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Research article

A fixed point result of weakly contractive operators in generalized metric spaces

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Abstract: In this note, by using basic properties of the recently introduced concepts of generalized metric spaces, new conditions for the existence of a fixed point for weakly type contractive operator which sends a closed subset into the ambient space under consideration are examined. Our obtained result extends and unifies its corresponding ideas in metric and modular spaces. A comparative non-trivial example is provided to show the novelty and preeminence of our proposed notion.

Keywords: fixed point; generalized metric space; modular space; weakly contractive
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1. Introduction and preliminaries

For about a century, there has been enormous investigations concerning the existence of fixed points of nonexpansive operators. For some examples, we refer [4, 5, 15, 23]. This research is motivated by the contraction mapping principle due to Banach [3]. Following the publication of Banach fixed point (FP) theorem, more than a handful of related developments have taken place (e.g., see [1, 6, 11, 19, 20, 25, 27]). Along the line, the examination of new spaces has been an active project in the mathematical community. In particular, the idea of modular spaces, as an extension of metric spaces (MS), was initiated by Nakano [22] and was built further by Koshi [13] and Yamamuro [28]. However, deeper developments of these concepts are in respect of Musielak [21], Mazur [17] and their co-investigators.

In 2008, Chisyakov [7] launched the notion of modular MS induced by F-modular, thereby, coming up with the theory of MS generated by modular, which was later named as modular MS in [8]. Recently, Reich and Alexander [24] coined an interesting concept of a generalized metric space (MS) which is an improvement of both the ideas of a modular space and of a MS. They established a new FP result for Rakotch type contractive operator which takes a closed subset into the ground space [24]. In general, the theory of modular metric spaces has enormous applications in areas such as electrorheological fluids, economy, engineering, approximation theory, and many emerging fields. For some known results on this topic, the reader can consult [10, 18] and the references therein.

Following the new idea in [24], we notice that a host of FP results for a significant number of known contractions are still awaiting to be investigated. Whence, by availing the properties of a generalized metric and modular spaces, we analyse in this paper, new criteria for the existence of a FP for weakly type contractive operators. Our obtained result complements the idea of [24, Theorem 4.1] and some references therein.

Hereafter, we gather specific basics of modular spaces, modular MS and related ideas needed in the sequel. For these preliminaries, the reader can follow [12, 14, 21, 24, 26]. Accordingly, let Υ be a vector space over the field Φ ($\Phi = \mathbb{R}$ or $\mathbb{C}$). The functional $\eta : \Upsilon \longrightarrow [0, +\infty]$ is called a modular (cf. [12, 14, 26]) if the following properties are obeyed:

($m1$) $\eta(x) = 0$ if and only if $x = 0$;
($m2$) $\eta(\lambda x) = \eta(x)$ for all $x \in \Upsilon$, and $\lambda \in \Phi$ with $|\lambda| = 1$;
($m3$) $\eta(\lambda x + \beta y) \leq \eta(x) + \eta(y)$ for each $x, y \in \Upsilon$, and each $\lambda, \beta \geq 0$ obeying $\lambda + \beta = 1$.

Remark 1. Note that η is a non-decreasing function. In other words, assume that $0 < \lambda < \beta$. Then, it follows from Property ($m3$) with $y = 0$ that $\eta(\lambda x) = \eta\left(\frac{\lambda}{\beta}(\beta x)\right) \leq \eta(\beta x)$.

A modular η defines a corresponding modular space, that is, the vector space Υ_η given by:

$$\Upsilon_\eta = \{x \in \Upsilon : \eta(\alpha x) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0\}.$$

Let η be a modular defined on a vector space Υ. The modular η is said to obey a Δ_2-type condition if we can find a number $\Omega > 0$ such that

$$\eta(2x) \leq \Omega\eta(x), x \in \Upsilon_\eta. \tag{1.1}$$

Khamsi et al. [12] studied a modular function L_η(which is a special case of a modular space) with a modular η fulfilling a Δ_2-type criterion. They ([12]) noted that if ω is a point-valued mapping of a closed subset F of L_η such that for some $\gamma \in [0, 1)$,

$$\eta(\omega(x), \omega(y)) \leq \gamma\eta(x, y) \ \forall x, y \in F,$$

and such that there exists $x_0 \in F$ obeying

$$\sup\left\{\eta(2\omega^k(x_0)) : k = 1, 2, \cdots\right\} < +\infty,$$

then ω possesses a FP.

