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Research article

A multiplicity result for double phase problem in the whole space

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Abstract: In the present paper, we discuss the solutions of the following double phase problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + |u|^{p-2}u + \mu(x)|u|^{q-2}u = f(x,u),\ x \in \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < q < N$ and $0 \leq \mu \in C^{0,\alpha}(\mathbb{R}^N)$, $\alpha \in (0,1]$. Based on the theory of the double phase Sobolev spaces $W^{1,H}(\mathbb{R}^N)$, we prove the existence of at least two non-trivial weak solutions.

Keywords: double phase operator; Musielak-Orlicz-Sobolev space; critical point
Mathematics Subject Classification: 35D30, 35J20, 35J60

1. Introduction

In recent years, the differential equations and variational problems driven by the so-called double phase operator have been greatly studied. The existence of solutions for double phase problems on bounded domains have been greatly discussed, see for example [1–9]. For unbounded domains, Liu and Dai [10], Liu and Winkert [11], Robert [12], Ge and Pucci [13] and Shen, Wang, Chi and Ge [14] investigated the existence and multiplicity of solutions for double phase problem.

In this paper we study the following double phase problem:

$$-\text{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + |u|^{p-2}u + \mu(x)|u|^{q-2}u = f(x,u),\ x \in \mathbb{R}^N, \tag{P}$$

where $1 < p < q < N$ and

$$\frac{q}{p} \leq 1 + \frac{\alpha}{N},\ 0 \leq \mu \in C^{0,\alpha}(\mathbb{R}^N),\ \alpha \in (0,1]. \tag{1.1}$$

The first work concerning the ground state solution for problem (P), was that of Liu and Dai [10]. More specifically, they studied the existence of at least three nontrivial solutions of (P) under the following assumption on f:

(h_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $\gamma \in (q, p^*)$ such that

$$|f(x,t)| \leq k(x)|t|^{\gamma-1}, \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $p^* = \frac{Np}{N-p}$, $k(x) \geq 0$, $k \in L^{\theta}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $\frac{1}{\theta} + \frac{\gamma}{\tau} = 1$, here $\theta > 1$ and $\tau \in (\gamma, p^*]$.

(h_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0$ uniformly in x.

(h_3) $\lim\limits_{t\to +\infty} \frac{F(x,t)}{|t|^{q}} = +\infty$ uniformly in x.

(h_4) $\frac{f(x,t)}{|t|^{q-1}}$ is strictly increasing on $(-\infty, 0)$ and $(0, +\infty)$.

It must be point out that (h_1) is subcritical growth condition, (h_3) means that $f(x,u)$ is superlinear at infinity; (h_4) is a well-known Nehari-type condition. In the present paper, we will further study the existence of two non-trivial weak solutions of (P) under the following sublinear growth condition:

$(h_1)'$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $\gamma \in (1, p)$ such that

$$|f(x,t)| \leq k(x)|t|^{\gamma-1}, \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $k(x) \geq 0$, $k \in L^{\theta}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $\frac{1}{\theta} + \frac{\gamma}{p^*} = 1$.

(h_5) There exists a $C > 1$ large enough, $c_0 > 0$, $x_0 \in \mathbb{R}^N$, $0 < r < 1$ such that $f(x,t) = 0$, for any $x \in \mathbb{R}^N$, $0 < |t| \leq \delta$ and

$$f(x,t) \geq c_0|t-\delta|^{\gamma-1}, \ \forall x \in B_r(x_0), t \in (\delta, 1],$$

where $0 < \delta < \min\left\{\frac{1}{2}\left(\frac{c_0 p r^q}{\gamma 2^{\gamma+1}(C^q + r^q) m_\mu}\right)^{\frac{1}{p-\gamma}}, \frac{1}{2}\right\}$ and $m_\mu = \max\{1, \sup\limits_{x \in B_{2r}(x_0)} \mu(x)\}$.

