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Research article

Existence of solutions for the boundary value problem of non-instantaneous impulsive fractional differential equations with p-Laplacian operator

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Abstract: In this paper, we consider a boundary value problem of impulsive fractional differential equations with the nonlinear p-Laplacian operator, where impulses are non-instantaneous. By converting the given problem into an equivalent integral form and applying the Schauder fixed point theorem, we obtain some sufficient conditions for the existence of solutions. An illustrative example is presented to demonstrate the validity of our results.

Keywords: fractional differential equations; non-instantaneous impulsive; p-Laplacian operator; fixed point theorem
Mathematics Subject Classification: 34A08, 34B37

1. Introduction

Fractional differential equations are considered as a generalization of ordinary differential equations and have attracted growing attention recently. The theory of fractional differential equations is generally considered to provide valuable tools for mathematical modeling of many real-world phenomena in physics [1, 2, 3, 4], polymer rheology [5], regular variation in thermodynamics, control theory, biology [6], blood flow phenomena, viscoelasticity, and damping. The main advantage of fractional derivatives is that they can describe the properties of heredity and memory of many materials. The fractional derivative of order α with $1 < \alpha \leq 2$ appears in several diffusion problems used in physical and engineering applications, for example, in the mechanism of superdiffusion [7]. Therefore, the research on the existence and controllability of the fractional evolution system with order $1 < \alpha \leq 2$ has become a hot research topic recently. We refer [8, 9, 10, 11, 12, 13] and the references cited therein for recent development on this topic.

Note that differential equations with p-Laplacian operator have been widely applied in many scientific fields including dynamical systems, and mathematical models of mechanics. Turbulent flow in a porous medium is a fundamental mechanics problem. To study this type of problems, Leibenson

[14] introduced the p-Laplacian equation as follows:

$$(\phi_p(u'(t)))' = f(t, x(t), u'(t)),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$. Recently, the existence of solutions to boundary value problems for the fractional differential equations with the p-Laplacian operator have received considerable attention [15, 16, 17]. Liu and Jia [18] considered the following fractional differential equation involving the p-Laplacian operator:

$$\begin{cases} (\phi_p({}^cD_{0^+}^{\alpha}u(t)))' = f(t, u(t)),\ t \in (0, 1), \\ u(0) = r_0 u(1),\ u'(0) = r_1 u'(1), \\ u^{(i)}(0) = 0,\ i = 1, 2, \ldots, [\alpha] - 1, \end{cases}$$

where ${}^cD_{0^+}^{\alpha}$ is the standard Caputo derivative, $1 < \alpha \in \mathbb{R}$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$. The existence of solutions of boundary value problem for the p-Laplacian operator equations was obtained by means of the Banach contraction mapping principle.

Impulse is a universal phenomenon in human social activities and nature. According to the duration of the change process, the impulse can be divided into instantaneous impulse and non-instantaneous impulse. The instantaneous impulse was first proposed by V. Milman et al. [19]. Instantaneous impulse effects appear in many practical systems and applications, for example, pharmacology and automatic control systems. However, not all the phenomena in real life such as the drug injection process could be described by instantaneous impulses. Therefore, the concept of non-instantaneous impulses was first proposed by Hernández and O'Regan [20]. This type of impulse is more advantageous in modeling the dynamics of evolution processes, and differential equations involving such have received extensive attention [21, 22, 23, 24, 25].

Agarwal et al. [26] investigated Caputo fractional differential equations with non-instantaneous impulses:

$$\begin{cases} {}_0^cD^q u(t) = f(t, u(t)), & t \in (t_k, s_k],\ k = 0, 1, \ldots, m+1, \\ u(t) = \phi_k(t, u(t), u(s_k - 0)), & t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases}$$

where u, $u_0 \in \mathbb{R}$, $f : \cup_{k=0}^{m+1}[t_k, s_k \times \mathbb{R} \to \mathbb{R}]$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(k = 0, 1, 2, \ldots, m)$. By using the iterative techniques combined with lower and upper solutions methods, some sufficient conditions for the existence of positive solutions to the above initial value problems were obtained.