Suppose that η is a modular defined on a vector space Υ. For each $x, y \in \Upsilon$, let

$$\mu(x, y) = \eta(x - y).$$

Obviously, $\mu(x,y)=0$ if and only if $x=y$ and $\mu(x,y)=\mu(y,x)$. Suppose that η satisfies the Δ_2-type condition (1.1) with a number $\Omega>0$. Then for each $x,y,z\in\Upsilon_\eta$, we get (cf. [24])

$$\begin{aligned}\mu(x,z)=\eta(x-z) &= \eta((x-y)+(y-z))\\ &= \eta(2(2^{-1}(x-y)+2^{-1}(y-z)))\\ &\leq \Omega\eta(2^{-1}(x-y)+2^{-1}(y-z))\\ &\leq \Omega(\eta(x-y)+\eta(y-z))\\ &\leq \Omega(\mu(x,y)+\mu(y,z)).\end{aligned}$$

Reich and Alexander [24] launched the idea of a generalized MS in the following way:

Definition 1.1. [24] Let Υ be a nonempty set, $\mu:\Upsilon^2\longrightarrow[0,+\infty],\Omega>0$, and for each $x,y\in\Upsilon$,

$(g1)$ $\mu(x,y)=0$ if and only if $x=y$;
$(g2)$ $\mu(x,y)=\mu(y,x)$;
$(g3)$ $\mu(x,z)\leq\Omega(\mu(x,y)+\mu(y,z))$.

Then, the pair (Υ,μ) is called a generalized MS.

For each $x\in\Upsilon$ and $r>0$, take

$$B_\mu(x,y)=\{y\in\Upsilon:\mu(x,y)\leq r\}.$$

We endow the space Υ which is uniformly determined by the base

$$\mho(\epsilon)=\{(x,y)\in\Upsilon^2:\mu(x,y)\leq\epsilon\,,\epsilon>0\}. \tag{1.2}$$

This space can be metricized (by a metric $\tilde{\mu}$ (cf. [24])). In like manner, we endow the space Υ with the topology generated by this uniformity and assume that the uniform space is complete.

Using the above preliminaries, Reich and Alexander [24] proved the following FP theorem for a Rakotch type contractive operator.

Theorem 1.2. *[24, 4.1] Let F be a nonempty closed subset of Υ and the mapping $\omega:F\longrightarrow\Upsilon$ obeys the conditions*

$$\mu(\omega(x),\omega(y))\leq\varphi(\mu(x,y))\mu(x,y),$$

for each $x,y\in\Upsilon$ fulfilling $\mu(x,y)<+\infty$, where the function $\varphi:[0,+\infty)\longrightarrow[0,+\infty)$ is non-increasing and satisfies $\varphi(t)<1$ for all $t>0$. Suppose that for each integer $n\geq1$, there exists a point $x_n\in F$ such that $\omega^n(x_n)$ exists and is contained in F, and that the set

$$G=\{\omega^i(x_n):n=1,2,\cdots\text{ and }i\in\{0,\cdots,n\}\}$$

is bounded (that is, $\sup\{\mu(x,y)<+\infty,\text{ for all }x,y\in G\}$). Then ω has at least one FP in F.

Definition 1.3. Let (Υ,μ) be a MS. A mapping $\omega:\Upsilon\longrightarrow\Upsilon$ is said to be weakly contractive, if for all $x,y\in\Upsilon$,

$$\mu(\omega x,\omega y)\leq\mu(x,y)-\vartheta(\mu(x,y)),$$

where $\vartheta:[0,+\infty)\longrightarrow[0,+\infty)$ is a continuous and non-decreasing function such that $\vartheta(0)=0$ and $\lim\limits_{t\longrightarrow+\infty}\vartheta(t)=+\infty$.