Remark 1.1. There are many functions $f(x,t)$ satisfying $(h_1)'$ and (h_5). For example,

$$f(x,t) = \begin{cases} 0, & \text{if } 0 \leq |t| < \delta, \\ k_1(x)(t-\delta)^{\gamma-1}, & \text{if } t \geq \delta, \\ k_1(x)(-t-\delta)^{\gamma-1}, & \text{if } t \leq -\delta, \end{cases}$$

where $k_1(x) \geq 0$, $k \in C(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $\frac{1}{\theta} + \frac{\gamma}{p^*} = 1$ and $\inf\limits_{x \in B_r(x_0)} k(x) \geq c_0 > 0$. Indeed,

$$|f(x,t)| \begin{cases} \leq k_1(x)|t|^{\gamma-1}, & \text{if } 0 \leq |t| \leq \delta, \\ = k_1(x)(t-\delta)^{\gamma-1} < k_1(x) < \dfrac{k_1(x)}{\delta^{\gamma-1}}|t|^{\gamma-1}, & \text{if } \delta < t < 1+\delta, \\ = k_1(x)(-t-\delta)^{\gamma-1} < k_1(x) < \dfrac{k_1(x)}{\delta^{\gamma-1}}|t|^{\gamma-1}, & \text{if } -1-\delta < t < -\delta, \\ = k_1(x) < k_1(x)|t|^{\gamma-1}, & \text{if } |t| = 1+\delta, \\ < k_1(x)|t|^{\gamma-1}, & \text{if } |t| > 1+\delta. \end{cases}$$

Hence, we have $|f(x,t)| \leq k(x)|t|^{\gamma-1}$ with $k(x) = k_1(x)(1 + \frac{1}{\delta^{\gamma-1}})$ and $f(x,t) = k_1(x)(t-\delta)^{\gamma-1} \geq (t-\delta)^{\gamma-1} \inf\limits_{x \in B_r(x_0)} k_1(x) = c_0(t-\delta)^{\gamma-1}$ for all $x \in B_r(x_0)$ and $\delta < t \leq 1$.

The main result of this paper establishes the following Theorem 1.2.

Theorem 1.2. *Assume that hypotheses (1.1), $(h_1)'$ and (h_5) hold. Then the problem (P) has at least two distinct nontrivial weak solutions $u_0, \widetilde{u}_0$ in $W^{1,H}(\mathbb{R}^N)$ and $\widetilde{u}_0(x) \leq u_0(x)$ for a.e. $x \in \mathbb{R}^N$.*

Sketch of the proof. We introduce the following functions

$$H(x,t) = t^p + \mu(x)t^q$$

for all $(x,t) \in \mathbb{R}^N \times [0,+\infty)$. Now, let us consider the Musielak-Orlicz space

$$L^{H}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_{\mathbb{R}^N} H(x,|u|)dx < +\infty\}$$

endowed with the norm

$$|u|_{H} = \inf\{\tau > 0 : \int_{\mathbb{R}^N} H(x, \frac{|u|}{\tau})dx \leq 1\}$$

and the usual Musielak-Orlicz Sobolev space

$$W^{1,H}(\mathbb{R}^N) = \{u \in L^{H}(\mathbb{R}^N) : |\nabla u| \in L^{H}(\mathbb{R}^N)\}$$

equipped with the Luxemburg norm given by

$$\|u\| = \inf\{\tau > 0 : \int_{\mathbb{R}^N} \Big(H(x, \frac{|\nabla u|}{\tau}) + H(x, \frac{|u|}{\tau})\Big)dx \leq 1\}.$$

Under Assumption 1.1, we have the following facts:

$$W^{1,H}(\mathbb{R}^N) \text{ is separable reflexive Banach space} \tag{1.2}$$

(see [10, Theorem 2.7 (ii)]) and the following continuous embedding hold

$$W^{1,H}(\mathbb{R}^N) \hookrightarrow L^{\vartheta}(\mathbb{R}^N) \text{ for all } \vartheta \in [p, p^*] \tag{1.3}$$

(see [10, Theorem 2.7 (iii)]); and from [10, Proposition 2.6] we directly obtain that

$$\min\{\|u\|^p, \|u\|^q\} \leq \rho(u) \leq \max\{\|u\|^p, \|u\|^q\}, \ \forall u \in W^{1,H}(\mathbb{R}^N), \tag{1.4}$$

where $\rho(u) := \int_{\mathbb{R}^N} [H(x,|\nabla u|) + H(x,|u|)]dx$.

We introduce the following two functionals in $W^{1,H}(\mathbb{R}^N)$:

$$J(u) = \int_{\mathbb{R}^N} \Big(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q + \frac{1}{p}|u|^p + \frac{\mu(x)}{q}|u|^q\Big)dx,$$
$$K(u) = \int_{\mathbb{R}^N} F(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$. Consider the C^1-functional $\varphi : W^{1,H}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\varphi(u) = J(u) - K(u).$$

We split the proof into several steps.