Motivated by the work above, in this work, we consider the boundary value problem of non-instantaneous impulsive fractional differential equations with the p-Laplacian operator:

$$\begin{cases} (\phi_p({}^cD_{0^+}^{\alpha}u(t)))' = f(t, u(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, m, \\ u(t) = g_k(t, u(t)),\ u'(t) = h_k(t, u(t)), & t \in (t_k, s_k],\ k = 1, 2, \ldots, m, \\ u(0) = u'(0) = 0,\ \phi_p({}^cD_{0^+}^{\alpha}u(s_k)) = 0, & k = 0, 1, \ldots, m, \end{cases} \tag{1.1}$$

where ${}^cD_{0^+}^{\alpha}$ denotes the Caputo fractional derivative, $1 < \alpha \le 2$, $J = [0, 1]$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = 1$ are pre-fixed numbers. The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_k, h_k : [t_k, s_k] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $k = 1, 2, \ldots, m$.

We point out, comparing the earlier work [3, 18, 24, 25, 26], our major contributions are:

i) We propose a new model based on [18, 25] and our model involves both the non-instantaneous impulsive effect, fractional order derivative, and the derivative term of p-Laplacian operator, while the models in [3, 18] do not consider non-instantaneous impulsive effects.

ii) Motivated by [24, 26], for the nonlinear system with Caputo fractional derivative of order $\alpha \in (1, 2]$ and non-instantaneous impulsive, we properly define the boundary value conditions and non-instantaneous impulsive conditions.

iii) We establish an integral equation equivalent to the problem (1.1) and develop new techniques different from [3, 18, 24, 25, 26] to obtain new results on the existence of solutions to non-instantaneous impulsive fractional differential equations with the p-Laplacian operator.

iv) The novelty of this work lies in the study of the class of non-instantaneous impulsive fractional order differential equations with the p-Laplacian operator and especially in introducing a non-instantaneous impulsive fractional differential system of order $\alpha \in (1, 2]$.

The rest of this paper is organized as follows. In Section 2, we give some basic definitions and fractional calculus. In Section 3, we use Schauder fixed point theorem to prove the existence results for the solutions to the above-proposed problem (1.1). In Section 4, as an application we give and analyze an important illustrative example. The conclusion is given in the last section.

2. Preliminaries

In this section, we present some useful lemmas, theorems, and some definitions, which are important in obtaining the existence of solutions.

Consider the piecewise continuous space $PC(J, \mathbb{R}) := \{u : J \to \mathbb{R} : u \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \cdots, m$ and there exist $u(t_k^-)$ and $u(t_k^+)$, $k = 1, 2, \cdots, m$, with $u(t_k^-) = u(t_k)\big\}$ with the norm $\|u\|_{PC} := \max\{|u(t)| : t \in J\}$. Obviously $(PC(J, \mathbb{R}), \|\cdot\|_{PC})$ is a Banach space. Set $PC^1(J, \mathbb{R}) := \{u \in PC(J, \mathbb{R}) : u' \in PC(J, \mathbb{R})\}$ with the norm $\|u\|_{PC^1} := \max\{\|u\|_{PC}, \|u'\|_{PC}\}$. Then, $(PC^1(J, \mathbb{R}), \|\cdot\|_{PC^1})$ is also a Banach space.

Definition 2.1. ([2]) *The fractional integral of order $\alpha (\alpha > 0)$ of function $u \in L^1([0, 1], \mathbb{R})$ is given by*

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) ds,$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. ([2]) *The Caputo fractional derivative of order $\alpha (\alpha > 0)$ of function $u \in C^n([0, 1], \mathbb{R})$ is given by*

$${}^c D^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} u^{(n)}(s) ds,$$

where $t > 0$, $n = [\alpha] + 1$, $\Gamma(\alpha)$ is the Gamma function.

Lemma 2.3. ([3]) *For $\alpha > 0$ and $u \in C(0, 1) \cap L^1(0, 1)$, the solution of fractional differential equation ${}^c D_{0^+}^{\alpha} u(t) = 0$ is given by $u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n - 1$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α.*

Lemma 2.4. ([2, 4]) *For $\alpha > 0$ and $u \in C(0, 1) \cap L^1(0, 1)$, then (i) $I_{0^+}^{\alpha} ({}^c D_{0^+}^{\alpha} u(t)) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $n = [\alpha] + 1$, (ii) ${}^c D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$.*

