AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 17612–17627.
DOI:10.3934/math.2022970
Received: 30 May 2022
Revised: 12 July 2022
Accepted: 21 July 2022
Published: 01 August 2022

Research article

New convergence analysis of a class of smoothing Newton-type methods for second-order cone complementarity problem

Li Dong and Jingyong Tang*

College of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

* **Correspondence:** Email: tangjy@xynu.edu.cn.

Abstract: In this paper we propose a class of smoothing Newton-type methods for solving the second-order cone complementarity problem (SOCCP). The proposed method design is based on a special regularized Chen-Harker-Kanzow-Smale (CHKS) smoothing function. When the solution set of the SOCCP is nonempty, our method has the following convergence properties: (i) it generates a bounded iteration sequence; (ii) the value of the merit function converges to zero; (iii) any accumulation point of the generated iteration sequence is a solution of the SOCCP; (iv) it has the local quadratic convergence rate under suitable assumptions. Some numerical results are reported.

Keywords: second-order cone complementarity problem; smoothing Newton-type method; global convergence; quadratic convergence
Mathematics Subject Classification: 65K05, 90C33

1. Introduction

We consider the second-order cone complementarity problem (SOCCP), which is to find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$x \in \mathcal{K},\ y \in \mathcal{K},\ x^T y = 0,\ y = F(x), \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, $\mathcal{K} \subset \mathbb{R}^n$ is the Cartesian product of second-order cones (SOCs), that is, $\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_r}$ with $r, n_1, ..., n_r \geq 1$ and $n = \sum_{i=1}^r n_i$, where $\mathcal{K}^{n_i}$ are the n_i-dimensional second-order cone defined by

$$\mathcal{K}^{n_i} := \{(x_1, \bar{x}^T)^T \in \mathbb{R} \times \mathbb{R}^{n_i - 1} : x_1 \geq \|\bar{x}\|\}. \tag{1.2}$$

Here and below, $\|\cdot\|$ denotes the 2-norm. Since $\mathcal{K}^1$ is the set of nonnegative real values $\mathbb{R}_+$ (the nonnegative orthant in $\mathbb{R}$), the SOCCP includes the well-known nonlinear complementarity problem as a special case, corresponding to $n_i = 1$ for all $i = 1, 2, ..., r$.

In the last few years, the SOCCP has attracted a lot of attention due to its wide applicability in many fields (e.g., [2–4, 14, 16]). A number of numerical methods have been proposed to solve the SOCCP among which the smoothing Newton-type method is one of the most effective methods (e.g., [1, 5, 7, 9, 11, 12, 15, 18, 20–23]). The main idea of these smoothing Newton-type methods is to use a smoothing function to reformulate the concerned SOCCP as a system of smooth nonlinear equations $H(z) = 0$ and then solve it by using Newton's method. In these smoothing Newton-type methods, it has been proved that any accumulation point z^* of the generated iteration sequence $\{z^k\}$ satisfies $H(z^*) = 0$. However, many papers do not analyze whether such an accumulation point exists (e.g., [5, 15]). To ensure that such an accumulation point exists, existing smoothing Newton-type methods usually require that the solution set of the SOCCP is nonempty and bounded (e.g., [7, 9, 11, 12, 18, 20–23]).

Recently, Huang, Hu and Han [10] presented a nonmonotone smoothing algorithm for solving the symmetric cone complementarity problem for which global convergence is established by just requiring that the solution set of the problem is nonempty. Motivated by their work, in this paper, we give a new convergence analysis of a class of smoothing Newton-type methods for the SOCCP. Specifically, we introduce a special regularized Chen-Harker-Kanzow-Smale (CHKS) smoothing function that is perturbed by μ^t where μ is the smoothing parameter and $t \in (0, 1/2]$ is a constant. By using this smoothing function, we reformulate the SOCCP as a system of smooth nonlinear equations $H_t(z) = 0$ (see, Section 3 below) and propose a class of smoothing Newton-type methods to solve it. We prove that, when the solution set of the SOCCP is only nonempty, the proposed method has the following convergence properties.

(i) It generates a bounded iteration sequence;

(ii) The value of the merit function converges to zero;

(iii) Any accumulation point of the generated iteration sequence is a solution of the SOCCP;

(iv) It has a local quadratic convergence rate under suitable assumptions.

The paper is organized as follows. In Section 2, we introduce Euclidean Jordan algebras associated with the SOC $\mathcal{K}^n$. In Section 3, based on a special regularized CHKS smoothing function, we reformulate the SOCCP as a family of parameterized smooth nonlinear equations. In Section 4, we give a class of smoothing Newton-type methods for solving the SOCCP. In Section 5, we investigate its global and local convergence properties respectively. The numerical results are reported in Section 6. Some conclusions are given in Section 7.

Throughout the paper, $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors. $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) denotes the nonnegative (positive) orthant in $\mathbb{R}^n$. For convenience, we write $(u_1^T, ..., u_m^T)^T$ as $(u_1, ..., u_m)$ for any vectors $u_i \in \mathbb{R}^{n_i}$. I_n represents the $n \times n$ dimension identity matrix; $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. int$\mathcal{K}$ denotes the interior of $\mathcal{K}$. For any $x, y \in \mathbb{R}^n$, we write $x \succeq_{\mathcal{K}} y$ ($x \succ_{\mathcal{K}} y$) if $x - y \in \mathcal{K}$ ($x - y \in \text{int}\mathcal{K}$). For any $\alpha, \beta > 0$, $\alpha = O(\beta)$ ($\alpha = o(\beta)$) means that α/β is uniformly bounded (tends to zero) as $\beta \to 0$.

2. Euclidean Jordan algebras

For any vectors $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, their Jordan product associated with the SOC $\mathcal{K}^n$ is defined by

$$x \circ y := (x^T y,\ x_1 \bar{y} + y_1 \bar{x}).$$

The identity element under this product is $\mathbf{e} := (1, 0, ..., 0)^T \in \mathbb{R}^n$. A vector $x \in \mathbb{R}^n$ is said to be invertible if there exists a unique $y \in \mathbb{R}^n$ such that $x \circ y = \mathbf{e}$. We shall call this y the inverse of x and denote it by x^{-1}. Moreover, if $x \in \mathcal{K}^n$, then there exists a unique vector in $\mathcal{K}^n$, which we denote by $\sqrt{x}$, such that $\sqrt{x} \circ \sqrt{x} = x$.

For any $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, its spectral decomposition with respect to the SOC $\mathcal{K}^n$ is

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2,$$

where $\lambda_1(x), \lambda_2(x)$ and c_1, c_2 are the spectral values and associated spectral vectors of x, respectively, that are given by

$$\lambda_i(x) = x_1 + (-1)^i \|\bar{x}\|, \quad c_i = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{\bar{x}}{\|\bar{x}\|}), & \bar{x} \neq 0, \\ \frac{1}{2}(1, (-1)^i \omega), & \bar{x} = 0, \end{cases} \quad i = 1, 2,$$

with any $\omega \in \mathbb{R}^{n-1}$ such that $\|\omega\| = 1$.