Step 1. The functional φ is weakly lower semi-continuous in $W^{1,H}(\mathbb{R}^N)$.

First, by Proposition 3.1 (ii) in [10], we known that K is weakly continuous in $W^{1,H}(\mathbb{R}^N)$. Thus, it is enough to show that functional J is weakly lower semi-continuous in $W^{1,H}(\mathbb{R}^N)$. Let $u_n \rightharpoonup u$ weakly in $W^{1,H}(\mathbb{R}^N)$. Since J is convex, we deduced that the following inequality holds:

$$\langle J'(u), u_n - u\rangle \leq J(u_n) - J(u).$$

Then we get that

$$\begin{aligned} 0 &= \liminf_{n\to+\infty}\langle J'(u), u_n - u\rangle \\ &\leq \liminf_{n\to+\infty}[J(u_n) - J(u)] \\ &= \liminf_{n\to+\infty} J(u_n) - J(u), \end{aligned}$$

which implies that

$$J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

Step 2. The functional φ is coercive.

Set $M = \max\left\{1, \left(\frac{2p|k|_\infty}{\gamma}\right)^{\frac{1}{p-\gamma}}\right\}$. Then for any $u \in W^{1,H}(\mathbb{R}^N)$, we have

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}^N}\Big(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q + \frac{1}{p}|u|^p + \frac{\mu(x)}{q}|u|^q\Big)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &= \int_{\mathbb{R}^N}\Big(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q + \frac{1}{2p}|u|^p + \frac{\mu(x)}{q}|u|^q\Big)dx \\ &+ \int_{\Omega_1}\Big(\frac{1}{2p}|u|^p - F(x,u)\Big)dx + \int_{\Omega_2}\Big(\frac{1}{2p}|u|^p - F(x,u)\Big)dx, \end{aligned} \tag{1.5}$$

where $\Omega_1 = \{x \in \mathbb{R}^N : |u(x)| \geq M\}$ and $\Omega_2 = \mathbb{R}^N\backslash\Omega_1$.

On the one hand, it is easy to compute directly that

$$\int_{\Omega_1}\Big(\frac{1}{2p}|u|^p - F(x,u)\Big)dx \geq \int_{\Omega_1}|u|^p\Big(\frac{1}{2p} - \frac{|k|_\infty}{\gamma}|u|^{\gamma-p}\Big)dx \geq 0. \tag{1.6}$$

On the other hand, by using Young's inequality, for $\varepsilon \in (0,1)$ we estimate

$$\frac{k(x)|u(x)|^\gamma}{\gamma} \leq \frac{1}{\theta\gamma}\Big(\frac{k(x)}{\varepsilon}\Big)^\theta + \frac{1}{p^*}(\varepsilon|u(x)|^\gamma)^{\frac{p^*}{\gamma}}.$$

Then we deduce that

$$\begin{aligned} &\int_{\Omega_2}\Big(\frac{1}{2p}|u|^p - F(x,u)\Big)dx \geq \int_{\Omega_2}\Big(\frac{|u|^p}{2p} - \frac{k(x)|u|^\gamma}{\gamma}\Big)dx \\ \geq &\int_{\Omega_2}\Big(\frac{|u|^p}{2p} - \frac{1}{\theta\gamma}\Big(\frac{k(x)}{\varepsilon}\Big)^\theta - \frac{1}{p^*}(\varepsilon|u(x)|^\gamma)^{\frac{p^*}{\gamma}}\Big)dx \\ = &\int_{\Omega_2}\Big(\frac{|u|^{p^*}|u|^{p-p^*}}{2p} - \frac{k(x)^\theta}{\theta\gamma\varepsilon^\theta} - \frac{1}{p^*}\varepsilon^{\frac{p^*}{\gamma}}|u(x)|^{p^*}\Big)dx \\ \geq &\int_{\Omega_2}\Big(\frac{|u|^{p^*}M^{p-p^*}}{2p} - \frac{k(x)^\theta}{\theta\gamma\varepsilon^\theta} - \frac{1}{p^*}\varepsilon^{\frac{p^*}{\gamma}}|u(x)|^{p^*}\Big)dx. \end{aligned}$$

