|^2\}, \ \mu > 0. \tag{2.8}$$

Now, we provide the flow framework of our algorithm as follows:

Algorithm 1: New hybrid three-term HS-PRP CG method (HTHP)

Input : Select the start point $x_0 \in \mathbb{R}^n$, the parameters $\epsilon > 0$, $0 < \delta < \sigma < 1$, $0 \leq \bar{c} < 1$, and $\mu > 0$.

Step 1 : Compute g_k, **if** $\|g_k\| \leq \epsilon$, **then**
  stop.
end
Step 2 : **if** $k = 0$, **then**
  set $d_k := -g_k$;
else
  Compute the search direction d_k using Eqs (2.4)–(2.8).
end
Step 3 : Compute the stepsize α_k using any line search strategy technique.
Step 4 : Compute the next iterate $x_{k+1} = x_k + \alpha_k d_k$.
Step 5 : Set $k := k + 1$ and go to Step 1.

3. Convergence result

In this section, we provide the global convergence result of the HTHP method under the following assumption.

Assumption 1. *The level set $\mathcal{B} := \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ is bounded, where x_0 is starting point.*

Assumption 2. *In some neighborhood $\mathcal{L}$ of $\mathcal{B}$, the gradient of the function f is Lipschitz continuous. That is, we can find $L > 0$, such that for all x*

$$\|g(x) - g(v)\| \leq L\|x - v\|, \ v \in \mathcal{L}.$$

In other words, Assumption 1 states that there exists a constant $T > 0$, such that:

$$\|x\| \leq T, \ \ \forall x \in \mathcal{B}.$$

Furthermore, observe that from Assumptions 1 and 2, we can obtain a positive constant F, such that:

$$\|g(x)\| \leq F, \ \ \forall x \in \mathcal{B}.$$

Next, we will present the sufficient descent condition for the HTHP method.

Lemma 1. *Let d_k be generated by* (2.5). *Then we obtain*

$$g_k^T d_k \leq -\left(1 - \frac{1}{4}(1+\bar{c})^2\right)\|g_k\|^2. \tag{3.1}$$

So, the search direction given by (2.5) *satisfies the sufficient descent condition* (1.4).

Proof. For $k = 0$, we have $g_0^T d_0 = -\|g_0\|^2$ and then the relation (3.1) is obvious since $0 \leq c_k \leq \bar{c} < 1$. Meanwhile, for $k \geq 1$, multiplying both sides (2.5) by g_k^T, we get

$$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + \frac{g_k^T r_{k-1}}{n_k} g_k^T d_{k-1} - \frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{n_k^2}(g_k^T d_{k-1}) + c_k \frac{g_k^T d_{k-1}}{n_k} g_k^T r_{k-1} \\ &= -\|g_k\|^2 + (1+c_k)\frac{g_k^T r_{k-1}}{n_k} g_k^T d_{k-1} - \frac{\|r_{k-1}\|^2}{n_k^2}(g_k^T d_{k-1})^2. \end{aligned} \tag{3.2}$$

Using the inequality $a_k^T b_k \leq \frac{1}{2}\left(\|a_k\|^2 + \|b_k\|^2\right)$ with

$$a_k = \frac{1}{\sqrt{2}}(1+c_k)g_k, \quad b_k = \frac{\sqrt{2}(g_k^T d_{k-1})r_{k-1}}{n_k},$$

we obtain

$$(1+c_k)\frac{g_k^T r_{k-1}}{n_k} g_k^T d_{k-1} \leq \frac{1}{4}(1+c_k)^2\|g_k\|^2 + \frac{(g_k^T d_{k-1})^2\|r_{k-1}\|^2}{n_k^2}. \tag{3.3}$$

Combining (3.2) with (3.3), we get

$$\begin{aligned} g_k^T d_k &\leq -\|g_k\|^2 + \frac{1}{4}(1+c_k)^2\|g_k\|^2 + \frac{(g_k^T d_{k-1})^2\|r_{k-1}\|^2}{n_k^2} - \frac{\|r_{k-1}\|^2}{n_k^2}(g_k^T d_{k-1})^2 \\ &= -\|g_k\|^2 + \frac{1}{4}(1+c_k)^2\|g_k\|^2 \\ &= -\left(1 - \frac{1}{4}(1+c_k)^2\right)\|g_k\|^2 \\ &\leq -\left(1 - \frac{1}{4}(1+\bar{c})^2\right)\|g_k\|^2. \end{aligned}$$

The proof is completed. □

Remark 1. *The lemma above indicates that the HTHP always satisfies the sufficient descent condition without depending on any line search.*

Next, we will establish the global convergence properties of the HTHP method.

Theorem 1. *Let Assumptions 1 and 2 hold, and assume conditions* (1.2) *and* (1.3) *are satisfied, then*

$$\liminf_{k\to\infty} \|g_k\| = 0. \tag{3.4}$$

Proof. We will prove the theorem by contradiction, that is, assume that (3.4) is not true. Then, there exists a constant ζ such that

$$||g_k|| \geq \zeta, \quad \text{for all } \; k \geq 0. \tag{3.5}$$

From (2.6), we have

$$\begin{aligned} \left|\beta_k^{HTHP}\right| &= \left|\frac{g_k^T r_{k-1}}{n_k} - \frac{||r_{k-1}||^2 g_k^T d_{k-1}}{n_k^2}\right| \\ &\leq \frac{||g_k||\, ||r_{k-1}||}{\mu||d_{k-1}||\, ||r_{k-1}||} + \frac{||r_{k-1}||^2\, ||g_k||\, ||d_{k-1}||}{(\mu||d_{k-1}||\, ||r_{k-1}||)^2} \\ &= \left(\frac{1}{\mu} + \frac{1}{\mu^2}\right)\frac{||g_k||}{||d_{k-1}||}. \end{aligned} \tag{3.6}$$

Next, from (2.7), we have

$$\begin{aligned} \left|\kappa_k^{HTHP}\right| &= \left|c_k \frac{g_k^T d_{k-1}}{n_k}\right| \\ &= c_k \left|\frac{g_k^T d_{k-1}}{n_k}\right| \\ &\leq \bar{c}\frac{||g_k||\, ||d_{k-1}||}{n_k} \\ &\leq \bar{c}\frac{||g_k||\, ||d_{k-1}||}{\mu||d_{k-1}||\, ||r_{k-1}||} \\ &= \bar{c}\frac{||g_k||}{\mu||r_{k-1}||}. \end{aligned} \tag{3.7}$$

Furthermore, from (2.5)–(2.8), (3.6) and (3.7), we have

$$\begin{aligned} ||d_k|| &= \left\|-g_k + \beta_k^{HTHP} d_{k-1} + \kappa_k^{HTHP} r_{k-1}\right\| \\ &\leq ||g_k|| + \left|\beta_k^{HTHP}\right| ||d_{k-1}|| + \left|\kappa_k^{HTHP}\right| ||r_{k-1}|| \\ &\leq ||g_k|| + \left(\frac{1}{\mu} + \frac{1}{\mu^2}\right)\frac{||g_k||}{||d_{k-1}||}||d_{k-1}|| + \bar{c}\frac{||g_k||}{\mu||r_{k-1}||}||r_{k-1}|| \\ &= \left(1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{\bar{c}}{\mu}\right)||g_k|| \\ &\leq \left(1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{\bar{c}}{\mu}\right)F. \end{aligned}$$

Hence, the sequence $\{||d_k||\}$ generated by the HTHP method has an upper bound, i.e.

$$||d_k|| \leq Y, \; \forall k \geq 0, \tag{3.8}$$

where $Y = \left(1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \frac{\bar{c}}{\mu}\right)F$.