For any $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values $\lambda_1(x), \lambda_2(x)$ and spectral vectors c_1, c_2, the following results hold:

(1) $x^2 := \lambda_1(x)^2 c_1 + \lambda_2(x)^2 c_2 \in \mathcal{K}^n$ and $x^2 = x \circ x$.

(2) If $x \in \mathcal{K}^n$, then $\lambda_2(x) \geq \lambda_1(x) \geq 0$ and $\sqrt{x} = \sqrt{\lambda_1(x)}c_1 + \sqrt{\lambda_2(x)}c_2$.

(3) If $x \in \text{int}\mathcal{K}^n$, then $\lambda_2(x) \geq \lambda_1(x) > 0$ and $x^{-1} = \lambda_1(x)^{-1}c_1 + \lambda_2(x)^{-1}c_2$.

Given an element $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define the symmetric matrix

$$L_x := \begin{bmatrix} x_1 & \bar{x}^T \\ \bar{x} & x_1 I_{n-1} \end{bmatrix}.$$

It is easy to verify that $L_x y = x \circ y, \ \forall y \in \mathbb{R}^n$. Moreover, if $x \in \text{int}\mathcal{K}^n$, then L_x is invertible with

$$L_x^{-1} = \frac{1}{\det(x)} \begin{bmatrix} x_1 & -\bar{x}^T \\ -\bar{x} & \frac{\det(x)}{x_1} I_{n-1} + \frac{\bar{x}\bar{x}^T}{x_1} \end{bmatrix},$$

where $\det(x) := x_1^2 - \|\bar{x}\|^2$ denotes the determinant of x.

We can also define the trace of $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ by $\mathbf{Tr}(x) := \lambda_1(x) + \lambda_2(x) = 2x_1$. Then, for any $x, y \in \mathbb{R}^n$, it holds that $\mathbf{Tr}(x \circ y) = 2x^T y$ and $\mathbf{Tr}(\mathbf{e}) = 2$.

3. Reformulation of the SOCCP

In the following analysis, we assume that $\mathcal{K} = \mathcal{K}^n$. This does not result in any loss of generality because our analysis can be easily extended to the general case. Smoothing Newton-type methods are typically designed based on a smoothing function. Up to now, many smoothing functions for the SOCCP have been proposed. Among them, the CHKS smoothing function

$$\varphi(\mu, x, y) = x + y - \sqrt{(x - y)^2 + 4\mu\mathbf{e}}, \quad \forall(\mu, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

is one of the most prominent smoothing functions which satisfies (see, [8, Proposition 4.1])

$$\varphi(0, x, y) = 0 \Longleftrightarrow x \in \mathcal{K}, \ y \in \mathcal{K}, \ x^T y = 0.$$

Chi and Liu [6] proposed a regularized CHKS smoothing function which is denoted as

$$\phi(\mu, x, y) = (1+\mu)(x+y) - \sqrt{(1-\mu)^2(x-y)^2 + 4\mu\mathbf{e}},\ \forall(\mu, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n.$$

Based on ϕ, Chi and Liu [6] proposed a non-interior continuation method for solving the SOC optimization problem.

In this paper, we introduce a special regularized CHKS smoothing function which is defined by

$$\phi_t(\mu, x, y) = (1+\mu^t)(x+y) - \sqrt{(1-\mu^t)^2(x-y)^2 + 4\mu\mathbf{e}},\ \forall(\mu, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{3.1}$$

where $t \in (0, 1/2]$ is a constant. As it will be shown later (see, Theorem 5.1 below), the parameter $t \in (0, 1/2]$ plays a key role in proving the boundedness of the generated iteration sequence.

The following theorem gives the continuously differentiable property of the function ϕ_t, which has a proof that is similar to Theorem 2.4 in [6].

Theorem 3.1. *Let $\phi_t(\mu, x, y)$ be defined by* (3.1). *Denote $w := \sqrt{(1-\mu^t)^2(x-y)^2 + 4\mu\mathbf{e}}$. Then $\phi_t(\mu, x, y)$ is continuously differentiable at any $(\mu, x, y) \in \mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^n$ with*

$$(\phi_t(\mu, x, y))'_\mu = t\mu^{t-1}(x+y) - L_w^{-1}[(\mu^t - 1)t\mu^{t-1}(x-y)^2 + 2\mathbf{e}],$$

$$(\phi_t(\mu, x, y))'_x = (1+\mu^t)I_n - (1-\mu^t)^2 L_w^{-1} L_{x-y},$$

$$(\phi_t(\mu, x, y))'_y = (1+\mu^t)I_n + (1-\mu^t)^2 L_w^{-1} L_{x-y}.$$

Moreover, $\lim_{\mu\to 0} \phi_t(\mu, x, y) = \phi_t(0, x, y)$ *and*

$$\phi_t(0, x, y) = 0 \Longleftrightarrow x \in \mathcal{K},\ y \in \mathcal{K},\ x^T y = 0.$$

To reformulate the SOCCP as a family of parameterized smooth equations, we introduce a class of single variable functions as follows.

Assumption 3.1. Assume that $h : \mathbb{R} \to \mathbb{R}$ is a function that satisfies the following conditions:

(a) h is continuously differentiable with $h'(\mu) > 0$ for any $\mu > 0$;
(b) $h(\mu) = 0$ implies that $\mu = 0$;
(c) $-\frac{h(\mu)}{h'(\mu)} \in [-\mu, 0)$ for any $\mu > 0$;
(d) $h(\mu) \geq \mu$ for any $\mu \geq 0$;
(e) there exist $\eta_1 \geq 0$ and $\eta_2 \geq 0$ such that $h'(\mu) \leq \eta_1 h(\mu) + \eta_2$ for any $\mu \geq 0$.

Regarding Assumption 3.1, we have the following remarks.

Remark 3.1. (i) The conditions (a)–(c) were introduced by Jiang [13].
(ii) There are many functions satisfying Assumption 3.1, for example, $h(\mu) = \mu$, $h(\mu) = e^\mu - 1$, $h(\mu) = e^\mu + \mu - 1$ and so on.
(iii) If h_1 and h_2 satisfy Assumption 3.1, then $\alpha h_1 + \beta h_2$ satisfies Assumption 3.1 for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

In the rest of this paper, we let $z := (\mu, x, y) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. We define the function $H_t : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ as

$$H_t(z) := \begin{pmatrix} h(\mu) \\ y - F(x) \\ \phi_t(\mu, x, y) \end{pmatrix}, \tag{3.2}$$

where ϕ_t is given in (3.1). Then, by Theorem 3.1 and Assumption 3.1 (b), we can immediately get the following theorem.