Lemma 2.5. ([20]) *Let ϕ_p be a p-Laplacian operator. Then*

(i) for $1 < p \leq 2$, $uv > 0$, and $|u|, |v| \geq \rho > 0$, then

$$\left|\phi_p(u)-\phi_p(v)\right| \le (p-1)\rho^{p-2}\,|u-v|.$$

(*ii*) *If $p>2$, and $|u|,|v|\le\rho^*$, then*

$$\left|\phi_p(u)-\phi_p(v)\right| \le (p-1)\rho^{*\,p-2}\,|u-v|.$$

Theorem 2.6. ([27])*(Schauder fixed point theorem) Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $T:\Omega\to\Omega$ be a continuous mapping such that the set $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω. That is, there exists an $x\in\Omega$ such that $Tx=x$.*

Lemma 2.7. *Let $y:[0,1]\to\mathbb{R}$ and $G_k,H_k:[t_k,s_k]\to\mathbb{R}$ are continuous functions. A function $u\in PC^1(J,\mathbb{R})$ is a solution of the impulsive problem*

$$\begin{cases}(\phi_p({}^cD^\alpha_{0^+}u(t)))'=y(t), & t\in(s_k,t_{k+1}],k=0,1,\dots,m,\\ u(t)=G_k(t),u'(t)=H_k(t), & t\in(t_k,s_k],k=1,2,\dots,m,\\ u(0)=u'(0)=0,\ \phi_p({}^cD^\alpha_{0^+}u(s_k))=0, & k=0,1,\dots,m,\end{cases} \tag{2.1}$$

if and only if u satisfies the following integral equation

$$u(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\int_0^s y(\tau)d\tau)ds, & t\in[0,t_1],\\ G_k(t), & t\in(t_k,s_k],k=1,2,\dots,m,\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds & \\ -\frac{1}{\Gamma(\alpha)}\int_0^{s_k}(s_k-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds & \\ +\frac{s_k}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds & \\ -\frac{t}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds & \\ +G_k(s_k)+H_k(s_k)(t-s_k), & t\in(s_k,t_{k+1}],k=1,2,\dots,m.\end{cases} \tag{2.2}$$

Proof. Assume that u satisfies (2.1). For $t\in[0,t_1]$, integrating the first equation of (2.1) from zero to t, we obtain $\phi_p({}^cD^\alpha_{0^+}u(t))-\phi_p({}^cD^\alpha_{0^+}u(0))=\int_0^t y(s)ds$. Using the boundary condition $\phi_p({}^cD^\alpha_{0^+}u(0))=0$, we have $\phi_p({}^cD^\alpha_{0^+}u(t))=\int_0^t y(s)ds$.

Applying $\phi_p^{-1}=\phi_q$ on both sides of the equation $\phi_p({}^cD^\alpha_{0^+}u(t))=\int_0^t y(s)ds$, we get

$${}^cD^\alpha_{0^+}u(t)=\phi_q(\int_0^t y(s)ds).$$

For $t\in[0,t_1]$, by Lemma 2.4, we have $u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\int_0^s y(\tau)d\tau)ds+a_0+b_0t$. Then, $u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\phi_q(\int_0^s y(\tau)d\tau)ds+b_0$.

Condition $u(0)=0$, $u'(0)=0$ further implies $a_0=0$ and $b_0=0$. Thus

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\int_0^s y(\tau)d\tau)ds.$$

For $t\in(t_k,s_k],\ k=1,2,\dots,m$, applying the impulsive condition of (2.1), we find that

$$u(t)=G_k(t).$$

In addition, for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, m$, integrating both sides from s_k to t, we obtain $\phi_p({}^cD_{0^+}^\alpha u(t)) - \phi_p({}^cD_{0^+}^\alpha u(s_k)) = \int_{s_k}^t y(s)ds$. Using the condition $\phi_p({}^cD_{0^+}^\alpha u(s_k)) = 0$, we get $\phi_p({}^cD_{0^+}^\alpha u(t)) = \int_{s_k}^t y(s)ds$.