Now, from (1.2) and using Lemma 1, $0 \leq \bar{c} < 1$, $\delta > 0$, $\alpha_k > 0$, we have

$$\begin{aligned} f(x_k+\alpha_k d_k) &\le f(x_k)+\delta\alpha_k g_k^T d_k \\ &\le f(x_k)-\delta\alpha_k\left(1-\frac{1}{4}(1+\bar{c})^2\right)\|g_k\|^2 \\ &\le f(x_k). \end{aligned}$$

If we expand the above result and together with Assumption 1, we obtain

$$f(x_k+\alpha_k d_k)=f(x_{k+1})\le f(x_k)\le f(x_{k-1})\le ...\le f(x_0)<+\infty. \tag{3.9}$$

Also, adding condition (1.3) by $-g_k^T d_k$ yields

$$g(x_k+\alpha_k d_k)^T d_k-g_k^T d_k\ge \sigma g_k^T d_k-g_k^T d_k=-(1-\sigma)g_k^T d_k.$$

Applying Lemma 1 and Assumption 2 to relation above, we now have

$$-(1-\sigma)g_k^T d_k\le (g_{k+1}-g_k)^T d_k\le \|g_{k+1}-g_k\|\|d_k\|\le L\|x_{k+1}-x_k\|\|d_k\|.$$

Using the equation $\|x_{k+1}-x_k\|=\|\alpha_k d_k\|=\alpha_k\|d_k\|$, then the above relation will be

$$\frac{-(1-\sigma)g_k^T d_k}{L\|d_k\|^2}\le \alpha_k. \tag{3.10}$$

Multiplying (3.10) by $-\delta g_k^T d_k\ge 0$ and combining with (1.2), we get

$$\frac{\delta(1-\sigma)(g_k^T d_k)^2}{L\|d_k\|^2}\le -\delta\alpha_k g_k^T d_k\le f(x_k)-f(x_{k+1})$$

or

$$\frac{\delta(1-\sigma)(g_k^T d_k)^2}{L\|d_k\|^2}\le f(x_k)-f(x_{k+1}). \tag{3.11}$$

Summing (3.11), and applying (3.9), we have

$$\frac{\delta(1-\sigma)}{L}\sum_{k=0}^{\infty}\frac{(g_k^T d_k)^2}{\|d_k\|^2}\le (f(x_0)-f(x_1))+(f(x_1)-f(x_2))+...\le f(x_0)<+\infty.$$

That implies,

$$\sum_{k=0}^{+\infty}\frac{(g_k^T d_k)^2}{\|d_k\|^2}<+\infty. \tag{3.12}$$

Now, from inequality (3.5) and (3.1) we get that

$$\begin{aligned} g_k^T d_k &\le -\left(1-\frac{1}{4}(1+\bar{c})^2\right)\|g_k\|^2 \\ &\le -\left(1-\frac{1}{4}(1+\bar{c})^2\right)\zeta^2. \end{aligned} \tag{3.13}$$

Upon squaring both sides of (3.13), then dividing by $\|d_k\|^2$ and also using (3.8), we obtain

$$\sum_{k=0}^{+\infty}\frac{(g_k^T d_k)^2}{\|d_k\|^2}\ge \left(1-\frac{1}{4}(1+\bar{c})^2\right)\zeta^2\sum_{k=0}^{+\infty}\frac{1}{\|d_k\|^2}=+\infty.$$

This result contradicts (3.12). Therefore, the condition (3.4) holds. □

4. Numerical experiments

This section analyzes the performance of the new HTHP CG algorithm on several benchmark test functions considered from Andrei [5] and Moré et al. [37], with dimensions ranging from 2 to 1,000,000 (see Table 1). To illustrate the efficiency, the proposed method was compared with other existing methods such as TTCDDY [12], HTT [1], and MPRP [52], based on the following metrics:

- Number of iterations denoted as NOI.
- Number of function evaluations presented as NOF.
- Central processing unit time denoted as CPU time.

Table 1. The problems and their dimensions.

No	Problem/Dimension	No	Problem/Dimension
1	COSINE 6000	67	Extended DENSCHNB 300,000
2	COSINE 100,000	68	Generalized Quartic 9000
3	COSINE 800,000	69	Generalized Quartic 90,000
4	DIXMAANA 2000	70	Generalized Quartic 500,000
5	DIXMAANA 30,000	71	BIGGSB1 110
6	DIXMAANB 8000	72	BIGGSB1 200
7	DIXMAANB 16,000	73	SINE 100,000
8	DIXMAANC 900	74	SINE 50,000
9	DIXMAANC 9000	75	FLETCBV 15
10	DIXMAAND 4000	76	FLETCBV 55
11	DIXMAAND 30,000	77	NONSCOMP 5000
12	DIXMAANE 800	78	NONSCOMP 80,000
13	DIXMAANE 16,000	79	POWER 150
14	DIXMAANF 5000	80	POWER 90
15	DIXMAANF 20,000	81	RAYDAN1 500
16	DIXMAANG 4000	82	RAYDAN1 5000
17	DIXMAANG 30,000	83	RAYDAN2 2000
18	DIXMAANH 2000	84	RAYDAN2 20,000
19	DIXMAANH 50,000	85	RAYDAN2 500,000
20	DIXMAANI 120	86	DIAGONAL1 800
21	DIXMAANI 12	87	DIAGONAL1 2000
22	DIXMAANJ 1000	88	DIAGONAL2 100
23	DIXMAANJ 5000	89	DIAGONAL2 1000
24	DIXMAANK 4000	90	DIAGONAL3 500
25	DIXMAANK 40	91	DIAGONAL3 2000
26	DIXMAANL 800	92	Discrete Boundary Value 2000
27	DIXMAANL 8000	93	Discrete Boundary Value 20,000
28	DIXON3DQ 150	94	Discrete Integral Equation 500
29	DIXON3DQ 15	95	Discrete Integral Equation 1500