Theorem 3.2. (i) (x, y) *is the solution of the* SOCCP *if* $H_t(z) = 0$.
(ii) $H_t(z)$ *is continuously differentiable at any* $z = (\mu, x, y) \in \mathbb{R}_{++}\times\mathbb{R}^n\times\mathbb{R}^n$ *with its Jacobian*

$$H_t'(z) = \begin{bmatrix} h'(\mu) & 0 & 0 \\ 0 & -F'(x) & I \\ (\phi_t(z))'_\mu & (\phi_t(z))'_x & (\phi_t(z))'_y \end{bmatrix},$$

where $(\phi_t(z))'_\mu$, $(\phi_t(z))'_x$ *and* $(\phi_t(z))'_y$ *are given in Theorem* 3.1.

In the case of smoothing Newton-type methods, it is essential that the Jacobian matrix $H_t'(z)$ is invertible since the direction of descent should be well defined and unique to solve $H_t(z) = 0$. To establish the nonsingularity of $H_t'(z)$, we need the monotonicity of F which has been extensively used in previous studies (e.g., [5,7,9,11,12,15,20–22]). The function F is said to be monotone if it satisfies

$$\langle x - y, F(x) - F(y)\rangle \geq 0, \quad \forall\ (x, y) \in \mathbb{R}^n\times\mathbb{R}^n.$$

Under this monotonicity assumption, similarly to the proof of Theorem 5.1 in [12], we can establish the nonsingularity of $H_t'(z)$ as follows.

Theorem 3.3. *If* F *is monotone, then* $H_t'(z)$ *is nonsingular for any* $z \in \mathbb{R}_{++}\times\mathbb{R}^n\times\mathbb{R}^n$.

4. Class of smoothing Newton-type methods

Let $H_t(z)$ be defined by (3.2). We denote the merit function $f_t : \mathbb{R}^{1+2n} \to \mathbb{R}_+$ by

$$f_t(z) := \|H_t(z)\|^2 = h(\mu)^2 + \|y - F(x)\|^2 + \|\phi_t(\mu, x, y)\|^2. \tag{4.1}$$

We now give our methods for solving the SOCCP.

Algorithm 4.1. (A class of smoothing Newton-type methods)
Step 0: Choose constants $\delta, \sigma \in (0, 1)$ and $\mu_0 > 0$. Choose a function $h(\mu)$ satisfying the conditions (a)–(e) in Assumption 3.1. Choose $\eta_1 \geq 0$ and $\eta_2 \geq 0$ such that $h'(\mu) \leq \eta_1 h(\mu) + \eta_2$ for any $\mu \geq 0$. Choose $\gamma \in (0, 1)$ such that $\gamma \leq \mu_0$ and $\gamma(\eta_1+\eta_2) < 1$. Choose $(x^0, y^0) \in \mathbb{R}^n\times\mathbb{R}^n$ and let $z^0 := (\mu_0, x^0, y^0)$. Let $\bar{z} := (1, 0, ..., 0)^T \in \mathbb{R}^{1+2n}$. Set $k := 0$.
Step 1: If $\|H_t(z^k)\| = 0$, then stop. Otherwise, compute

$$\zeta_k := \gamma \min\{1, f_t(z^k)\}. \tag{4.2}$$

Step 2: Compute the search direction $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta y^k) \in \mathbb{R}^{1+2n}$ by solving the following system

$$H_t'(z^k)\Delta z^k = -H_t(z^k) + \zeta_k h'(\mu_k)\bar{z}. \tag{4.3}$$

Step 3: Find a step-size $\alpha_k := \delta^{l_k}$, where l_k is the smallest nonnegative integer l satisfying

$$f_t(z^k + \delta^l \Delta z^k) \leq (1 - \sigma\delta^{2l}) f_t(z^k). \tag{4.4}$$

Step 4: Set $z^{k+1} := z^k + \alpha_k \Delta z^k$. Set $k := k + 1$ and go to Step 1.

Theorem 4.1. *If F is monotone, then Algorithm* 4.1 *is well-defined and can generate an infinite sequence $\{z^k = (\mu_k, x^k, y^k)\}$ with $\mu_k > 0$ for any $k \geq 0$.*

Proof. Suppose that $z^k = (\mu_k, x^k, y^k) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ for some k. Then, it follows from Theorem 3.3 that $H_t'(z^k)$ is nonsingular. Hence, the system (4.3) is solvable. Moreover, by (4.3), we have that

$$\begin{aligned} f_t'(z^k)\Delta z^k &= 2H_t(z^k)^T H_t'(z^k)\Delta z^k \\ &= 2H_t(z^k)^T[-H_t(z^k) + \zeta_k h'(\mu_k)\bar{z}] \\ &= -2f_t(z^k) + 2h(\mu_k)h'(\mu_k)\zeta_k \\ &\leq -2f_t(z^k) + 2h(\mu_k)\zeta_k[\eta_1 h(\mu_k) + \eta_2] \\ &= -2[f_t(z^k) - \eta_1\zeta_k h(\mu_k)^2 - \eta_2\zeta_k h(\mu_k)], \end{aligned} \tag{4.5}$$

where the first inequality follows from Assumption 3.1(e). By the definition of ζ_k in (4.2), also notice that $\min\{1, a^2\} \leq a$ for any $a > 0$; additionally, we have that $\zeta_k \leq \gamma$ and $\zeta_k \leq \gamma\|H_t(z^k)\|$. This together with $h(\mu_k) \leq \|H_t(z^k)\|$ gives

$$\zeta_k h(\mu_k)^2 \leq \gamma f_t(z^k) \quad \text{and} \quad \zeta_k h(\mu_k) \leq \gamma f_t(z^k). \tag{4.6}$$

By Assumption 3.1(d), we have that $f_t(z^k) \geq h(\mu_k)^2 \geq \mu_k^2 > 0$. So, by (4.5) and (4.6), it holds

$$f_t'(z^k)\Delta z^k \leq -2[1 - \gamma(\eta_1 + \eta_2)]f_t(z^k) < 0. \tag{4.7}$$

This implies that Δz^k is the direction of descent of $f_t(z)$ at z^k. Next we show that there exists at least one nonnegative integer l that satisfies (4.4). On the contrary, we suppose that for any nonnegative integer l,

$$f_t(z^k + \delta^l \Delta z^k) > (1 - \sigma\delta^{2l}) f_t(z^k),$$

i.e.,

$$\frac{f_t(z^k + \delta^l \Delta z^k) - f_t(z^k)}{\delta^l} > -\sigma\delta^l f_t(z^k). \tag{4.8}$$

Since $\mu_k > 0$, $f_t(z)$ is continuously differentiable at z^k. So, by letting $l \to \infty$ on both sides of (4.8), we have that $f_t'(z^k)\Delta z^k \geq 0$. This contradicts (4.7). Hence, we can find the step size $\alpha_k \in (0, 1]$ in Step 3 and get the $(k+1)$th iteration point $z^{k+1} = z^k + \alpha_k \Delta z^k$. Moreover, by the first equation in (4.3), we have that $\Delta\mu_k = -\frac{h(\mu_k)}{h'(\mu_k)} + \zeta_k$ which together with Assumption 3.1(c) gives

$$\mu_{k+1} = \mu_k + \alpha_k \Delta\mu_k = \mu_k - \alpha_k \frac{h(\mu_k)}{h'(\mu_k)} + \alpha_k\zeta_k \geq (1 - \alpha_k)\mu_k + \alpha_k\zeta_k > 0. \tag{4.9}$$

Thus, we can conclude that if $z^k \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ for some k, then z^{k+1} can be generated by Algorithm 4.1 with $z^{k+1} \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$. Since $z^0 = (\mu_0, x^0, y^0) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$, by mathematical induction, we prove the theorem. □

5. Convergence analysis

In this section, we establish the global and local quadratic convergence of Algorithm 4.1 under the assumption that the solution set of the SOCCP is nonempty, without requiring its boundedness.