In 1997, Alber and Guerre [2] established that every weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhodes [16] showed that the corresponding result is also true when Hilbert space is replaced with a complete MS. Dutta et al. [9] improved the weak contractive condition and presented a FP theorem for a single-valued mapping which, in turn, extended the principal results in [2, 16].

2. Main results

We employ the fundamental concepts recorded in Section 1. Suppose that $\vartheta : [0,+\infty) \longrightarrow [0,+\infty)$ is a non-decreasing function such that

$$\vartheta(0) = 0 \text{ and } \lim_{t\longrightarrow+\infty} \vartheta(t) = +\infty.$$

Hereunder, we establish a FP result for a weakly type contractive operator which takes a closed subset into the ambient space.

Theorem 2.1. *Let F be a nonempty closed subset of Υ and let the mapping $\omega : F \longrightarrow \Upsilon$ satisfies*

$$\mu(\omega(x), \omega(y)) \leq \mu(x, y) - \vartheta(\mu(x, y)), \tag{2.1}$$

for each $x, y \in F$. Suppose further that for each integer $n \geq 1$,there exists a point $x_n \in F$ such that $\omega^n(x_n)$ exists and is contained in F, and the set

$$G = \{\omega^i(x_n) : n = 1, 2, \cdots, i \in \{0, \cdots, n\}\}$$

is bounded. Then there exists a point $u^* \in F$ such that $\omega(u^*) = u^*$. This invariant point is unique provided $\mu(x, y) < +\infty$.

Proof. Set $\omega^0(x) = x, x \in F$, and let

$$\Omega_0 = \sup\{\mu(y, z) : y, z \in G\}. \tag{2.2}$$

Observe that the assumption $\omega^n(x_n) \in F$ implies that the set G is well-defined. Let $\epsilon > 0$ be given, and pick a natural number k such that

$$k(\epsilon) \geq 1 + \Omega_0\vartheta(\epsilon)^{-1}. \tag{2.3}$$

Let $n_i, k_i, i = 1, 2$ be integers fulfilling

$$k(\epsilon) \leq k_i,\ i = 1, 2. \tag{2.4}$$

Obviously, $\omega^{k_i}(x_{n_i}), i = 1, 2$ are well-defined. By (2.4), we get

$$\mu(\omega^{k_1}(x_{n_1}), \omega^{k_2}(x_{n_2})) = \mu(\omega^{k(\epsilon)}(\omega^{k_1-k(\epsilon)}(x_{n_1})), \omega^{k_2-k(\epsilon)}(x_{n_2})). \tag{2.5}$$

Claim: We can find an integer $j \in \{0, \cdots, k(\epsilon)\}$ such that

$$\mu(\omega^j(\omega^{k_1-k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2-k(\epsilon)}(x_{n_2}))) \leq \epsilon. \tag{2.6}$$

Assume contrary that (2.6) does not hold. Then, for all $j \in \{0, \cdots, k(\epsilon) - 1\}$,

$$\mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) > \epsilon. \tag{2.7}$$

By (2.1), (2.2) and (2.7), we obtain

$$\begin{aligned}
&\mu(\omega^{j+1}(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^{j+1}(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) \\
\leq\ & \mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) \\
& - \vartheta\left(\mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2})))\right) \\
\leq\ & \mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) - \vartheta(\epsilon)
\end{aligned}$$

and

$$\mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) - \mu(\omega^{j+1}(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^{j+1}(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) \geq \vartheta(\epsilon). \tag{2.8}$$

From (2.2) and (2.8), we have

$$\begin{aligned}
\Omega_0 \ &\geq\ \mu\left(\omega^{k_1 - k(\epsilon)}(x_{n_1}), \omega^{k_2 - k(\epsilon)}(x_{n_2})\right) \\
&\geq\ \mu\left(\omega^{k_1 - k(\epsilon)}(x_{n_1}), \omega^{k_2 - k(\epsilon)}(x_{n_2})\right) - \mu\left(\omega^{k_1}(x_{n_1}), \omega^{k_2}(x_{n_2})\right) \\
&=\ \sum_{j=0}^{k(\epsilon)-1} [\mu(\omega^j(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^j(\omega^{k_2 - k(\epsilon)}(x_{n_2}))) \\
&\quad - \mu(\omega^{j+1}(\omega^{k_1 - k(\epsilon)}(x_{n_1})), \omega^{j+1}(\omega^{k_2 - k(\epsilon)}(x_{n_2})))] \\
&\geq\ \vartheta(\epsilon) k(\epsilon),
\end{aligned}$$