Continued on next page

N0	Problem/Dimension	NO	Problem/Dimension
30	DQDRTIC 9000	96	Extended Powell Singular 1000
31	DQDRTIC 90,000	97	Extended Powell Singular 2000
32	QUARTICM 5000	98	Linear Full Rank 100
33	QUARTICM 150,000	99	Linear Full Rank 500
34	EDENSCH 7000	100	Osborne 2 11
35	EDENSCH 40,000	101	Penalty1 200
36	EDENSCH 500,000	102	Penalty1 1000
37	EG2 100	103	Penalty2 100
38	EG2 35	104	Penalty2 110
39	FLETCHCR 1000	105	Extended Rosenbrock 500
40	FLETCHCR 50,000	106	Extended Rosenbrock 1000
41	FLETCHCR 200,000	107	Broyden Tridiagonal 500
42	Freudenstein & Roth 460	108	Broyden Tridiagonal 50
43	Freudenstein & Roth 10	109	HIMMELH 70,000
44	Generalized Rosenbrock 10,000	110	HIMMELH 240,000
45	Generalized Rosenbrock 100	111	Brown Badly Scaled 2
46	HIMMELBG 70,000	112	Brown and Dennis 4
47	HIMMELBG 240,000	113	Biggs EXP6 6
48	LIARWHD 15	114	Osborne1 5
49	LIARWHD 1000	115	Extended Beale 5000
50	Extended Penalty 1000	116	Extended Beale 10,000
51	Extended Penalty 8000	117	HIMMELBC 500,000
52	QUARTC 4000	118	HIMMELBC 1,000,000
53	QUARTC 80,000	119	ARWHEAD 100
54	QUARTC 500,000	120	ARWHEAD 1000
55	TRIDIA 300	121	ENGVAL1 500,000
56	TRIDIA 50	122	ENGVAL1 1,000,000
57	Extended Woods 150,000	123	DENSCHNA 500,000
58	Extended Woods 200,000	124	DENSCHNA 1,000,000
59	BDEXP 5000	125	DENSCHNB 500,000
60	BDEXP 50,000	126	DENSCHNB 1,000,000
61	BDEXP 500,000	127	DENSCHNC 10
62	DENSCHNF 90,000	128	DENSCHNC 500
63	DENSCHNF 280,000	129	DENSCHNF 500,000
64	DENSCHNF 600,000	130	DENSCHNF 1,000,000
65	DENSCHNB 6000	131	ENGVAL8 500,000
66	DENSCHNB 24,000	132	ENGVAL8 1,000,000

The executions were carried out under the weak Wolfe line search. For the proposed method, the parameters' values $\sigma = 0.009,\ \delta = 0.0001,\ \mu = 0.02,\ \bar{c} = 0.105$ were considered for the numerical experimentation. While, for the TTCDDY, HTT, and MPRP methods, the parameter values defined in the study were maintained. The termination criteria for all algorithms were set as $\|g_k\| \leq 10^{-6}$, and an algorithm is said to fail (failure point denoted as "NaN") if any of the following conditions hold:

- The stopping condition $||g_k|| \leq 10^{-6}$ is not satisfied.
- The NOI exceeds 2000.

All algorithms are coded on MATLAB as in [27], and the host computer is an Intel Core i7 processor with the following specifications: 16 GB RAM, 64-bit Windows 10 Pro operating system. The complete experimental results for the TTCDDY, HTT, MPRP, and HTHP methods are provided in https://github.com/malik1106/HTHP.git, and the graphical representation of the results are further evaluated using the performance profile tool introduced by Dolan and Moré [14], as shown in Figures 1 (for NOI), 2 (for NOF), and 3 (for CPU time) respectively. Based on the performance profile rule, the algorithm with the highest curve illustrates the efficiency of that algorithm over the others considered for comparison.

Referring to our plots, it is clear from Figures 1–3 that the curve of the proposed HTHP method lies above that of TTCDDY, HTT, and MPRP for all the three performance metrics, which include NOI, NOF, and well as the CPU time. This implies that it is the most efficient algorithm among the related TTCDDY, HTT, and MPRP methods.

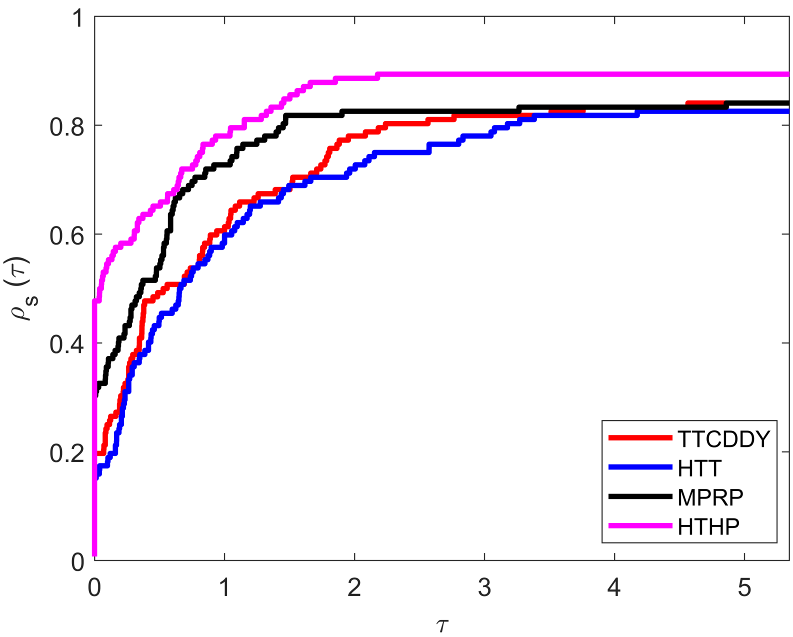

Figure 1. Performance profiles on NOI.

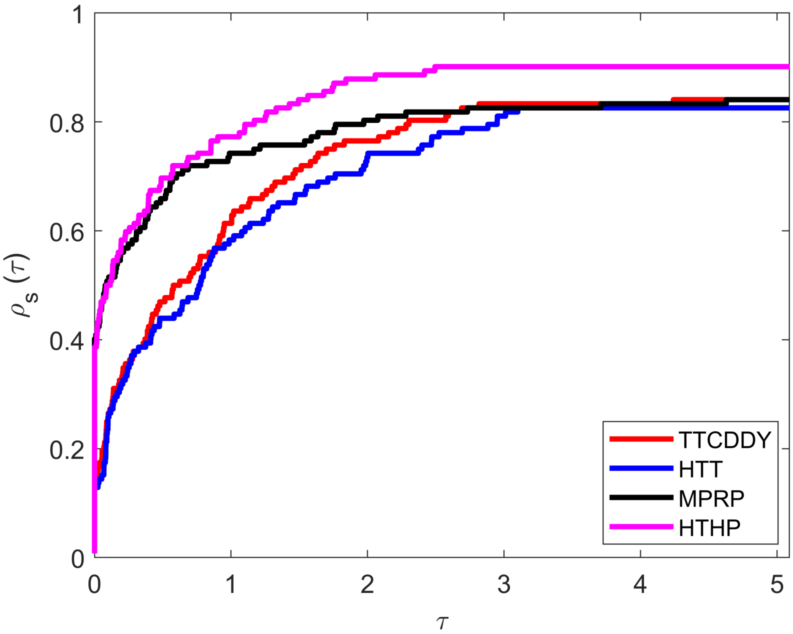

Figure 2. Performance profiles on NOF.