Applying $\phi_p^{-1} = \phi_q$ on both sides of the equation $\phi_p({}^cD_{0^+}^\alpha u(t)) = \int_{s_k}^t y(s)ds$, we have

$$ {}^cD_{0^+}^\alpha u(t) = \phi_q(\int_{s_k}^t y(s)ds). $$

By (i) of Lemma 2.4, we have $u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds + a_k + b_k t$. Then, $u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds + b_k$. Substituting $u(s_k)$, $u'(s_k)$ into the second equation of (2.1), we obtain

$$ \begin{aligned} a_k = {} & G_k(s_k) - H_k(s_k)s_k - \frac{1}{\Gamma(\alpha)}\int_0^{s_k}(s_k-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds \\ & + \frac{s_k}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds, \\ b_k = {} & H_k(s_k) - \frac{1}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds. \end{aligned} $$

Consequently, we have

$$ \begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds - \frac{1}{\Gamma(\alpha)}\int_0^{s_k}(s_k-s)^{\alpha-1}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds \\ & + \frac{s_k}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s y(\tau)d\tau)ds - \frac{t}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2} \\ & \times \phi_q(\int_{s_k}^s y(\tau)d\tau)ds + G_k(s_k) + H_k(s_k)(t-s_k). \end{aligned} $$

This accomplish the proof. □

3. Main results

We first pose the following assumptions:

(H_1) There exist a function $\varpi \in L^1([s_k, t_{k+1}], \mathbb{R}^+)$, and a non-decreasing function $\nu \in C([0,\infty), \mathbb{R}^+)$ such that $\| f(t,u) \| \leq \varpi(t)\nu(\| u \|_{PC})$, for each $t \in [s_k, t_{k+1}]$, and all $u \in \mathbb{R}$.

(H_2) There exist a function $\psi_k \in C([t_k, s_k], \mathbb{R}^+)$, and a non-decreasing function $\eta \in C([0,\infty), \mathbb{R}^+)$ such that $\| g_k(t,u) \| \leq \psi_k(t)\eta(\| u \|_{PC})$, for each $t \in [t_k, s_k]$, and all $u \in \mathbb{R}$.

(H_3) There exist a function $\varphi_k \in C([t_k, s_k], \mathbb{R}^+)$, and a non-decreasing function $\theta \in C([0,\infty), \mathbb{R}^+)$ such that $\| h_k(t,u) \| \leq \varphi_k(t)\theta(\| u \|_{PC})$, for each $t \in [t_k, s_k]$, and all $u \in \mathbb{R}$.

Define an operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as follows:

$$Tu(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_q(\int_0^s f(\tau,u(\tau))d\tau)ds, & t \in [0,t_1], \\ g_k(t,u(t)), & t \in (t_k,s_k], k=1,2,\cdots,m, \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)ds & \\ -\frac{1}{\Gamma(\alpha)} \int_0^{s_k} (s_k-s)^{\alpha-1}\phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)ds & \\ +\frac{s_k}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)ds & \\ -\frac{t}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)ds & \\ +g_k(s_k,u(s_k)) + h_k(s_k,u(s_k))(t-s_k), & t \in (s_k,t_{k+1}], k=1,2,\dots,m. \end{cases} \tag{3.1}$$

Let $M = \max_{k=1,2,\dots,m}\psi_k(s_k)$, $N = \max_{k=1,2,\dots,m}\varphi_k(s_k)$, $K = \int_0^1 \varpi(t)dt = \|\varpi\|_{L^1_{[0,1]}}$ and $L_k = s_k^{\alpha-1}((\alpha+1)s_k + \alpha t_{k+1}) + t_{k+1}^{\alpha}(k=1,2,\dots,m)$. Now we are ready to state the following results.

Theorem 3.1. *Assume that $(H_1)-(H_3)$ are satisfied. Setting $B_r = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC} \le r\}$, where*

$$r \ge \max\left\{\frac{\phi_q\big(K\nu(r)\big)t_1^{\alpha}}{\Gamma(\alpha+1)},\ \frac{L_k\phi_q\big(K\nu(r)\big)}{\Gamma(\alpha+1)} + M\eta(r) + N\theta(r)(t_{k+1}+s_k)\right\},$$

then the problem (1.1) has at least one solution.

Proof. The proof is based on the steps given below:

Step 1. We show that $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $PC(J,\mathbb{R})$.