from which we have $k(\epsilon) \leq \Omega_0 \vartheta(\epsilon)^{-1}$, a contradiction to (2.3). Hence, we deduce that there exists $j \in \{0, \cdots, k(\epsilon)\}$ such that (2.6) is valid. Now, take

$$\vartheta(t) = \frac{t}{2} \text{ for all } t \geq 0. \tag{2.9}$$

Clearly, $\vartheta(0) = 0, \lim_{t \longrightarrow +\infty} \vartheta(t) = +\infty$ and $\vartheta(t)$ is non-decreasing for all $t \geq 0$. Therefore, (2.1) and (2.6) give

$$\mu(\omega^{k_1}(x_{n_1}), (\omega^{k_2}(x_{n_2}))) \leq \epsilon - \frac{\epsilon}{2} < \epsilon,$$

for all integers $k_i, n_i, i = 1, 2$, obeying $k(\epsilon) \leq k_i \leq n_i, i = 1, 2$.

For each integer $k \geq 1$, let G_k be the closure of the set

$$\{\omega^p(x_n) : n \geq k \text{ is an integer and } p \in \{k, \cdots, n\}\}. \tag{2.10}$$

We have demonstrated that the diameter of G_k (with respect to the metric $\tilde{\mu}$) approaches zero as $k \longrightarrow +\infty$. It follows that there exists $u^* \in F$ such that

$$\bigcap_{k=1}^{+\infty} G_k = \{u^*\}. \tag{2.11}$$

We now show that u^* is a FP of ω. For this, let $\epsilon > 0$ be chosen. Then, by (1.2), there exists $k \geq 1$ such that

$$G_k \times G_k \subset \mho(\epsilon). \tag{2.12}$$

Using (2.11) and (2.12),

$$\mu(u^*, z) \leq \epsilon, \ \forall z \in G_k. \tag{2.13}$$

Availing (2.10) and (2.13), yields

$$\mu(u^*, \omega^k(x_{k+1})) \leq \epsilon, \ \mu(u^*, \omega^{k+1}(x_{k+1})) \leq \epsilon. \tag{2.14}$$

Considering (2.1), (2.9) and (2.14), we get

$$\begin{aligned} \mu(\omega(u^*), \omega^{k+1}(x_{k+1})) &\leq \mu(u^*, \omega^k(x_{k+1})) - \vartheta(\mu(u^*, \omega^k(x_{k+1}))) \\ &\leq \epsilon - \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Whence,

$$\begin{aligned} \mu(u^*, \omega(u^*)) &\leq \Omega\left(\mu(u^*, \omega^{k+1}(x_{k+1})) + \mu(\omega^{k+1}(x_{k+1}), \omega(u^*))\right) \\ &\leq 2\epsilon\Omega. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we infer that $\omega(u^*) = u^*$.

By applying (2.9), if $u \neq u^*$ are two FPs of ω, then we realize that

$$\begin{aligned} \mu(u, u^*) &= \mu(\omega(u), \omega(u^*)) \leq \mu(u, u^*) - \vartheta(\mu(u, u^*)) \\ &\leq \frac{\mu(u, u^*)}{2}, \end{aligned}$$

is a contradiction. Consequently, u^* is the unique FP of ω provided $\mu(u, u^*) < +\infty$ for all $u, u^* \in F$. □

In the study of weak contractions, the following observation is common.