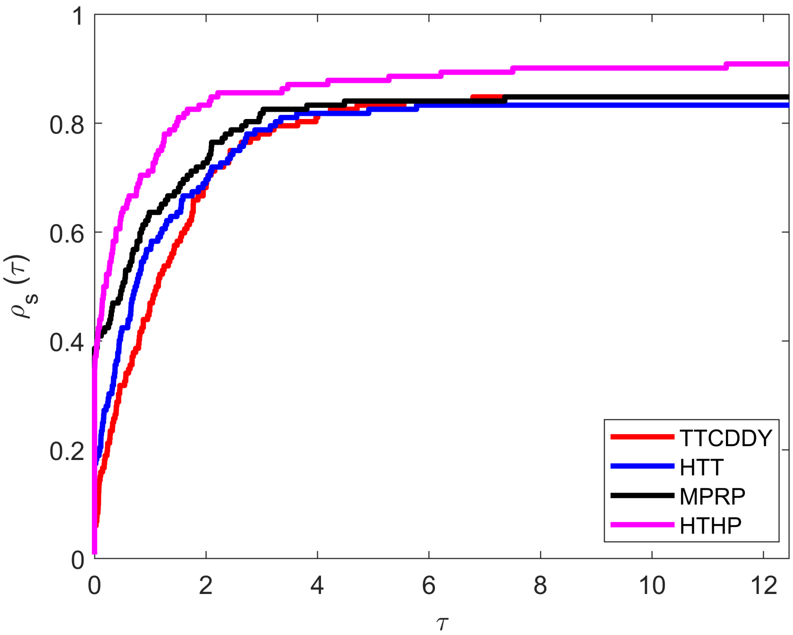

Figure 3. Performance profiles on CPU time.

5. Application to image restoration and portfolio selection

Until now, the CG method have been widely used in solving some problems, such as the regression analysis problems [47, 48], image restoration problems [10, 27], motion control [2, 3], and portfolio selection problems [1, 6, 12, 13, 31, 32]. In this section, we apply the proposed method for solving image restoration and portfolio selection problems.

5.1. Image restoration

The problem of restoring images that have been corrupted by noise in the transmission or acquisition process are among the difficult optimization problems due to their nonsmooth properties. Most of the available gradient-based algorithms are unable to solve these problems directly due to the nature of the problems. With recent advances in gradient-based methods, more efficient and reliable noise suppression process capable of producing better and more accurate results can be achieved. One of the classical noise models considered by several researchers is impulse noise. Lately, researchers have investigated the performance of some gradient-based methods on image restoration problems (see [7, 50]).

In this section, we demonstrate the performance of the proposed HTHP CG method in recovering the original Camera, Lena, and Goldhill 256 × 256 grey level images (x) that have been corrupted by salt-and-pepper impulse noise. For this purpose, we first consider the index set of the noise candidate as:

$$K = \{(i, j) \in W | \bar{\xi}_{ij} \neq \xi_{ij},\ \xi_{ij} = s_{\min} \text{ or } s_{\max}\},$$

where $x_{i,j}$ represent the grey level of the true image x at the pixel location (i, j), $W = \{1, 2, \cdot, M\} \times \{1, 2, \cdot, N\}$ and $\bar{\xi}$ is an adaptive median filter of the observed noisy image ξ of x corrupted by salt-and-pepper impulse noise. Also, $s_{\min}$ and $s_{\max}$ denotes the minimum and maximum of a noisy pixel respectively. Based on the above, we defined the image restoration problem as follows:

$$\min \mathcal{G}(u),$$

where

$$\mathcal{G}(u) = \sum_{(i,j)\in K} \left\{ \sum_{(m,n)\in V_{i,j}/K} \phi_\alpha(u_{i,j} - \xi_{m,n}) + \frac{1}{2} \sum_{(m,n)\in V_{i,j}\cap K} \phi_\alpha(u_{i,j} - u_{m,n}) \right\},$$

where $V_{i\ j} = \{(i,\ j-1), (i, j+1), (i-1, j), (i+1, j)\}$ is the neighborhood of (i, j). From the above equation, it is obvious that the regularity of $\mathcal{G}$ relies on the Huber function ϕ which is chosen as the edge-preserving potential function with $\phi_\alpha(t) = \sqrt{t^2 + \alpha}$ with $\alpha = 1$.

To demonstrate the suitability of the proposed HTHP method, we compare the performance result with that of TTCDDY, HTT, and MPRP methods based on three metrics which include CPU time (CPUT), relative error (RelErr), and peak signal-to-noise ratio (PSNR). All the methods were implemented on MATLAB software installed on an Intel Core i7 computer with 16 GB RAM. The quality of the images restored is based on 30, 50, and 80 percent noise degrees respectively.

From results presented in Tables 2–7, we can see that the proposed method outperformed the other methods considered in the study based on all the three metrics employed which includes CPUT, RelErr, as well as PSNR. In addition, Figures 4–6 show that the proposed method was able to remove noise from the corrupted Camera, Lena, and Goldhill images with a better accuracy compare to the other methods. Based on these results, we can conclude that the proposed HTHP CG method is suitable and effective.