Let $0 < \varepsilon < \min\left\{1, (\frac{p^* M^{p-p^*}}{2p})^{\frac{\gamma}{p^*}}\right\}$. Then

$$\int_{\Omega_2} \Big(\frac{1}{2p}|u|^p - F(x,u)\Big)dx \geq -\int_{\Omega_2} \frac{k(x)^\theta}{\theta\gamma\varepsilon^\theta}dx \geq -C_0. \tag{1.7}$$

Consequently, using (1.6) and (1.7) in (1.5) finally yields we obtain that

$$\varphi(u) \geq \frac{1}{2q}\int_{\mathbb{R}^N} \Big(|\nabla u|^p + \mu(x)|\nabla u|^q + |u|^p + \mu(x)|u|^q\Big)dx - C_0,$$

so that by (1.4) it follows that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$.

Therefore, using Steps 1 and 2, and applying the Weierstrass Theorem, we deduce that there exists a global minimizer $u_0 \in W^{1,H}(\mathbb{R}^N)$ of φ. The following Step 3 to show that $u_0 \neq 0$.

Step 3. We have $\varphi(u_0) = \inf\limits_{u \in W^{1,H}(\mathbb{R}^N)} \varphi(u) < 0$.

Let $\xi \in C_0^\infty(B_{2r}(x_0))$ such that $\xi(x) \equiv 1$, $x \in B_r(x_0)$; $0 \leq \xi(x) \leq 1$, $|\nabla\xi(x)| \leq \frac{C}{r}$, $x \in \mathbb{R}^N$. Denote $t = 2\delta$, then by assumption (h_5), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,t\xi)dx = \int_{B_{2r}(x_0)} F(x,t\xi)dx = \int_{B_{2r}(x_0)} \int_0^{t\xi} f(x,s)dsdx \\ \geq c_0 \int_{B_{2r}(x_0)} \int_\delta^{2\delta} (s-\delta)^{\gamma-1} dsdx \\ = c_0 \int_{B_{2r}(x_0)} \frac{1}{\gamma}\Big(\frac{t}{2}\Big)^\gamma dx = \frac{c_0}{\gamma 2^\gamma} t^\gamma |B_{2r}(x_0)|,\end{aligned}$$

and so

$$\begin{aligned}\varphi(t\xi) = &\int_{B_{2r}(x_0)} \Big(\frac{1}{p}|\nabla t\xi|^p + \frac{\mu(x)}{q}|\nabla t\xi|^q + \frac{1}{p}|t\xi|^p + \frac{\mu(x)}{q}|t\xi|^q\Big)dx \\ &- \int_{B_{2r}(x_0)} F(x,t\xi)dx \\ \leq &\frac{t^p}{p} m_\mu \int_{B_{2r}(x_0)} (|\nabla\xi|^p + |\nabla\xi|^q + |\xi|^p + |\xi|^q)dx - \frac{c_0}{\gamma 2^\gamma} t^\gamma |B_{2r}(x_0)| \\ \leq &\frac{2t^p}{p}\Big(1 + \frac{C^q}{r^q}\Big)m_\mu |B_{2r}(x_0)| - \frac{c_0}{\gamma 2^\gamma} t^\gamma |B_{2r}(x_0)| < 0.\end{aligned}$$

It follows from Step 3 that $u_0 \in W^{1,H}(\mathbb{R}^N)$ is a non-trivial weak solution of problem (P). It remains to show that there exists another non-trivial weak solution of problem (P).

Step 4. There exists a critical point $\widetilde{u}_0 \in W^{1,H}(\mathbb{R}^N)$ of φ.

Let

$$\widetilde{f}(x,t) = \begin{cases} f(x,t), & \text{if } |t| \leq |u_0(x)|, \\ f(x,u_0(x)), & \text{if } |t| > |u_0(x)|, \end{cases}$$

and $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds$. Then it follows from $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ that $\widetilde{f}(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

$$|\widetilde{f}(x,t)| \leq k(x)|t|^{\gamma-1}.$$

Similarly to Proposition 3.1 (i) in [10], we get that the functional

$$\widetilde{K}(u) = \int_{\mathbb{R}^N} \widetilde{F}(x,u)dx$$

is of class $C^1(W^{1,H}(\mathbb{R}^N),\mathbb{R})$, and

$$\langle \widetilde{K}'(u), v\rangle = \int_{\mathbb{R}^N} \widetilde{f}(x,u)v dx$$

for all $u, v \in W^{1,H}(\mathbb{R}^N)$. Next, we define the functional $\widetilde{\varphi} : W^{1,H}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\widetilde{\varphi}(u) = J(u) - \widetilde{K}(u).$$