Corollary 1. *Let F be a nonempty closed subset of Υ and assume that there exists $\lambda \in [0, 1)$ such that the mapping $\omega : F \longrightarrow \Upsilon$ satisfies*

$$\mu(\omega(x), \omega(y)) \leq \lambda\mu(x, y), \tag{2.15}$$

for each $x, y \in F$. Suppose further that for each integer $n \geq 1$, there exists a point $x_n \in F$ such that $\omega^n(x_n)$ exists and is contained in F, and the set

$$G = \{\omega^i(x_n) : n = 1, 2, \cdots, i \in \{0, \cdots, n\}\}$$

is bounded. Then ω has a FP in F. Moreover, this FP is unique provided $\mu(x, y) < +\infty$.

Proof. Take $\vartheta(t) = (1 - \lambda)t$ for all $t \geq 0$ with $\lambda \in [0, 1)$ in Theorem 2.1. □

In the following, we provide a comparative example to support the hypotheses of Theorem 2.1.

Example 2.2. Let

$$\Upsilon = \left\{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{2}\right\} \bigcup \left\{x \in \mathbb{R} : \frac{1}{2} < x \leq 1\right\}$$

and $F = \Upsilon \setminus \left(\frac{1}{2}, 1\right]$. Suppose that $\mu(x, y) = \frac{\eta(x-y)}{\zeta^2}$, for all $x, y \in \Upsilon$, where $\zeta \in (0, 1)$ and the modular η obeys the Δ_2-condition with the constant ζ. Define the mappings $\omega : F \longrightarrow \Upsilon$ and $\vartheta : [0, +\infty) \longrightarrow [0, +\infty)$, respectively as follows: $\omega x = x^2$, for all $x \in F$, and $\vartheta(t) = (1 - \zeta)t$ for all $t \geq 0$. Clearly, $0 = \vartheta(0)$ and $\lim_{t \longrightarrow +\infty} \vartheta(t) = +\infty$. Note that if $x = 0$, the validity of (2.1) holds trivially. So, for $0 < x \leq \frac{1}{2}$, we have

$$\begin{aligned}
\mu(\omega x, \omega y) &= \frac{1}{\zeta^2}[\eta(\omega x - \omega y)] = \frac{1}{\zeta^2}[\eta(x^2 - y^2)] \\
&= \frac{1}{\zeta^2}[\eta((x - y)(x + y))] \leq \frac{1}{\zeta^2}[\eta(x - y)] \\
&\leq \frac{1}{\zeta^2}[\eta(2(2^{-1}(x - y) + 2^{-1}(y - y)))] \\
&\leq \frac{\zeta}{\zeta^2}[\eta(2^{-1}(x - y) + 2^{-1}(y - y))] \\
&\leq \frac{1}{\zeta}\eta(x - y) = \zeta\mu(x, y) \\
&= \mu(x, y) - (1 - \zeta)\mu(x, y) \\
&= \mu(x, y) - \vartheta(\mu(x, y)).
\end{aligned}$$

That is, Eq (2.1) holds good. It is now easy to see that all the conditions of Theorem 2.1 and Corollary 1 are satisfied. Therefore, there exists a unique $u^* = 0 \in F$ such that $\omega(0) = 0$.

We observe that in Corollary 1, the mapping ω is not a contraction mapping on Υ, in the sense of Banach [3], since with the Euclidean metric μ on $\left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
\mu(\omega x, \omega y) &= |x^2 - y^2| = |(x - y)(x + y)| \\
&= |x - y| > \lambda\mu(x, y), \ \forall \lambda \in [0, 1).
\end{aligned}$$

We notice also that Theorems 2.1 and 1.2 do not coincide, since, by our definition, there exists some $t_* > 0$ such that $\vartheta(t_*) > 1$.

3. Conclusions

In this article, a fixed point result (Theorem 2.1) for weakly contractive type operators defined on a closed subset of a generalized MS has been discussed. A comparative illustration (Example 2.2) is set up to support our assumptions, and to indicate that Theorem 2.1 does not coincide with its analogues in the existing literature. The obtained result herein can be investigated and advanced via other contractions such as Reich, Meir-Keeler, Caristi, Kannan, Chatterjea, Zamfirescu, Hardy-Rogers, Pre*šić* contractions, to mention but a few. In addition, the contractive inequality (2.1) can be employed to examine new criteria for existence of solutions to integral/differential equations of either integer or non-integer orders.

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Conflict of interest

The authors declare that there is no competing interest.

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