Case1. For $t \in [0,t_1]$, we have

$$\begin{aligned} \|(Tu_n)(t) - (Tu)(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\|\phi_q(\int_0^s f(\tau,u_n(\tau))d\tau) - \phi_q(\int_0^s f(\tau,u(\tau))d\tau)\right\| ds \\ &\le \frac{(q-1)\rho^{q-2}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\int_0^s \|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\| d\tau)ds. \end{aligned}$$

Case2. For $t \in (t_k,s_k],\ k=1,2,\dots,m$, we have

$$\|(Tu_n)(t) - (Tu)(t)\| \le \|g_k(\cdot,u_n(\cdot)) - g_k(\cdot,u(\cdot))\|.$$

Case3. For $t \in (s_k,t_{k+1}],\ k=1,2,\dots,m$, we get

$$\begin{aligned} &\|(Tu_n)(t) - (Tu)(t)\| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\|\phi_q(\int_{s_k}^s f(\tau,u_n(\tau))d\tau) - \phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)\right\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{s_k} (s_k-s)^{\alpha-1} \left\|\phi_q(\int_{s_k}^s f(\tau,u_n(\tau))d\tau) - \phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)\right\| ds \\ &\quad + \frac{s_k}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k-s)^{\alpha-2} \left\|\phi_q(\int_{s_k}^s f(\tau,u_n(\tau))d\tau) - \phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)\right\| ds \\ &\quad + \frac{t}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k-s)^{\alpha-2} \left\|\phi_q(\int_{s_k}^s f(\tau,u_n(\tau))d\tau) - \phi_q(\int_{s_k}^s f(\tau,u(\tau))d\tau)\right\| ds \end{aligned}$$

$$
\begin{aligned}
&+ \|g_k(s_k, u_n(s_k)) - g_k(s_k, u(s_k)\| + \|h_k(s_k, u_n(s_k)) - h_k(s_k, u(s_k))\| (t + s_k) \\
&\le \frac{(q-1)\rho^{q-2}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\int_{s_k}^s \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\| d\tau) ds \\
&+ \frac{(q-1)\rho^{q-2}}{\Gamma(\alpha)} \int_0^{s_k} (s_k - s)^{\alpha-1} (\int_{s_k}^s \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\| d\tau) ds \\
&+ \frac{s_k(q-1)\rho^{q-2}}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k - s)^{\alpha-2} (\int_{s_k}^s \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\| d\tau) ds \\
&+ \frac{t_{k+1}(q-1)\rho^{q-2}}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k - s)^{\alpha-2} (\int_{s_k}^s \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\| d\tau) ds \\
&+ \|g_k(\cdot, u_n(\cdot)) - g_k(\cdot, u(\cdot)\| + \|h_k(\cdot, u_n(\cdot)) - h_k(\cdot, u(\cdot))\| (t_{k+1} + s_k).
\end{aligned}
$$

Thus, we conclude from the above cases that $\|Tu_n - Tu\|_{PC} \to 0$, as $n \to \infty$. This implies that T is continuous.

Step 2. The operator T is uniformly bounded.

Case 1. For $u \in B_r$ and when $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|Tu(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(\int_0^s \|f(\tau, u(\tau))\| d\tau) ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(\int_0^s \varpi(\tau) \nu(\|u\|_{PC}) d\tau) ds \\
&\le \frac{\phi_q(K\nu(r)) t_1^\alpha}{\Gamma(\alpha+1)} \le r.
\end{aligned}
$$

Case 2. For $u \in B_r$ and when $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we obtain

$$
\|Tu(t)\| \le \|g_k(t, u(t))\| \le \psi_k(s_k)\eta(r) \le r.
$$

Case 3. For $u \in B_r$ and when $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we get