The same arguments as those used for functional φ imply that $\widetilde{\varphi} \in C^1(W^{1,H}(\mathbb{R}^N),\mathbb{R})$ and $\widetilde{\varphi}$ is coercive. And by the definition of $\widetilde{\varphi}$, we get

$$\widetilde{\varphi}(u_0) = \varphi(u_0) < 0.$$

In the following, we determine a critical point $\widetilde{u}_0 \in W^{1,H}(\mathbb{R}^N)$ of $\widetilde{\varphi}$, such that $\widetilde{\varphi}(\widetilde{u}_0) > 0$ via the Mountain Pass Theorem.

First, we will show that there exists $0 < r_0 < \min\{1, \|u_0\|\}$ such that

$$\inf_{v\in W^{1,H}(\mathbb{R}^N);\|v\|=r_0} \widetilde{\varphi}(v) > 0 = \widetilde{\varphi}(0). \tag{1.8}$$

Using $(h_1)'$ and (h_5), for any $u \in W^{1,H}(\mathbb{R}^N)$ with $0 < \|u\| < \min\{1, \|u_0\|\}$ we have

$$\begin{aligned}
\widetilde{\varphi}(u) &= \int_{\mathbb{R}^N} \Big(\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q + \frac{1}{p}|u|^p + \frac{\mu(x)}{q}|u|^q\Big)dx - \int_{\mathbb{R}^N} \widetilde{F}(x,u)dx \\
&\geq \frac{1}{q}\|u\|^q - \int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} \widetilde{F}(x,u)dx \\
&\geq \frac{1}{q}\|u\|^q - \int_{\Omega_3} \widetilde{F}(x,u(x))dx - \int_{\Omega_4} \widetilde{F}(x,u(x))dx \\
&\geq \frac{1}{q}\|u\|^q - \int_{\Omega_3} \frac{k(x)}{\gamma}|u(x)|^\gamma dx - \int_{\Omega_4} \frac{k(x)}{\gamma}|u_0(x)|^\gamma dx \\
&\geq \frac{1}{q}\|u\|^q - \frac{2\delta^{\gamma-q}}{\gamma}\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} k(x)|u(x)|^q dx,
\end{aligned} \tag{1.9}$$

where $\Omega_3 = \{x \in \mathbb{R}^N : |u(x)| \leq |u_0(x)|\} \cap \{x \in \mathbb{R}^N : |u(x)| > \delta\}$, $\Omega_4 = \{x \in \mathbb{R}^N : |u(x)| > |u_0(x)|\} \cap \{x \in \mathbb{R}^N : |u(x)| > \delta\}$. Since $q < p^*$, then there exists $q < \tau < p^*$ such that $W^{1,H}(\mathbb{R}^N)$ is continuously embedded in $L^\tau(\mathbb{R}^N)$. Thus, there exists a positive constant C_τ such that

$$|u|_\tau \leq C_\tau\|u\|, \ \forall u \in W^{1,H}(\mathbb{R}^N).$$

Using Hölder's inequality and the above estimate, we obtain

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} k(x)|u(x)|^q dx \\
\leq&\Big(\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |k(x)|^{\tau'} dx\Big)^{\frac{1}{\tau'}}\Big(\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |u(x)|^\tau dx\Big)^{\frac{q}{\tau}} \\
\leq&\Big(\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |k(x)|^{\tau'} dx\Big)^{\frac{1}{\tau'}} C_\tau^q\|u\|^q,
\end{aligned} \tag{1.10}$$

where $\frac{1}{\tau'} + \frac{q}{\tau} = 1$.