Lemma 5.1. *Let ϕ_t be defined by* (3.1). *For any $(\mu, x, y, c) \in \mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, one has*

$$\phi_t(\mu, x, y) = c \Longleftrightarrow x + \mu^t y - \frac{c}{2} \succ_{\mathcal{K}} 0, \mu^t x + y - \frac{c}{2} \succ_{\mathcal{K}} 0$$

$$\text{and} \left(x + \mu^t y - \frac{c}{2}\right) \circ \left(\mu^t x + y - \frac{c}{2}\right) = \mu \mathbf{e}.$$

Proof. By Lemma 4.1 in [10], for any $(\mu, a, b, c) \in \mathbb{R}_{++} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, we have that

$$a + b - \sqrt{(a-b)^2 + 4\mu \mathbf{e}} = c \Longleftrightarrow a - \frac{c}{2} \succ_{\mathcal{K}} 0, \;\; b - \frac{c}{2} \succ_{\mathcal{K}} 0$$

$$\text{and} \left(a - \frac{c}{2}\right) \circ \left(b - \frac{c}{2}\right) = \mu \mathbf{e}.$$

Since ϕ_t can be rewritten as

$$\phi_t(\mu, x, y) = (x + \mu^t y) + (\mu^t x + y) - \sqrt{[(x + \mu^t y) - (\mu^t x + y)]^2 + 4\mu \mathbf{e}},$$

we obtain the desired result. □

Lemma 5.2. *Suppose that F is monotone and $\{z^k = (\mu_k, x^k, y^k)\}$ is the iteration sequence generated by Algorithm* 4.1. *Then for all $k \geq 0$,*

$$\mu_k \geq \gamma \min\{1, f_t(z^k)\}. \tag{5.1}$$

Proof. By Step 0, we have that $\mu_0 \geq \gamma \geq \gamma \min\{1, f_t(z^0)\}$. Suppose that $\mu_k \geq \gamma \min\{1, f_t(z^k)\}$ for some k. Then, from (4.2) and (4.9) it follows that

$$\begin{aligned} \mu_{k+1} &\geq (1 - \alpha_k)\mu_k + \alpha_k \zeta_k \\ &\geq (1 - \alpha_k)\gamma \min\{1, f_t(z^k)\} + \alpha_k \zeta_k \\ &= \gamma \min\{1, f_t(z^k)\} \\ &\geq \gamma \min\{1, f_t(z^{k+1})\}, \end{aligned}$$

where the last inequality holds because $\{f_t(z^k)\}$ is monotonically decreasing as given by Step 3. So, by mathematical induction, we prove the lemma. □

Theorem 5.1. *Suppose that F is monotone and the solution set of the* SOCCP *is nonempty. Then the iteration sequence $\{z^k = (\mu_k, x^k, y^k)\}$ generated by Algorithm* 4.1 *is bounded.*

Proof. By Theorem 4.1, we have that $z^k = (\mu_k, x^k, y^k) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ for all $k \geq 0$. Moreover, since the sequence $\{f_t(z^k)\}$ is monotonically decreasing, we have that $f_t(z^k) \leq f_t(z^0)$ for any $k \geq 0$. Now we assume $\|z^k\| \to \infty$ as $k \to \infty$ and we will thus derive a contradiction. Since

$$0 < \mu_k \leq h(\mu_k) \leq \|H_t(z^k)\| = \sqrt{f_t(z^k)} \leq \sqrt{f_t(z^0)}, \tag{5.2}$$

where the second inequality holds by Assumption 3.1 (d), we have that $\|(x^k, y^k)\| \to \infty$ as $k \to \infty$. For any $t \in (0, 1/2]$, we define

$$a^k := \frac{1}{\mu_k^t}(y^k - F(x^k)), \quad b^k := \frac{1}{2\mu_k^t}\phi_t(\mu_k, x^k, y^k), \quad \forall k \geq 0. \tag{5.3}$$

Then, from (4.1), we have the following for any $k \geq 0$

$$\|a^k\|^2 + \|b^k\|^2 = \frac{1}{\mu_k{}^{2t}}\|y^k - F(x^k)\|^2 + \frac{1}{4\mu_k^{2t}}\|\phi_t(\mu_k, x^k, y^k)\|^2 \leq 2\frac{f_t(z^k)}{\mu_k^{2t}}. \tag{5.4}$$

For any $k \geq 0$, by (5.1), if $f_t(z^k) \geq 1$, then $\mu_k \geq \gamma$; therefore,

$$\frac{f_t(z^k)}{\mu_k^{2t}} \leq \frac{f_t(z^0)}{\gamma^{2t}}. \tag{5.5}$$

And if $f_t(z^k) < 1$, then $\mu_k \geq \gamma f_t(z^k)$ which gives

$$\frac{f_t(z^k)}{\mu_k^{2t}} \leq \frac{f_t(z^k)^{1-2t}}{\gamma^{2t}} \leq \frac{f_t(z^0)^{1-2t}}{\gamma^{2t}}, \quad t \in (0, 1/2]. \tag{5.6}$$

By (5.4), (5.5) and (5.6), the sequence $\{(a^k, b^k)\}$ is uniformly bounded. Since the solution set of the SOCCP is nonempty, there exists a solution of the SOCCP, denoted by $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$, such that

$$x^* \succeq_{\mathcal{K}} 0, \;\; y^* \succeq_{\mathcal{K}} 0, \;\; \langle x^*, y^* \rangle = 0, \;\; y^* = F(x^*). \tag{5.7}$$

Now we construct another sequence $\{(\hat{x}^k, \hat{y}^k)\}$ given

$$\hat{x}^k := x^k + \mu_k^t y^k - \mu_k^t b^k, \;\; \hat{y}^k := \mu_k^t x^k + y^k - \mu_k^t b^k. \tag{5.8}$$

By (5.3), we have that $\phi_t(\mu_k, x^k, y^k) = 2\mu_k^t b^k$. Thus, from Lemma 5.1 it follows that

$$\hat{x}^k \succ_{\mathcal{K}} 0, \quad \hat{y}^k \succ_{\mathcal{K}} 0 \quad \text{and} \quad \hat{x}^k \circ \hat{y}^k = \mu_k \mathbf{e}. \tag{5.9}$$