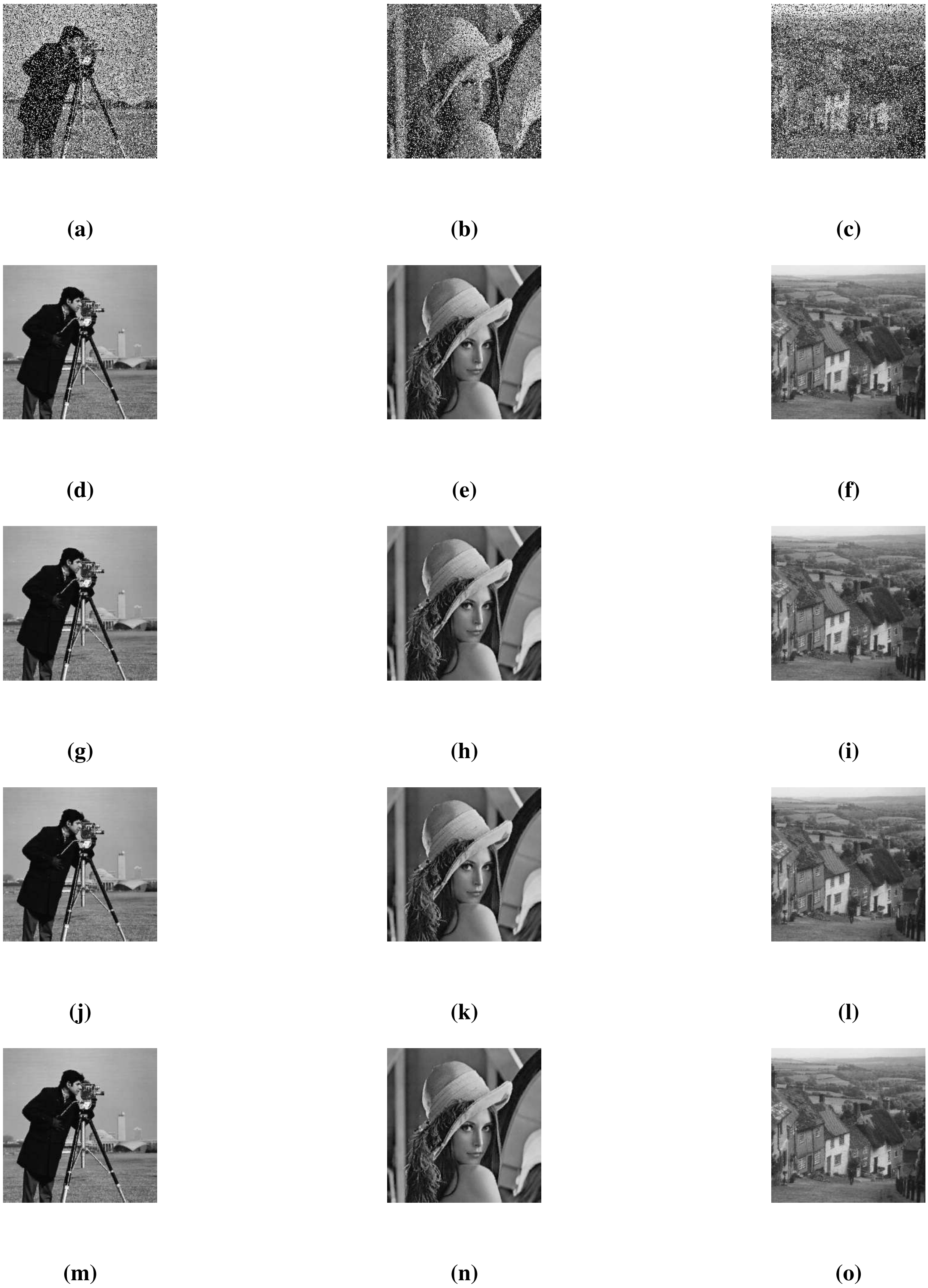

Figure 4. Images corrupted by 30% salt-and-pepper noise: (a–c), the restored images via TTCDDY: (d–f), the restored images via HTT: (g–i), the restored images via HTHP: (j–l), and the restored images via MPRP: (m–o).

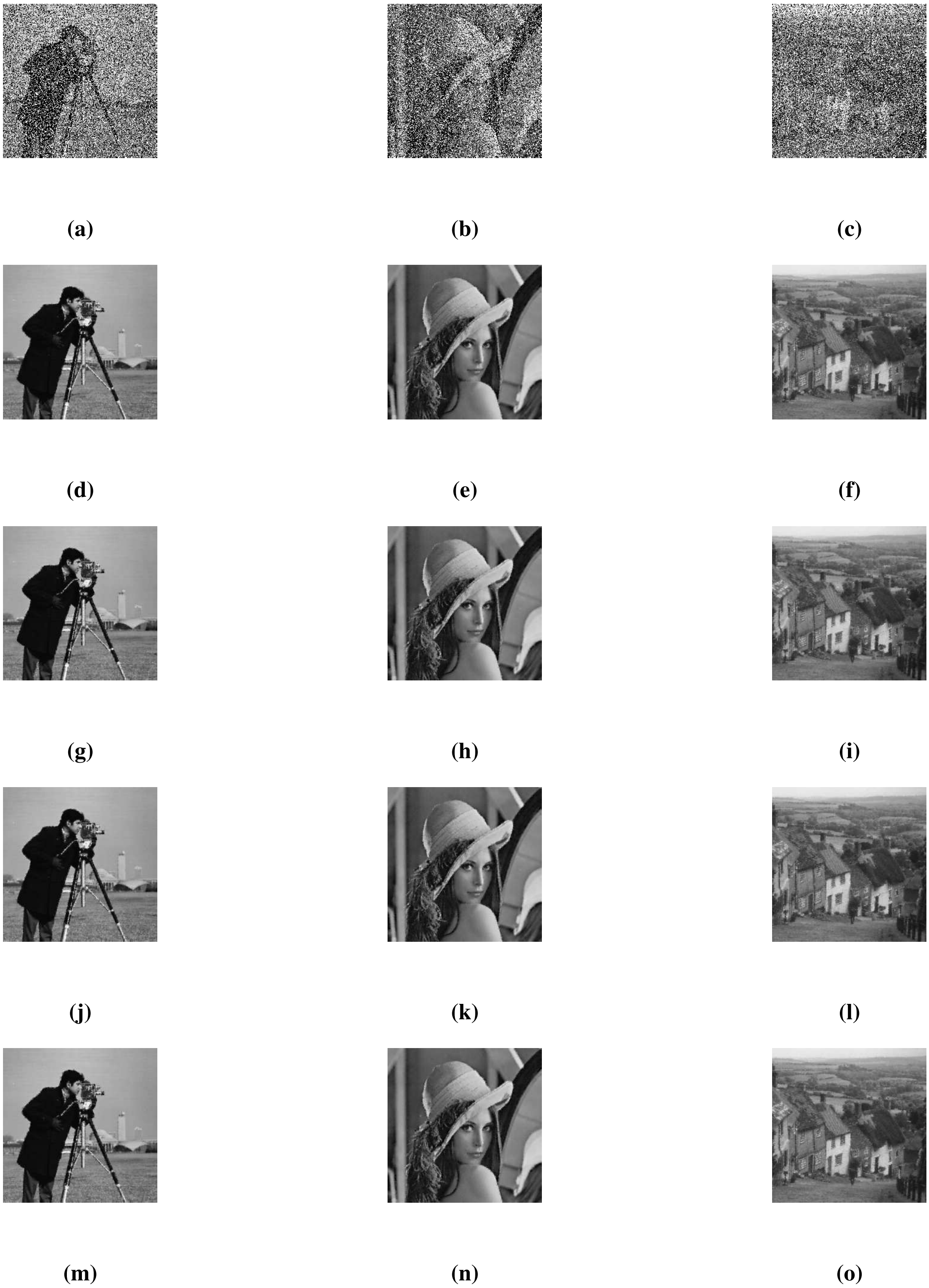

Figure 5. Images corrupted by 50% salt-and-pepper noise: (a–c), the restored images via TTCDDY: (d–f), the restored images via HTT: (g–i), the restored images via HTHP: (j–l), and the restored images via MPRP: (m–o).