$$
\begin{aligned}
\|Tu(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(\int_{s_k}^s \|f(\tau, u(\tau))\| d\tau) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{s_k} (s_k - s)^{\alpha-1} \phi_q(\int_{s_k}^s \|f(\tau, u(\tau))\| d\tau) ds \\
&+ \frac{s_k}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k - s)^{\alpha-2} \phi_q(\int_{s_k}^s \|f(\tau, u(\tau))\| d\tau) ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_0^{s_k} (s_k - s)^{\alpha-2} \phi_q(\int_{s_k}^s \|f(\tau, u(\tau))\| d\tau) ds \\
&+ \|g_k(s_k, u(s_k))\| + \|h_k(s_k, u(s_k))\| (t + s_k) \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q(\int_{s_k}^s \varpi(\tau) \nu(\|u\|_{PC}) d\tau) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{s_k} (s_k - s)^{\alpha-1} \phi_q(\int_{s_k}^s \varpi(\tau) \nu(\|u\|_{PC}) d\tau) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{s_k}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^{s}\varpi(\tau)\nu(\|u\|_{PC})d\tau)ds\\
&+\frac{t}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^{s}\varpi(\tau)\nu(\|u\|_{PC})d\tau)ds\\
&+\psi_k(s_k)\eta(r)+\varphi_k(s_k)\theta(r)(t_{k+1}+s_k)\\
\le&\frac{\phi_q(K\nu(r))t_{k+1}^{\alpha}}{\Gamma(\alpha+1)}+\frac{\phi_q(K\nu(r))s_k^{\alpha}}{\Gamma(\alpha+1)}+\frac{\phi_q(K\nu(r))s_k^{\alpha}}{\Gamma(\alpha)}+\frac{\phi_q(K\nu(r))s_k^{\alpha-1}t_{k+1}}{\Gamma(\alpha)}\\
&+\psi_k(s_k)\eta(r)+\varphi_k(s_k)\theta(r)(t_{k+1}+s_k)\\
\le&\frac{L_k\phi_q(K\nu(r))}{\Gamma(\alpha+1)}+M\eta(r)+N\theta(r)(t_{k+1}+s_k)\le r.
\end{aligned}
$$

Therefore, $T(B_r)$ is uniformly bounded.

Step 3. T maps a bounded set into an equicontinuous set of $B(r)$.

Case 1. For $e_1, e_2 \in [0, t_1]$, with $e_1 < e_2,\ e_1 < \xi < e_2,\ u \in B(r)$, we get

$$
\begin{aligned}
\|(Tu)(e_2)-(Tu)(e_1)\|\le&\Big\|\frac{1}{\Gamma(\alpha)}\int_0^{e_1}(e_2-s)^{\alpha-1}\phi_q(\int_0^{s}f(\tau,u(\tau))d\tau)ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{e_1}(e_1-s)^{\alpha-1}\phi_q(\int_0^{s}f(\tau,u(\tau))d\tau)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{e_1}^{e_2}(e_2-s)^{\alpha-1}\phi_q(\int_0^{s}f(\tau,u(\tau))d\tau)ds\Big\|\\
\le&\frac{1}{\Gamma(\alpha)}\int_0^{e_1}\left\|(e_2-s)^{\alpha-1}-(e_1-s)^{\alpha-1}\right\|\phi_q(\int_0^{s}\|f(\tau,u(\tau))\|d\tau)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{e_1}^{e_2}(e_2-s)^{\alpha-1}\phi_q(\int_0^{s}\|f(\tau,u(\tau))\|d\tau)ds\\
\le&\frac{\phi_q(K\nu(r))}{\Gamma(\alpha+1)}(e_2^{\alpha}-e_1^{\alpha})\le\frac{\phi_q(K\nu(r))\alpha\xi^{\alpha-1}}{\Gamma(\alpha+1)}(e_2-e_1).
\end{aligned}
$$

Case 2. For $e_1, e_2 \in (t_k, s_k],\ k = 1, 2, \ldots, m$, with $e_1 < e_2,\ u \in B(r)$, we have

$$
\|(Tu)(e_2)-(Tu)(e_1)\|=\|g_k(e_2,u(e_2))-g_k(e_1,u(e_1))\|\to 0\ as\ e_1\to e_2.
$$

Case 3. For $e_1, e_2 \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, m$, with $e_1 < e_2,\ e_1 < \xi < e_2,\ u \in B(r)$, we have

$$
\begin{aligned}
\|(Tu)(e_2)-(Tu)(e_1)\|\le&\Big\|\frac{1}{\Gamma(\alpha)}\int_0^{e_1}((e_2-s)^{\alpha-1}-(e_1-s)^{\alpha-1})\phi_q(\int_{s_k}^{s}f(\tau,u(\tau))d\tau)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{e_1}^{e_2}(e_2-s)^{\alpha-1}\phi_q(\int_{s_k}^{s}f(\tau,u(\tau))d\tau)ds\Big\|\\
&+\frac{(e_2-e_1)}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}\phi_q(\int_{s_k}^{s}\|f(\tau,u(\tau))\|d\tau)ds\\
&+\|h_k(s_k,u(s_k))\|(e_2-e_1)\\
\le&\frac{\phi_q(K\nu(r))}{\Gamma(\alpha+1)}(e_2^{\alpha}-e_1^{\alpha})+\frac{s_k^{\alpha-1}\phi_q(K\nu(r))(e_2-e_1)}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \varphi_k(s_k)\theta(r)(e_2 - e_1) \\
&\leq [\frac{\phi_q(K\nu(r))\alpha(\xi^{\alpha-1} + s_k^{\alpha-1})}{\Gamma(\alpha+1)} + N\theta(r)](e_2 - e_1).
\end{aligned}
$$