By inequalities (1.9) and (1.10), we infer that it is enough to show that

$$\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |k(x)|^{\tau'} dx \to 0, \text{ as } \|u\| \to 0$$

in order to prove (1.8). Indeed, taking into account the fact that $k \in L^\infty(\mathbb{R}^N)$, yields

$$\begin{aligned}\delta^q \int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} (k(x))^{\tau'} dx \le& \int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} (k(x))^{\tau'} |u(x)|^q dx \\ \le& |k|_\infty^{\tau'} \int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |u(x)|^q dx \\ \le& |k|_\infty^{\tau'} \int_{\mathbb{R}^N} |u(x)|^q dx \le |k|_\infty^{\tau'} C_q^q \|u\|^q,\end{aligned}$$

which implies that

$$\int_{\{x\in\mathbb{R}^N:|u(x)|>\delta\}} |k(x)|^{\tau'} dx \to 0, \text{ as } \|u\| \to 0.$$

In view of Mountain Pass Theorem (see Ambrosetti-Rabinowitz [15] with the variant given by Theorem 1.15 in Willem [16]), there exists a sequence $\{u_n\} \subset W^{1,H}(\mathbb{R}^N)$, such that

$$\widetilde{\varphi}(u_n) \to c > 0 \text{ and } \widetilde{\varphi}'(u_n) \to 0,$$

where $c = \inf_{\lambda\in\Gamma} \max_{t\in[0,1]} \widetilde{\varphi}(\lambda(t))$, and

$$\Gamma = \{\lambda \in C([0, 1], W^{1,H}(\mathbb{R}^N)) : \lambda(0) = 0, \lambda(1) = u_0\}.$$

Since the functional $\widetilde{\varphi}$ is coercive, we obtain that $\{u_n\}$ is bounded in $W^{1,H}(\mathbb{R}^N)$, and passing to a subsequence, still denoted by $\{u_n\}$, we may assume that there exists a $\widetilde{u}_0 \in W^{1,H}(\mathbb{R}^N)$, such that $u_n \rightharpoonup \widetilde{u}_0$ weakly in $W^{1,H}(\mathbb{R}^N)$. By (1.3), we deduce that

$$W^{1,H}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

Thus, there is a positive constant $M > 0$ such that

$$\max\{|u_n|_\gamma, |u_n|_{p^*}, |\widetilde{u}_0|_\gamma, |\widetilde{u}_0|_{p^*}\} \le M.$$

We first will prove that the $u_n \to \widetilde{u}_0$ in $W^{1,H}(\mathbb{R}^N)$. Recall that

$$\begin{aligned}\langle J'(u_n) - J'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle =& \langle\widetilde{\varphi}'(u_n) - \widetilde{\varphi}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle \\ &+ \langle\widetilde{K}'(u_n) - \widetilde{K}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle.\end{aligned}$$

Then it is enough to show that

$$\lim_{n\to+\infty} \langle\widetilde{K}'(u_n) - \widetilde{K}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle = 0.$$

Denote $\Omega_j = \{x \in \mathbb{R}^N : |x| \leq j\}$ and $\Omega_j^c = \mathbb{R}^N \backslash \Omega_j$, $j \in \mathbb{N}$. Then by the fact that $k \in L^\theta(\mathbb{R}^N)$, we deduce that

$$|k|_{L^\theta(\Omega_j^c)} \to 0 \text{ as } j \to +\infty,$$

and so for given $\varepsilon \in (0,1)$, there exists $j_0 > 0$ big enough such that

$$|k|_{L^\theta(\Omega_{j_0}^c)} < \frac{\varepsilon}{8M^\gamma}.$$

We also known that $u_n \to \widetilde{u}_0$ in $L^\gamma(\Omega_{j_0})$ because the embedding $W^{1,H}(\Omega_{j_0}) \hookrightarrow L^\gamma(\Omega_{j_0})$ is compact. It follows that there exists $n_0 > 0$, such that

$$|u_n - \widetilde{u}_0|_{L^\gamma(\Omega_{j_0})} < \frac{\varepsilon}{4|k|_\infty M^{\gamma-1}}, \forall n > n_0.$$

By a straightforward computation we deduce that

$$\begin{aligned}
&|\langle \widetilde{K}'(u_n) - \widetilde{K}'(\widetilde{u}_0), u_n - \widetilde{u}_0 \rangle| \\
=&\Big| \int_{\mathbb{R}^N} (\widetilde{f}(x,u_n) - \widetilde{f}(x,\widetilde{u}_0)(u_n - \widetilde{u}_0)dx \Big| \\
\leq& \int_{\Omega_{j_0}} k(x)(|u_n|^{\gamma-1} + |\widetilde{u}_0|^{\gamma-1})|u_n - \widetilde{u}_0|dx \\
&+ \int_{\Omega_{j_0}^c} k(x)(|u_n|^{\gamma-1} + |\widetilde{u}_0|^{\gamma-1})|u_n - \widetilde{u}_0|dx \\
=&: I_1 + I_2.
\end{aligned}$$