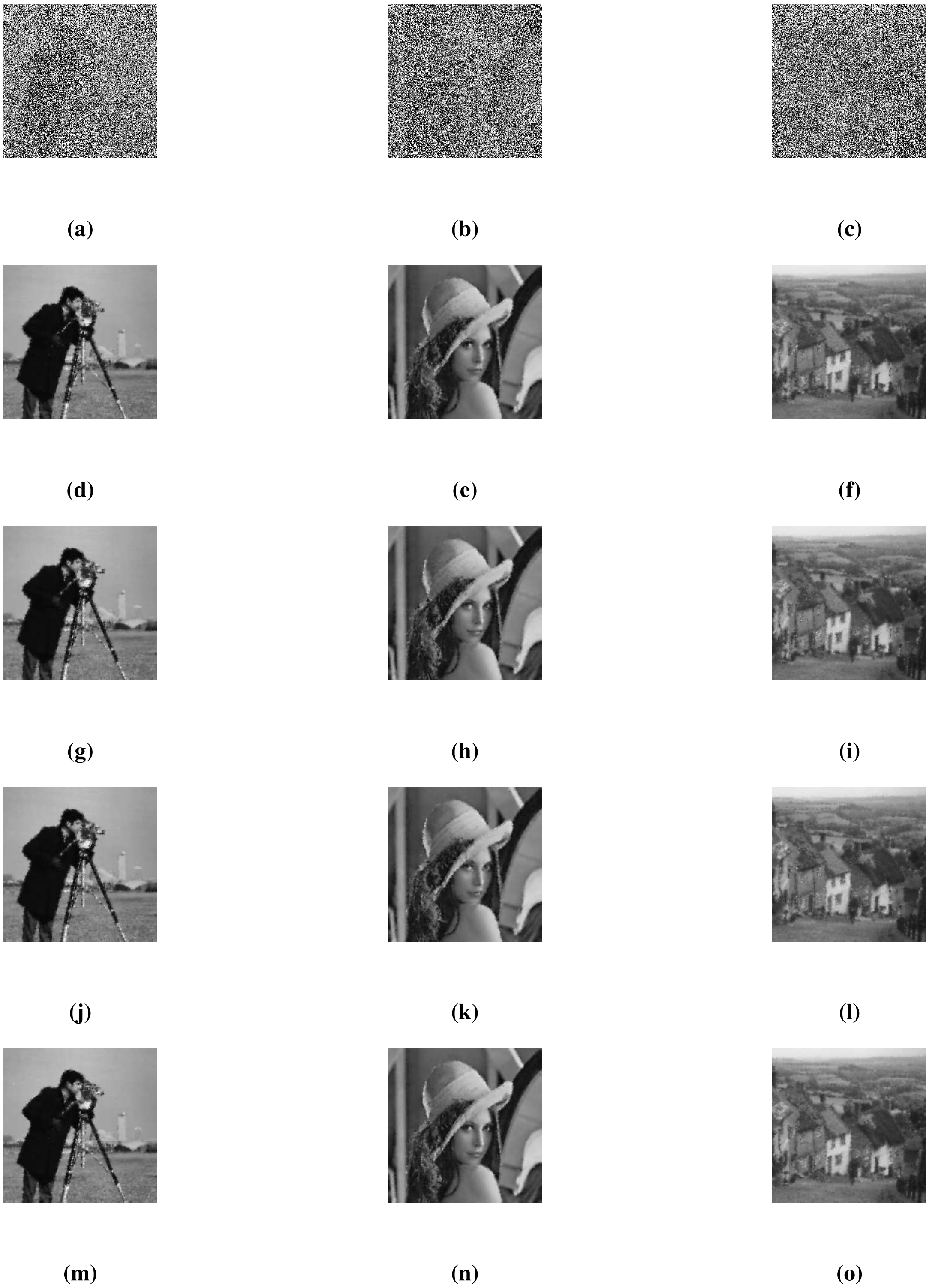

Figure 6. Images corrupted by 70% salt-and-pepper noise: (a–c), the restored images via TTCDDY: (d–f), the restored images via HTT: (g–i), the restored images via HTHP: (j–l), and the restored images via MPRP: (m–o).

Table 2. Image restoration outputs for Camera based on CPUT, RelErr, and PSNR.

Noise	TTCDDY			HTT		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	48.7915	1.05125	30.7558	48.5623	1.05126	30.7558
50%	116.132	1.71497	27.4747	117.190	1.71497	27.4747
80%	190.563	3.15721	23.5289	186.039	2.99665	23.6932

Table 3. Image restoration outputs for Camera based on CPUT, RelErr, and PSNR.

Noise	HTHP			MPRP		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	48.5041	1.05112	30.7567	59.1099	1.2126	30.7433
50%	117.053	1.71051	27.3803	114.1785	1.8109	27.2048
80%	189.843	2.93330	23.8340	182.7823	3.2413	23.7697

Table 4. Image restoration outputs for Lena based on CPUT, RelErr, and PSNR.

Noise	TTCDDY			HTT		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	37.1322	0.89482	33.8083	37.2949	0.89494	33.8069
50%	86.0965	1.34941	30.2385	81.7588	1.34964	30.2370
80%	185.258	25.3976	26.0779	149.421	2.48394	26.3320

Table 5. Image restoration outputs for Lena based on CPUT, RelErr, and PSNR.

Noise	HTHP			MPRP		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	37.4014	0.89483	33.8086	46.7401	0.9161	33.6496
50%	80.0735	1.34933	30.2309	110.7538	1.4301	30.2379
80%	150.912	2.62686	25.9123	145.4011	2.4176	26.4453

Table 6. Image restoration outputs for Goldhill based on CPUT, RelErr, and PSNR.

Noise	TTCDDY			HTT		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	47.9128	0.98980	32.0271	48.1301	0.88681	32.2260
50%	80.1133	1.49985	29.4045	81.8965	1.44497	29.2995
80%	125.720	2.63321	25.9414	123.6915	2.60388	25.8423

Table 7. Image restoration outputs for Goldhill based on CPUT, RelErr, and PSNR.

Noise	HTHP			MPRP		
	CPUT	RelErr	PSNR	CPUT	RelErr	PSNR
30%	48.0866	0.85528	32.1903	46.2320	0.8965	32.0978
50%	81.8959	1.52717	29.2131	78.0435	1.4165	29.4420
80%	124.789	2.59839	25.9093	142.8682	2.6428	25.8946

5.2. Portfolio selection

When investing in bonds or stocks, investors must pay attention to two basic components related to financial instruments, namely risk and return. Risk is the possibility of loss that occurs when an investment is made, while the return is the possible profit that can be obtained when investing. In practice, there is a positive correlation between the expected return and the risk that must be borne; hence, the greater the expected return, the greater the risk obtained, and vice versa [15]. One way to make the right investment decisions is to build a portfolio. An investment portfolio is a collection of investment instruments in several financial securities, which may or may not be the same to minimize risk and/or maximize returns. By creating a portfolio, investors can identify which securities to choose and how much capital to invest in the selected securities. Surely, investors will choose an efficient portfolio to invest [34].

An efficient portfolio aims to minimize risk or maximize return. This study focuses on selecting an efficient portfolio by minimizing risk. Here, portfolio risk is measured using a risk measuring instrument. Several risk measurement tools can be used for this purpose, of which a variance risk measurement tool has been used here [9]. To minimize portfolio variance, several ways in the optimization mathematical theory can be used, one of which is the CG method. This method is an optimization method that is still widely used and being developed by researchers. Among the advantages of using the CG method are low memory usage and high convergence speed [5].