By (5.7), (5.8) and (5.9), and also using the fact that $\langle p, q \rangle \geq 0$ holds for any $p \succeq_{\mathcal{K}} 0$ and $q \succeq_{\mathcal{K}} 0$ (see, [10, Lemma 2.3]), we have that

$$\begin{aligned}
2\mu_k &= \mathbf{Tr}(\mu_k \mathbf{e}) = 2\langle \hat{x}^k, \hat{y}^k \rangle \\
&\geq 2[\langle \hat{x}^k, \hat{y}^k \rangle - \langle x^*, \hat{y}^k \rangle - \langle \hat{x}^k, y^* \rangle] \\
&= 2\langle \hat{x}^k - x^*, \hat{y}^k - y^* \rangle \\
&= 2\langle x^k - x^* + \mu_k^t y^k - \mu_k^t b^k, y^k - y^* + \mu_k^t x^k - \mu_k^t b^k \rangle,
\end{aligned}$$

which together with (5.3), (5.7) and the monotonicity of F gives

$$\begin{aligned}
\mu_k &\geq \langle x^k - x^*, y^k - y^* \rangle + \mu_k^t \langle x^k - x^*, x^k - b^k \rangle \\
&\quad + \mu_k^t \langle y^k - b^k, y^k - y^* \rangle + \mu_k^{2t} \langle y^k - b^k, x^k - b^k \rangle \\
&= \langle x^k - x^*, F(x^k) - F(x^*) + \mu_k^t a^k \rangle + \mu_k^t \langle x^k - x^*, x^k - b^k \rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \mu_k^t\langle y^k - b^k, y^k - y^*\rangle + \mu_k^{2t}\langle F(x^k) - b^k + \mu_k^t a^k, x^k - b^k\rangle \\
&\geq \mu_k^t\langle x^k - x^*, a^k\rangle + \mu_k^t\|x^k\|^2 + \mu_k^t\langle x^k, -b^k\rangle + \mu_k^t\langle -x^*, x^k\rangle + \mu_k^t\langle x^*, b^k\rangle \\
&+ \mu_k^t\|y^k\|^2 + \mu_k^t\langle y^k, -y^*\rangle + \mu_k^t\langle -b^k, y^k\rangle + \mu_k^t\langle b^k, y^*\rangle \\
&+ \mu_k^{2t}\langle F(x^k) - F(b^k), x^k - b^k\rangle + \mu_k^{2t}\langle F(b^k) + \mu_k^t a^k - b^k, x^k - b^k\rangle \\
&\geq \mu_k^t[\|(x^k, y^k)\|^2 + \langle x^k, p(\mu_k^t, a^k, b^k)\rangle - \langle y^k, y^* + b^k\rangle + q(\mu_k^t, a^k, b^k)],
\end{aligned}
\tag{5.10}
$$

where

$$
p(\mu_k^t, a^k, b^k) := a^k - b^k - x^* + \mu_k^t(F(b^k) + \mu_k^t a^k - b^k),
$$

$$
q(\mu_k^t, a^k, b^k) := \langle x^*, b^k - a^k\rangle + \langle b^k, y^*\rangle + \mu_k^t\langle F(b^k) + \mu_k^t a^k - b^k, -b^k\rangle.
$$

Furthermore, by (5.2) and (5.10), we have that

$$
\|(x^k, y^k)\|^2 + \langle x^k, p(\mu_k^t, a^k, b^k)\rangle - \langle y^k, y^* + b^k\rangle + q(\mu_k^t, a^k, b^k) \leq \mu_k^{1-t} \leq f_t(z^0)^{\frac{1-t}{2}},
$$

which together with $\|(x^k, y^k)\| \to \infty$ as $k \to \infty$ gives

$$
\lim_{k\to\infty}\left[1 + \frac{\langle x^k, p(\mu_k^t, a^k, b^k)\rangle}{\|(x^k, y^k)\|^2} - \frac{\langle y^k, y^* + b^k\rangle}{\|(x^k, y^k)\|^2} + \frac{q(\mu_k^t, a^k, b^k)}{\|(x^k, y^k)\|^2}\right] \leq \lim_{k\to\infty}\frac{f_t(z^0)^{\frac{t-1}{2}}}{\|(x^k, y^k)\|^2} = 0. \tag{5.11}
$$

Since $\{(a^k, b^k)\}$ is bounded and $0 < \mu_k \leq \sqrt{f_t(z^0)}$ for any $k \geq 0$, the sequences $\{p(\mu_k^t, a^k, b^k)\}$ and $\{q(\mu_k^t, a^k, b^k)\}$ are all bounded. Also notice that the sequences $\left\{\frac{x^k}{\|(x^k,y^k)\|}\right\}$ and $\left\{\frac{y^k}{\|(x^k,y^k)\|}\right\}$ are all bounded. Thus, we have that

$$
\begin{aligned}
&\lim_{k\to\infty}\left[\frac{\langle x^k, p(\mu_k^t, a^k, b^k)\rangle}{\|(x^k, y^k)\|^2} - \frac{\langle y^k, y^* + b^k\rangle}{\|(x^k, y^k)\|^2} + \frac{q(\mu_k^t, a^k, b^k)}{\|(x^k, y^k)\|^2}\right] \\
&= \lim_{k\to\infty}\left[\left\langle\frac{x^k}{\|(x^k, y^k)\|}, \frac{p(\mu_k^t, a^k, b^k)}{\|(x^k, y^k)\|}\right\rangle - \left\langle\frac{y^k}{\|(x^k, y^k)\|}, \frac{y^* + b^k}{\|(x^k, y^k)\|}\right\rangle + \frac{q(\mu_k^t, a^k, b^k)}{\|(x^k, y^k)\|^2}\right] = 0,
\end{aligned}
$$

which is in contradiction to (5.11). The proof is completed. □

Remark 5.1. In the proof of Theorem 5.1, the second inequality in (5.6) is essential and it only holds for $t \in (0, 1/2]$. This is why we replace the parameter μ by μ^t ($t \in (0, 1/2]$) for the CHKS function in (3.1).

Theorem 5.2. *Suppose that F is monotone and the solution set of the* SOCCP *is nonempty. Let $\{z^k = (\mu_k, x^k, y^k)\}$ be the iteration sequence generated by Algorithm* 4.1. *Then*

$$
\lim_{k\to\infty} f_t(z^k) = 0. \tag{5.12}
$$

Proof. Since $\{f_t(z^k)\}$ is monotonically decreasing, it is convergent. So, there exists a constant $f^* \geq 0$ such that

$$
\lim_{k\to\infty} f_t(z^k) = f^*, \quad \lim_{k\to\infty} \zeta_k = \zeta^* := \gamma\min\{1, f^*\}.
$$

Now we assume $f^* > 0$ and will thus derive a contradiction. Since the sequence $\{z^k = (\mu_k, x^k, y^k)\}$ is bounded, it has at least one accumulation point, denoted by $z^* := (\mu^*, x^*, y^*)$. Then there exists an infinite subsequence $\{z^k\}_{k\in K} \subset \{z^k\}$ such that $\lim\limits_{k\in K, k\to\infty} z^k = z^*$. We now divide the proof into the following two parts.