Applying Hölder's inequality and condition $(h_1)'$, we have

$$\begin{aligned}
I_1 \leq& |k|_\infty \int_{\Omega_{j_0}} (|u_n|^{\gamma-1} + |\widetilde{u}_0|^{\gamma-1})|u_n - \widetilde{u}_0|dx \\
\leq& |k|_\infty \Big[\big| |u_n|^{\gamma-1} \big|_{L^{\frac{\gamma}{\gamma-1}}(\Omega_{j_0})} + \big| |u_n|^{\gamma-1} \big|_{L^{\frac{\gamma}{\gamma-1}}(\Omega_{j_0})} \Big] |u_n - \widetilde{u}_0|_{L^\gamma(\Omega_{j_0})} \\
\leq& |k|_\infty \Big[|u_n|^{\gamma-1}_{L^\gamma(\mathbb{R}^N)} + \big|u_n\big|^{\gamma-1}_{L^\gamma(\mathbb{R}^N)} \Big] |u_n - \widetilde{u}_0|_{L^\gamma(\Omega_{j_0})} \\
\leq& 2|k|_\infty M^{\gamma-1} |u_n - \widetilde{u}_0|_{L^\gamma(\Omega_{j_0})} < \frac{\varepsilon}{2}
\end{aligned}$$

and

$$\begin{aligned}
I_2 \leq& \int_{\Omega_{j_0}^c} k(x)\big(|u_n|^{\gamma-1} + |\widetilde{u}_0|^{\gamma-1}\big)|u_n - \widetilde{u}_0|dx \\
\leq& |k|_{L^\theta(\Omega_{j_0}^c)} \Big[\big| |u_n|^{\gamma-1} \big|_{L^{\frac{p^*}{\gamma-1}}(\mathbb{R}^N)} + \big| |u_n|^{\gamma-1} \big|_{L^{\frac{p^*}{\gamma-1}}(\mathbb{R}^N)} \Big] |u_n - \widetilde{u}_0|_{L^{p^*}(\mathbb{R}^N)} \\
\leq& |k|_{L^\theta(\Omega_{j_0}^c)} \Big[|u_n|^{\gamma-1}_{L^{p^*}(\mathbb{R}^N)} + \big|u_n\big|^{\gamma-1}_{L^{p^*}(\mathbb{R}^N)} \Big] (|u_n|_{L^{p^*}(\mathbb{R}^N)} + |\widetilde{u}_0|_{L^{p^*}(\mathbb{R}^N)}) \\
\leq& 4|k|_{L^\theta(\Omega_{j_0}^c)} M^\gamma \\
<& \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently, we obtain that
$$|\langle \widetilde{K}'(u_n) - \widetilde{K}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle| < \varepsilon,$$
when $n \geq n_0$. By the arbitrariness of ε, we get
$$\lim_{n\to+\infty} \langle \widetilde{K}'(u_n) - \widetilde{K}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle = 0.$$
Noting that
$$\lim_{n\to+\infty} \langle \widetilde{\varphi}'(u_n) - \widetilde{\varphi}'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle = 0.$$
Then we obtain
$$\lim_{n\to+\infty} \langle J'(u_n) - J'(\widetilde{u}_0), u_n - \widetilde{u}_0\rangle = 0.$$
Due to Proposition 1.2 (ii) in [10], we have that $u_n \to \widetilde{u}_0$ in $W^{1,H}(\mathbb{R}^N)$. Since $\widetilde{\varphi} \in C^1(W^{1,H}(\mathbb{R}^N), \mathbb{R}^N)$, we observe that $\widetilde{u}_0$ is a non-trivial critical point of $\widetilde{\varphi}$ because $\widetilde{\varphi}(\widetilde{u}_0) = c > 0$ and $\widetilde{\varphi}'(\widetilde{u}_0) = 0$.