The development of technology and civilization has resulted in a better understanding of the benefits of short-term and long-term investments. The main purpose of investment is to obtain larger funds in the future. The capital market in every country is one of the places where investment activities are carried out by investors. One of the investment products that can be found in the capital market is stocks. Simply, stocks are proof of ownership of a small part of a company. This evidence entitles the shareholder to a stock of the company's assets and profits by the number of stocks owned [35]. There are at least two things that need to be considered when investing in stocks, namely risk and rate of return.

Returns on investment is a financial measure that is widely used to measure the probability of obtaining a return on investment. Returns can be in the form of returns that have occurred (realized returns) or expected returns, which have not yet occurred but are expected to occur in the future. Realized returns can be calculated using historical data, which is quite important because it is used as a measure of the performance of an asset. In addition, realized returns are also the basis for determining expected returns in the future [18].

For stock A, the realized returns can be defined as

$$r_{i,A} = \frac{p_i - p_{i-1}}{p_{i-1}}, \tag{5.1}$$

where, $r_{i,A}$ is the value of return of stock A at time i, p_i is stock price at time i, and p_{i-1} is stock price at time $i-1$ [46]. While the expected return can be formulated as follows:

$$E(R_{i,A}) = \mu_A = \sum_{i=1}^{N} r_{i,A} f(r_{i,A}), \tag{5.2}$$

where, $R_{i,A}$ is a random variable of the return of stock A at time i, $E(R_{i,A}) = \mu_A$ is the expected return of stock A, $R_{i,A}$ follows a certain distribution with the density function expressed as $f(r_{i,A})$, and $i = 1, 2, ..., N$ [20, 21].

In addition to returns, the risk also needs to be considered when investing. Risk is often associated with deviations from the expected outcomes. One way to calculate risk is using the standard deviation method, where the method is used to measure the deviation of values that have occurred with the expected value. The variance of stock A can be expressed as follows [20, 21]:

$$\sigma_A^2 = E\left[(R_{i,A} - \mu_A)^2\right] = \sum_{i=1}^{N} (r_{i,A} - \mu_A)^2 f(r_{i,A}). \tag{5.3}$$

In investing, Harry Markowitz advises to not put all the capital you have in just one asset because if that asset fails, all the capital invested in that asset will disappear. Thus, one way that investors can minimize risk is to diversify investments in the form of a portfolio. The formation of a portfolio is one way an investor can employ to maximize the expected return or/and minimize the level of risk that will be faced. Of course, the portfolio formed must be optimal. To form an optimal portfolio, Harry Markowitz proposed a method known as the mean-variance method, which uses the average and variance of historical stock price data [34]. The main result of this method is that the proportion of each stock is obtained so that an optimal portfolio can be formed.

Markowitz's portfolio theory works on how to diversify a stock portfolio to minimize risk. Portfolio risk is not just the weighted average in the portfolio but must also consider the relationship between the stocks. This relationship is known as covariance. Covariance is a measurement that expresses the joint variance of two random variables, defined as follows [20, 21]:

$$\sigma_{AB} = E\left[(R_{i,A} - \mu_A)(R_{i,B} - \mu_B)\right] = \sum_{i=1}^{N} \left[(r_{i,A} - \mu_A)(r_{i,B} - \mu_B)\right] f(r_{i,A}, r_{i,B}), \tag{5.4}$$

where $\sigma_{A,B}$ is the covariance of return between stocks A and B. Next, we will explore returns, expected returns, and variance and covariance of return of the portfolio. Suppose an investment portfolio consists K stocks, the return of each stock is $r_{i,1}, r_{i,2}, ..., r_{i,K}$. If r^T is the vector of return stocks in the investment portfolio, then we can express it as follows: $r^T = (r_{i,1}, r_{i,2}, ..., r_{i,K})$. It is assumed that the first and second moments of return on these assets exist. Let μ^T and w^T represent the transposing of the mean vector and weight vector, respectively, which can be expressed as follows: $\mu^T = (\mu_1, \mu_2, ..., \mu_K)$, $w^T = (w_1, w_2, ..., g_k)$ and $\mu_j = E[r_{i,j}]$, w_j is weight/proportion of funds allocated to the stocks j, $j = 1, 2, ..., K$. Based on these notations, the return of the portfolio can be formulated as follows [46]:

$$r_p = \sum_{j=1}^{K} w_j r_{i,j} = w^T r. \tag{5.5}$$

According to (5.5), the expected return of the portfolio can be expressed as the following equation:

$$\mu_p = E[r_p] = E[w^T r] = w^T E[r] = w^T \mu. \tag{5.6}$$

By using (5.5), the variance of portfolio return can be expressed as follows:

$$\sigma_p^2 = Var(r_p) = \sum_j^K \sum_l^K w_j w_l \sigma_{jl} = w^T \Sigma w, \tag{5.7}$$

where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1K} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{K1} & \sigma_{K2} & \dots & \sigma_{KK} \end{pmatrix},$$

and $\sigma_{11}, \sigma_{12}, ..., \sigma_{KK}$ can be determined by (5.4).

After knowing the expected return and variance of the portfolio return, the next problem becomes how to choose an efficient portfolio, namely a portfolio that has high return expectations with low risk as measured by variance. We know that Markowitz [34] popularized the method of selecting an efficient portfolio by minimizing portfolio risk, measured by variance. Therefore, the optimization problem of portfolio selection by minimizing risk to be solved is as follows:

$$\begin{cases} \text{minimize} : \sigma_p^2 = \sum_j^K \sum_l^K w_j w_l \sigma_{jl} = w^T \Sigma w. \\ \text{subject to} : \sum_{j=1}^K w_j = 1. \end{cases} \tag{5.8}$$

Now, we will consider the problem of determining the proportion of stock in a portfolio by applying the proposed CG method, such that it produces an optimal portfolio. The stock data analyzed in this portfolio problem is stock data traded on the capital market in Indonesia through the Indonesian Stock Exchange (IDX). The historical stock data used is the daily closing price of the stocks included in the IDX30 stock list and accessed through the website http://finance.yahoo.com. The names of the top 20 stocks are listed in Table 8.

Furthermore, from the 20 stocks that have been selected in the formation of the portfolio, daily historical data will is sought for the period from June 1, 2020, to May 31, 2022. The historical data for these stocks contain the opening price, highest price, lowest price and closing price, respectively. For analysis purposes, in this research, we only need the daily closing price of the stock. After selecting several stocks to be included in the portfolio formation, they were estimated for their distribution model, expectations, and return variance. Identification of the distribution model for the return of each stock was done by finding the return using the formula (5.1), and then fitting the distribution using EasyFit software. After obtaining the estimated distribution function, the next step is calculating the expected return and variance by using (5.2) and (5.3), respectively. In addition, we also calculate ratio between expected return and variance. Thus, we get the estimated distribution, expected return (μ), variance (σ^2), and ratio (μ/σ^2) of each return stock in Table 9.

Table 8. List of stocks included in the IDX30 index for the period February 2022–July 2022.