Part 1. Assume that $\alpha_k \geq c$ for all $k \in K$, where $c > 0$ is a fixed constant. Then from Steps 3 and 4, we have the following for all $k \in K$

$$f_t(z^{k+1}) \leq (1 - \sigma\alpha_k^2)f_t(z^k) \leq (1 - \sigma c^2)f_t(z^k). \tag{5.13}$$

By letting $k \to \infty$ with $k \in K$ on both sides of the inequality (5.13), we have that $f^* \leq (1 - \sigma c^2)f^*$ which gives $f^* = 0$. This contradicts the assumption $f^* > 0$.

Part 2. Assume that there exists an infinite subset $\bar{K} \subset K$ such that $\lim\limits_{k\in\bar{K}, k\to\infty} \alpha_k = 0$. Then, by the line search criterion (4.4), we have the following for all $k \in \bar{K}$

$$f_t(z^k + \delta^{-1}\alpha_k\Delta z^k) > (1 - \sigma(\delta^{-1}\alpha_k)^2)f_t(z^k),$$

i.e.,

$$\frac{f_t(z^k + \delta^{-1}\alpha_k\Delta z^k) - f_t(z^k)}{\delta^{-1}\alpha_k} > -\sigma\delta^{-1}\alpha_k f_t(z^k). \tag{5.14}$$

By (5.1), we have that $\mu^* \geq \gamma \min\{1, f^*\} > 0$. Thus, $f_t(z)$ is continuously differentiable at z^*. By letting $k \to \infty$ with $k \in \bar{K}$ on both sides of the inequality (5.14), we have that

$$f_t'(z^*)\Delta z^* \geq 0, \tag{5.15}$$

where $\Delta z^* := H_t'(z^*)^{-1}[-H_t(z^*) + \zeta^* h'(\mu^*)\bar{z}]$. On the other hand, from Step 3 it follows that

$$\frac{f_t(z^k + \alpha_k\Delta z^k) - f_t(z^k)}{\alpha_k} \leq -\sigma\alpha_k f_t(z^k). \tag{5.16}$$

By letting $k \to \infty$ with $k \in \bar{K}$ on both sides of the inequality (5.16), we have that

$$f_t'(z^*)\Delta z^* \leq 0. \tag{5.17}$$

Hence, we can conclude from (5.15) and (5.17) that $f_t'(z^*)\Delta z^* = 0$. By (4.3), we have the following for all $k \geq 0$

$$f_t'(z^k)\Delta z^k \leq -2[1 - \gamma(\eta_1 + \eta_2)]f_t(z^k). \tag{5.18}$$

By letting $k \to \infty$ with $k \in \bar{K}$ on both sides of the inequality (5.18), and also using $f_t'(z^*)\Delta z^* = 0$, we have that $2[1-\gamma(\eta_1+\eta_2)]f^* \leq 0$ which together with $\gamma(\eta_1+\eta_2) < 1$ yields $f^* = 0$. This also contradicts the assumption $f^* > 0$. We complete the proof. □

Theorem 5.3. *Suppose that F is monotone and the solution set of the* SOCCP *is nonempty. Then any accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm* 4.1 *is a solution of $f_t(z) = 0$.*

Proof. The theorem holds under the conditions of (5.12) and the continuity of f_t. □

Similar to the proof of Theorem 8 in [19], we obtain the local superlinear and quadratic convergence of Algorithm 4.1 as follows.

Theorem 5.4. *Suppose that F is monotone and the solution set of the* SOCCP *is nonempty. Suppose that z^* is an accumulation point of the infinite sequence $\{z^k\}$ generated by Algorithm* 4.1*. Suppose that H is semismooth at z^* and that all $V \in \partial H(z^*)$ are nonsingular. Then the whole sequence $\{z^k\}$ converges to z^* and $\|z^{k+1} - z^*\| = o(\|z^k - z^*\|)$. Furthermore, if H is strongly semismooth at z^*, then $\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^2)$.*

6. Numerical experiments

In this section, we discuss how we applied Algorithm 4.1 to solve some SOCCPs. All experiments were performed on a personal computer with 1.96 GB of memory and a Pentium(R) Dual-Core processor 2.93 GHz . The program codes were written in Matlab and run in a Matlab 7.1 environment. For the experiments, we chose $h(\mu) = e^{\mu} - 1, \mu_0 = 10^{-2}, \gamma = 10^{-4}, \sigma = 10^{-3}, \delta = 0.8$ and $t = 0.5$. We applied $\|H_t(z^k)\| \leq 10^{-5}$ as the stopping criterion.

Example 6.1. Consider the following linear SOCCP:

$$x \in \mathcal{K}^2, \;\; y \in \mathcal{K}^2, \;\; x^T y = 0, \;\; F(x) = Mx + q,$$

where

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Obviously, F is monotone. It is easy to see that $x = (\alpha + 1, \alpha)^T$ and $y = (0, 0)^T$ is the solution of the SOCCP for any $\alpha \geq -0.5$. Thus, the solution set of the SOCCP is unbounded. For the experiments, we chose the starting points as follows: (1) $x^0 = \textbf{rand}(2, 1), \; y^0 = \textbf{rand}(2, 1)$; (2) $x^0 = 100 \times \textbf{rand}(2, 1), \; y^0 = 100 \times \textbf{rand}(2, 1)$; (3) $x^0 = 10 \times \textbf{rand}(2, 1), \; y^0 = Mx^0 + q$; (4) $x^0 = -10 \times \textbf{rand}(2, 1), \; y^0 = Mx^0 + q$. The numerical results are listed in Table 1 where **ST** denotes the starting point, **IT** denotes the number of iterations, **HK** denotes the value of $\|H_t(z)\|$ when Algorithm 4.1 terminates and **SOL** denotes the solution obtained via Algorithm 4.1.

Table 1. Numerical results for Example 6.1.

ST	IT	HK	SOL
(1)	5	3.0356×10^{-6}	$((0.5100, -0.4900), (0, 0))$
	6	9.2030×10^{-7}	$((0.5153, -0.4847), (0, 0))$
	5	4.4773×10^{-6}	$((0.5068, -0.4932), (0, 0))$
(2)	8	9.5651×10^{-6}	$((0.5018, -0.4982), (0, 0))$
	6	9.8135×10^{-6}	$((0.8930, -0.1070), (0, 0))$
	6	1.3431×10^{-8}	$((0.5082, -0.4918), (0, 0))$
(3)	6	4.0106×10^{-8}	$((0.5177, -0.4823), (0, 0))$
	5	3.8350×10^{-6}	$((0.6641, -0.3359), (0, 0))$
	5	5.8296×10^{-6}	$((0.7465, -0.2535), (0, 0))$
(4)	4	1.5372×10^{-6}	$((0.5034, -0.4966), (0, 0))$
	5	8.7951×10^{-6}	$((0.5024, -0.4976), (0, 0))$
	5	9.0901×10^{-6}	$((0.5024, -0.4976), (0, 0))$

Example 6.2. Consider the following linear SOCCP:

$$x \in \mathcal{K}^3, \;\; y \in \mathcal{K}^3, \;\; x^T y = 0, \;\; F(x) = Mx + q,$$

where

$$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this example, F is also monotone. It is easy to see that $x = (\alpha, \alpha, 0)$ and $y = (0, 0, 0)$ is the solution of the SOCCP for any $\alpha \geq 0$. Thus, the solution set of the SOCCP is also unbounded. A main difference between Example 6.1 and Example 6.2 is that the SOCCP considered in Example 6.2 does not have a strictly complementary solution. For the experiments, the starting points were chosen as follows: (1) $x^0 = y^0 = (1, 1, 1)$; (2) $x^0 = y^0 = (10, 10, 10)$; (3) $x^0 = y^0 = (1000, 1000, 1000)$; (4) $x^0 = (1, 1, 1),\ y^0 = Mx^0 + q$; (5) $x^0 = -(1, 1, 1),\ y^0 = Mx^0 + q$; (6) $x^0 = (10, 10, 10),\ y^0 = Mx^0 + q$; (7) $x^0 = -(10, 10, 10),\ y^0 = Mx^0 + q$. The testing results are listed in Table 2.