As $e_2 \to e_1$, the right-hand side of the above inequality is independent of u and tends to zero, which means that $T(B_r)$ is equicontinuous. From Arzela-Ascoli theorem, T is compact. By using Schauder fixed point theorem, T has at least one fixed point $u \in B_r$. Therefore, the problem (1.1) has at least one positive solution u in B_r. □

4. An example

In this section, we give an example to illustrate our main results.

Example 4.1 Consider the following equation

$$
\begin{cases}
(\phi_{\frac{3}{2}}({}^cD_{0^+}^{\frac{4}{3}}u(t)))' = \frac{|u(t)|}{(1+e^t)(1+|u(t)|)}, & t \in (0, \frac{1}{4}] \cup (\frac{1}{2}, \frac{3}{4}], \\
u(t) = \frac{|u(t)|}{(6+16t^2)(1+|u(t)|)}, u'(t) = \frac{|u(t)|}{(8+4t)(1+|u(t)|)}, & t \in (\frac{1}{4}, \frac{1}{2}], \\
u(0) = u'(0) = 0, \ \phi_p({}^cD_{0^+}^{\alpha}u(s_k)) = 0, & k = 0, 1, 2,
\end{cases}
\tag{4.1}
$$

where $\alpha = \frac{4}{3},\ p = \frac{3}{2},\ q = 3,\ J = [0, 1]$ and $0 = s_0 < t_1 = \frac{1}{4} < s_1 = \frac{1}{2} < t_2 = \frac{3}{4} = s_2 < t_3 = 1, m = 2$.

Set $f(t, u(t)) = \frac{|u(t)|}{(1+e^t)(1+|u(t)|)},\ g_1(t, u(t)) = \frac{|u(t)|}{(6+16t^2)(1+|u(t)|)},\ h_1(t, u(t)) = \frac{|u(t)|}{(8+4t)(1+|u(t)|)}$.

For all $u \in \mathbb{R}$ and each $t \in (0, \frac{1}{4}] \cup (\frac{1}{2}, \frac{3}{4}]$, then we have $\|f(t, u)\| \leq \frac{1}{2} \parallel u \parallel_{PC},\ \varpi(t) = \frac{1}{2},\ \nu(t) = \sqrt{t}$. For all $u \in \mathbb{R}$ and each $t \in (\frac{1}{4}, \frac{1}{2}]$, we have $\|g_1(t, u)\| \leq \frac{1}{7} \parallel u \parallel_{PC},\ \|h_1(t, u)\| \leq \frac{1}{9} \parallel u \parallel_{PC},\ \psi_1(t) = \frac{1}{7} := M,\ \varphi_1(t) = \frac{1}{9} := N,\ \eta(t) = \theta(t) = t$.

Obviously, the inequality $r \geq \max\left\{\frac{\phi_q(K\nu(r))t_1^{\alpha}}{\Gamma(\alpha+1)}, \frac{L_1\phi_q(K\nu(r))}{\Gamma(\alpha+1)} + M\eta(r) + N\theta(r)(t_2 + s_1)\right\}$ reduces to $r \geq \max\{0.0331r,\ 0.7976r\} = 0.7976r$, which holds for every $r > 0$.

Thus, all the assumptions in Theorem 3.1 are satisfied, our results can be applied to the problem (1.1).

5. Conclusions

In this paper, we studied the existence of solutions for multipoint boundary value problems of impulsive fractional differential equations with p-Laplacian operator. By using fixed point theorem, we obtained some sufficient conditions for the existence of a solution for the boundary value problem of non-instantaneous impulsive fractional differential equations with p-Laplacian operator and Caputo derivative of order $\alpha \in (1, 2]$. We also presented an example to illustrate that our established results are applicable. In future studies, the concept can be applied to highly complex problems, including solving high order non-instantaneous impulsive differential equations.

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Conflict of interest

The authors declare that they have no competing interests.

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