No	Code	Name
1	ADRO	Adaro Energy Tbk.
2	ANTM	Aneka Tambang (Persero) Tbk
3	ASII	Astra International Tbk.
4	BBCA	Bank Central Asia Tbk.
5	BBNI	Bank Negara Indonesia (Persero) Tbk.
6	BBRI	Bank Rakyat Indonesia (Persero) Tbk.
7	BBTN	Bank Tabungan Negara (Persero) Tbk.
8	BMRI	Bank Mandiri (Persero) Tbk.
9	BRPT	Barito Pacific Tbk.
10	CPIN	Charoen Pokphand Indonesia Tbk.
11	ICBP	Indofood CBP Sukses Makmur Tbk.
12	INCO	Vale Indonesia Tbk.
13	INDF	Indofood Sukses Makmur Tbk.
14	KLBF	Kalbe Farma Tbk.
15	PGAS	Perusahaan Gas Negara (Persero) Tbk.
16	SMGR	Semen Indonesia (Persero) Tbk.
17	PTBA	Tambang Batubara Bukit Asam (Persero) Tbk.
18	TLKM	Telekomunikasi Indonesia (Persero) Tbk.
19	WSKT	Waskita Karya (Persero) Tbk.
20	UNVR	Unilever Indonesia Tbk.

Suppose that the investor forms a portfolio consisting of the five best stocks. Therefore, from the 20 stocks in Table 9, the five best stocks will be selected based on the largest ratio value. The covariance among five selected stocks is summarized in Table 10. Since five stocks were selected for the portfolio, the optimization problem (5.8) becomes

$$\begin{cases} \text{minimize} : \sigma_p^2 = \sum_{j=1}^{5} \sum_{l=1}^{5} w_j w_l \sigma_{jl} = w^T \Sigma w. \\ \text{subject to} : w_1 + w_2 + ... + w_5 = 1. \end{cases} \tag{5.9}$$

Note that the proposed HTHP method is to solve the optimization problem without constraints, so to solve the problem (5.9) by using the HTHP method, we need to convert it to an unconstrained problem. Suppose that $w_5 = 1 - w_1 - w_2 - w_3 - w_4$, where w_1, w_2, w_3, w_4 and w_5 are proportional to UNVR, SMGR, BRPT, WSKT and CPIN stocks, respectively. Furthermore, by using values of covariance among the selected stocks in Table 10, we have an unconstrained optimization portfolio selection problem as follows:

$$\begin{aligned} &\min_{(w_1,w_2,w_3,w_4)\in\mathbb{R}^4} (w_1 + w_2 + w_3 + w_4 - 1)((41w_1)/10^5 + w_2/3125 + (29w_3)/10^5 \\ &+ (41w_4)/10^5 - 51/10^5) + w_4(w_2/6250 - (3w_1)/10^5 + (3w_3)/25000 + (27w_4)/25000 + 1/10^4) \\ &+ w_1((29w_1)/10^5 + w_2/50000 - w_3/50000 - (3w_4)/10^5 + 1/10^4) + w_2(w_2/2500 - (7w_1)/10^5 \\ &+ w_3/25000 + (7w_4)/10^5 + 19/10^5) + w_3(w_2/10^5 - (7w_1)/50000 + (37w_3)/50000 + 11/50000). \end{aligned}$$

Table 9. Estimated distribution, expected return, variance, and ratio of each stock.

Stock	Estimated distribution	μ	σ^2	(μ/σ^2)
ADRO	Dagum (4P)	-0.00167	0.00076	-2.18694
ANTM	Dagum (4P)	-0.00278	0.00136	-2.0364
ASII	Dagum (4P)	-0.00056	0.00043	-1.29461
BBCA	Gen. Logistic	-0.00075	0.00024	-3.06997
BBNI	Gen. Logistic	-0.00144	0.00053	-2.71164
BBRI	Gen. Logistic	-0.00067	0.00045	-1.49148
BBTN	Gen. Logistic	-0.00106	0.00074	-1.43336
BMRI	Gen. Logistic	-0.00098	0.00043	-2.26977
BRPT	Log-Logistic (3P)	0.00145	0.00101	1.440135
CPIN	Gen. Logistic	0.00029	0.00051	0.576654
ICBP	Laplace	0.00012	0.00027	0.433568
INCO	Gen. Logistic	-0.00165	0.00085	-1.93436
INDF	Burr (4P)	-0.00002	0.00025	-0.06185
KLBF	Burr (4P)	-0.00014	0.00039	-0.35195
PGAS	Gen. Logistic	-0.00105	0.00079	-1.32354
SMGR	Burr (4P)	0.00093	0.00059	1.569875
PTBA	Gen. Logistic	-0.00130	0.00052	-2.50288
TLKM	Gen. Logistic	-0.00037	0.00037	-1.00194
WSKT	Gen. Logistic	0.00090	0.00118	0.761808
UNVR	Gen. Logistic	0.00135	0.00039	3.48021

Table 10. Covariance among the selected stocks.

Stock	UNVR	SMGR	BRPT	WSKT	CPIN
UNVR	0.00039	0.00012	0.00008	0.00007	0.00010
SMGR	0.00012	0.00059	0.00023	0.00026	0.00019
BRPT	0.00008	0.00023	0.00096	0.00022	0.00022
WSKT	0.00007	0.00026	0.00022	0.00118	0.00010
CPIN	0.00010	0.00019	0.00022	0.00010	0.00051

Applying Algorithm 1 to solve the above problems, we obtain $w_1 = 0.4347, w_2 = 0.1349, w_3 = 0.0858, w_4 = 0.0973$ and $w_5 = 0.2473$. After obtaining the weight value of each stock in the formation of an efficient portfolio, the next step is to calculate the expected return of the portfolio using (5.6) and to calculate the portfolio variance using (5.7). The expected value of portfolio return is $\mu_p = 0.000949$ and portfolio variance is $\sigma_p^2 = 0.000224$. From the results of the analysis, it can be seen that the optimal portfolio composed of five stocks is a portfolio with the composition of each stock as in Table 11.

From Table 11, the UNVR is 0.4347. This value indicates that the proportion of UNVR in the formed portfolio is 43.47% of the total allocation of funds. The second proportion, namely SMGR, is 0.1349, so the amount of funds that will be allocated is 13.49%. The third proportion, namely BRPT, is 0.0858, so the funds allocated are 8.58%. The fourth proportion for WSKT is 9.73% of funds allocated, and the fifth proportion for CPIN is 24.73%. By allocating each stock based on the portion in Table 11, the investment will provide a rate of return of 0.0949% for the total allocated funds and the risk of 0.00224% on the total funds.

Table 11. Optimal portfolio weight composition.

Stock	UNVR	SMGR	BRPT	WSKT	CPIN
Proportion	0.4347	0.1349	0.0858	0.0973	0.2473

6. Conclusions

We have presented a hybrid three-term CG method for solving unconstrained optimization problems. The method is a combination of HS and PRP three-term types. Under some conditions, the global convergence properties of the method were established. By using some test functions, the numerical results showed that the method is most efficient compared to the TTCDDY, HTT, and MPRP methods. Moreover, our method was able to solve the image restoration and portfolio selection problems.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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