Table 2. Numerical results for Example 6.2.

ST	IT	HK	SOL
(1)	3	8.5878×10^{-6}	$((0.0390, 0.0390, 0), (0, 0, 0))$
(2)	4	1.2772×10^{-6}	$((0.0673, 0.0673, 0), (0, 0, 0))$
(3)	5	7.2163×10^{-7}	$((1.7992, 1.7992, 0), (0, 0, 0))$
(4)	3	8.8100×10^{-6}	$((0.1278, 0.1278, 0), (0, 0, 0))$
(5)	3	8.7263×10^{-6}	$((0.0052, 0.0052, 0), (0, 0, 0))$
(6)	4	1.4706×10^{-6}	$((0.3179, 0.3179, 0), (0, 0, 0))$
(7)	4	1.2731×10^{-6}	$((0.0056, 0.0056, 0), (0, 0, 0))$

Example 6.3. Consider the following nonlinear SOCCP:

$$x \in \mathcal{K},\ y \in \mathcal{K},\ x^T y = 0,\ y = F(x),$$

where $\mathcal{K} = \mathcal{K}^3 \times \mathcal{K}^2$ and $F : \mathbb{R}^5 \to \mathbb{R}^5$ is given by

$$F(x) = \begin{pmatrix} 24(2x_1 - x_2)^3 + exp(x_1 - x_3) - 4x_4 + x_5 \\ -12(2x_1 - x_2)^3 + 3(3x_2 + 5x_3)/\sqrt{1 + (3x_2 + 5x_3)^2} - 6x_4 - 7x_5 \\ -exp(x_1 - x_3) + 5(3x_2 + 5x_3)/\sqrt{1 + (3x_2 + 5x_3)^2} - 3x_4 + 5x_5 \\ 4x_1 + 6x_2 + 3x_3 - 1 \\ -x_1 + 7x_2 - 5x_3 + 2 \end{pmatrix}.$$

From [9], F is monotone. Via Algorithm 4.1, we obtain one solution $x^* \approx (0.2324, -0.0731, 0.2206, 0.5339, -0.5339)^T$. We test this problem by implementing the starting point $x^0 = y^0$ as follows: (1) $(0, ..., 0)^T$; (2) $(1, ..., 1)^T$; (3) $(-1, ..., -1)^T$; (4) $(10, ..., 10)^T$; (5) $(-10, ..., -10)^T$; (6) $(100, ..., 100)^T$; (7) $(-100, ..., -100)^T$. For the purpose of comparison, we also implemented the smoothing Newton-type method developed by Narushima, Sagara and Ogasawara [17] to solve this test problem, which has been designed based on the following Fischer-Burmeister smoothing function:

$$\varphi(\mu, x, y) = x + y - \sqrt{x^2 + y^2 + 2\mu^2\mathbf{e}}, \quad \forall(\mu, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n.$$

The numerical results are listed in Table 3.

Table 3. Numerical results for Example 6.3.

	Algorithm 4.1		Algorithm in [17]	
ST	IT	HK	IT	HK
(1)	8	5.6960×10^{-6}	12	3.0873×10^{-6}
(2)	10	3.3814×10^{-6}	12	4.5373×10^{-6}
(3)	14	1.1748×10^{-6}	9	3.1992×10^{-7}
(4)	16	3.4836×10^{-6}	18	3.0818×10^{-6}
(5)	17	2.8765×10^{-7}	19	3.3922×10^{-10}
(6)	25	6.3377×10^{-7}	26	3.1577×10^{-6}
(7)	21	5.8425×10^{-6}	25	5.2742×10^{-6}

Example 6.4. Consider the following nonlinear SOCCP:

$$x \in \mathcal{K},\ y \in \mathcal{K},\ x^T y = 0,\ y = F(x),$$

where $\mathcal{K} = \mathcal{K}^4$ and $F : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} e^{x_1} + x_1^2 \\ e^{x_2} + x_2^2 \\ e^{x_3} + x_3^2 \\ e^{x_4} + x_4^2 \end{pmatrix}.$$

Via Algorithm 4.1, we obtain one solution $x^* \approx (0.3278, -0.1893, -0.1893, -0.1893)^T$. We tested this problem by implementing the starting point $x^0 = y^0$ as follows: (1) $(0, ..., 0)^T$; (2) $(1, ..., 1)^T$; (3) $(-1, ..., -1)^T$; (4) $(2, ..., 2)^T$; (5) $(-2, ..., -2)^T$; (6) $(4, ..., 4)^T$; (7) $(-4, ..., -4)^T$; (8) $(5, ..., 5)^T$; (9) $(-5, ..., -5)^T$; (10) $(10, ..., 10)^T$; (11) $(-10, ..., -10)^T$. The numerical results are listed in Table 4, where $*$ denotes that the algorithm fails to get an optimizer in the short CPU time.

Table 4. Numerical results for Example 6.4.

	Algorithm 4.1		Algorithm in [17]	
ST	IT	HK	IT	HK
(1)	10	3.4465×10^{-6}	7	5.7100×10^{-7}
(2)	10	6.1962×10^{-6}	10	3.8288×10^{-10}
(3)	8	1.6898×10^{-6}	*	*
(4)	13	6.4343×10^{-6}	10	7.3642×10^{-8}
(5)	9	3.7606×10^{-6}	11	6.9450×10^{-8}
(6)	11	6.9029×10^{-6}	13	1.2211×10^{-11}
(7)	9	2.8446×10^{-6}	56	3.0036×10^{-8}
(8)	12	2.8408×10^{-6}	10	9.6157×10^{-7}
(9)	11	5.0492×10^{-7}	47	2.3039×10^{-7}
(10)	15	5.2043×10^{-6}	19	5.6091×10^{-8}
(11)	17	2.4327×10^{-6}	*	*

From Tables 1 and 2, we can find that Algorithm 4.1 can solve all of the tested problems in very few iterations. Moreover, the numbers of iterations were slightly different for different starting points. From Tables 3 and 4, we may see that our algorithm has some advantages over the smoothing Newton-type algorithm presented in [17]. Although the reported numerical results are preliminary, they demonstrate that the proposed algorithm is promising for solving SOCCPs even though the solution sets of these problems are unbounded.