Finally, we will show that $\widetilde{u}_0(x) \leq u_0(x)$ for a.e. $x \in \mathbb{R}^N$. Indeed, it is easy to check that
$$\begin{aligned}
0 =& \langle \widetilde{\varphi}'(\widetilde{u}_0) - \varphi'(u_0), (\widetilde{u}_0 - u_0)^+\rangle \\
=& \int_{\mathbb{R}^N} \Big([|\nabla \widetilde{u}_0|^{p-2}\nabla \widetilde{u}_0 - |\nabla u_0|^{p-2}\nabla u_0]\nabla(\widetilde{u}_0 - u_0)^+ \\
&+\mu[|\nabla \widetilde{u}_0|^{q-2}\nabla \widetilde{u}_0 - |\nabla u_0|^{q-2}\nabla u_0]\nabla(\widetilde{u}_0 - u_0)^+ \\
&+[|\widetilde{u}_0|^{p-2}\widetilde{u}_0 - |u_0|^{p-2}u](\widetilde{u}_0 - u_0)^+ \\
&+\mu[|\widetilde{u}_0|^{q-2}\widetilde{u}_0 - |u_0|^{q-2}u_0](\widetilde{u}_0 - u_0)^+\Big)dx \\
&- \int_{\mathbb{R}^N} (\widetilde{f}(x,\widetilde{u}_0) - f(x,u_0))(\widetilde{u}_0 - u_0)^+dx \\
=& \int_{[\widetilde{u}_0 \geq u_0]} \Big((|\nabla \widetilde{u}_0|^{p-2}\nabla \widetilde{u}_0 - |\nabla u_0|^{p-2}\nabla u_0)\nabla(\widetilde{u}_0 - u_0)^+ \\
&+\mu(|\nabla \widetilde{u}_0|^{q-2}\nabla \widetilde{u}_0 - |\nabla u_0|^{q-2}\nabla u_0)\nabla(\widetilde{u}_0 - u_0)^+ \\
&+(|\widetilde{u}_0|^{p-2}\widetilde{u}_0 - |u_0|^{p-2}u])(\widetilde{u}_0 - u_0)^+ \\
&+\mu(|\widetilde{u}_0|^{q-2}\widetilde{u}_0 - |u_0|^{q-2}u_0)(\widetilde{u}_0 - u_0)^+\Big)dx,
\end{aligned}$$
where $(\widetilde{u}_0 - u_0)^+ = \max\{0, \widetilde{u}_0 - u_0\}$ and $[\widetilde{u}_0 \geq u_0] = \{x \in \mathbb{R}^N : \widetilde{u}_0(x) \geq u_0(x)\}$. Obviously, the each term on the right hand side of above equality is non-negative, then we conclude that
$$\int_{[\widetilde{u}_0 \geq u_0]} (|\widetilde{u}_0|^{p-2}\widetilde{u}_0 - |u_0|^{p-2}u)(\widetilde{u}_0 - u_0)dx = 0,$$
which implies that $\widetilde{u}_0(x) = u_0(x)$ for a.e. $x \in \{x \in \mathbb{R}^N : \widetilde{u}_0(x) \geq u_0(x)\}$. Consequently, $\widetilde{u}_0(x) \leq u_0(x)$, for a.e. $x \in \mathbb{R}^N$. This immediately yields
$$\widetilde{f}(x,\widetilde{u}_0) = f(x,\widetilde{u}_0) \text{ and } \widetilde{K}(\widetilde{u}_0) = K(\widetilde{u}_0).$$
Then we obtain
$$\varphi(\widetilde{u}_0) = \widetilde{\varphi}(\widetilde{u}_0) \text{ and } \varphi'(\widetilde{u}_0) = \widetilde{\varphi}'(\widetilde{u}_0),$$
which yields that $\widetilde{u}_0$ is a critical point of φ, and so a weak solution of problem (P). Recall that $\varphi(\widetilde{u}_0) = c > 0 > \varphi(u_0)$. Thus we see that $\widetilde{u}_0$ is non-trivial. Therefore, $\widetilde{u}_0 \neq u_0$ and this completes the proof of Theorem 1.2. □

2. Conclusions

In this paper, we have discussed a class of sublinear double phase problem in $\mathbb{R}^N$. Some new criteria to guarantee that the existence of two non-trivial weak solutions for the considered problem (P) is established by using the Weierstrass Theorem and Mountain Pass Theorem. Our results are obtained to improve and supplement some corresponding results.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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