7. Concluding remarks

Based on the regularized CHKS smoothing function ϕ_t in (3.1), we have proposed a class of smoothing Newton-type methods for solving the SOCCP. Unlike many smoothing Newton-type methods, which usually require the boundedness of the solution set, our method is globally and locally superlinearly/quadratically convergent when the solution set of the SOCCP is only nonempty, and it does not require its boundedness.

In the reformulation of the SOCCP, we need a function h that satisfies the conditions (a)–(e) in Assumption 3.1. Some comments on these conditions are explained as follows. The condition (a) ensures that $H_t(z)$ is smooth and $\frac{h(\mu)}{h'(\mu)}$ is well-defined. The condition (b) guarantees that $H_t(z) = 0$ gives $\mu = 0$ and that (x, y) is a solution of the SOCCP. The conditions (c) and (d) ensure that the sequence $\{\mu_k\}$ generated by Algorithm 4.1 is positive and bounded. The condition (e) ensures that the search direction is the direction of descent of the merit function.

Acknowledgements

This work has been supported by the Natural Science Foundation of Henan Province (222300420520) and Key Scientific Research Projects of Higher Education of Henan Province (22A110020). We are very grateful to the referees for their valuable comments and suggestions on the paper as they have considerably improved the paper.

Conflict of interest

The authors declare no conflict of interest.

References

1. L. J. Chen, C. F. Ma, A modified smoothing and regularized Newton method for monotone second-order cone complementarity problems, *Comput. Math. Appl.*, **61** (2011), 1047–1418. https://doi.org/10.1016/j.camwa.2011.01.009

2. J. S. Chen, Two classes of merit functions for the second-order cone complementarity problem, *Math. Meth. Oper. Res.*, **64** (2006), 495–519. https://doi.org/10.1007/s00186-006-0098-9

3. J. S. Chen, S. H. Pan, A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs, *Pac. J. Optim.*, **8** (2012), 33–74.

4. J. S. Chen, S. H. Pan, A descent method for a reformulation of the second-order cone complementarity problem, *J. Comput. Appl. Math.*, **213** (2008), 547–558. https://doi.org/10.1016/j.cam.2007.01.029

5. X. D. Chen, D. F. Sun, J. Sun, Complementarity functions and numerical experiments on some smoothing Newton methods for second-order-cone complementarity problems, *Comput. Optim. Appl.*, **25** (2003), 39–56. https://doi.org/10.1023/A:1022996819381

6. X. N. Chi, S. Y. Liu, A non-interior continuation method for second-order cone optimization, *Optimization*, **58** (2009), 965–979. https://doi.org/10.1080/02331930701763421

7. L. Dong, J. Y. Tang, J. C. Zhou, A smoothing Newton algorithm for solving the monotone second-order cone complementarity problems, *J. Appl. Math. Comput.*, **40** (2012), 45–61. https://doi.org/10.1007/s12190-012-0550-3

8. M. Fukushima, Z. Luo, P. Tseng, Smoothing functions for second-order-cone complementarity problems, *SIAM J. Optim.*, **12** (2002), 436–460. https://doi.org/10.1137/S1052623400380365

9. S. Hayashi, N. Yamashita, M. Fukushima, A combined smoothing and regularization method for monotone second-order cone complementarity problems, *SIAM J. Optim.*, **15** (2005), 593–615. https://doi.org/10.1137/S1052623403421516

10. Z. H. Huang, S. L. Hu, J. Y. Han, Convergence of a smoothing algorithm for symmetric cone complementarity problems with a nonmonotone line search, *Sci. China Ser. A-Math.*, **52** (2009), 833–848. https://doi.org/10.1007/s11425-008-0170-4

11. N. Huang, C. F. Ma, A regularized smoothing Newton method for solving SOCCPs based on a new smoothing C-function, *Appl. Math. Comput.*, **230** (2014), 315–329. https://doi.org/10.1016/j.amc.2013.12.116

12. Z. H. Huang, T. Ni, Smoothing algorithms for complementarity problems over symmetric cones, *Comput. Optim. Appl.*, **45** (2010), 557–579. https://doi.org/10.1007/s10589-008-9180-y

13. H. Jiang, Smoothed Fischer-Burmeister equation methods for the complementarity problem, Department of Mathematics, The University of Melbourne, Parille, Victoria, Australia, Technical Report, June, 1997.

14. Y. Kanno, J. Martins, A. Costa, Three-dimensional quasi-static frictional contact by using second-order cone linear complementarity problem, *Int. J. Numer. Meth. Eng.*, **65** (2006), 62–83. https://doi.org/10.1002/nme.1493

15. L. X. Liu, S. Y. Liu, A smoothing Newton method based on a one-parametric class of smoothing function for SOCCP, *J. Appl. Math. Comput.*, **36** (2011), 473–487. https://doi.org/10.1007/s12190-010-0414-7

16. N. Lu, Z. H. Huang, Solvability of Newton equations in smoothing-type algorithms for the SOCCP, *J. Comput. Appl. Math.*, **235** (2011), 2270–2276. https://doi.org/10.1016/j.cam.2010.10.025

17. Y. Narushima, N. Sagara, H. Ogasawara, A smoothing Newton method with Fischer-Burmeister function for second-order cone complementarity problems, *J. Optim. Theory Appl.*, **149** (2011), 79–101. https://doi.org/10.1007/s10957-010-9776-0

18. S. H. Pan, J. S. Chen, A regularization method for the second-order cone complementarity problem with the Cartesian P_0-property, *Nonlinear Anal. Theor.*, **70** (2009), 1475–1491. https://doi.org/10.1016/j.na.2008.02.028
19. L. Qi, D. Sun, G. Zhou, A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequality problems, *Math. Program.*, **87** (2000), 1–35. https://doi.org/10.1007/s101079900127
20. S. P. Rui, C. X. Xu, An inexact smoothing method for SOCCPs based on a one-parametric class of smoothing function, *Appl. Math. Comput.*, **241** (2014), 167–182. https://doi.org/10.1016/j.amc.2014.05.007
21. J. Y. Tang, L. Dong, J. C. Zhou, L. Sun, A smoothing-type algorithm for the second-order cone complementarity problem with a new nonmonotone line search, *Optimization*, **64** (2015), 1935–1955. https://doi.org/10.1080/02331934.2014.906595
22. J. Y. Tang, J. C. Zhou, L. Fang, A non-monotone regularization Newton method for the second-order cone complementarity problem, *Appl. Math. Comput.*, **271** (2015), 743–756. https://doi.org/10.1016/j.amc.2015.09.017
23. X. S. Zhang, S. Y. Liu, Z. H. Liu, A regularization smoothing method for second-order cone complementarity problem, *Nonlinear Anal. Real*, **12** (2011), 731–740. https://doi.org/10.1016/j.nonrwa.2010.08.001

AIMS Press