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Research article

Numerical solution of fractional variational and optimal control problems via fractional-order Chelyshkov functions

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Abstract: In this paper, we present a new numerical method based on the fractional-order Chelyshkov functions (FCHFs) for solving fractional variational problems (FVPs) and fractional optimal control problems (FOCPs). The fractional derivatives are considered in the Caputo sense. The operational matrix of fractional integral for FCHFs, together with the Lagrange multiplier method, are used to reduce the fractional optimization problem into a system of algebraic equations. Some results concerning the approximation errors are discussed and the convergence of the presented method is also demonstrated. The performance of the introduced method is tested through several examples. Some comparisons with recent numerical methods are introduced to show the accuracy and effectiveness of the presented method.

Keywords: fractional variational problems; fractional optimal control problems; fractional-order Chelyshkov functions; operational matrix; Lagrange multiplier method
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1. Introduction

Fractional calculus treats derivatives and integrals of non-integer order [18, 47]. The history of the development of fractional calculus and some of its applications can be found in [43]. Differential and integral equations of integer-order have played an increasingly important role in modeling many physical phenomena. However, they cannot present acceptable results for complex physical systems. Therefore, fractional differential and integral equations (FDIEs), have been used to model these systems [15, 22]. Solving FDIEs are a very important topic to be considered. For most of the FDIEs, exact solutions are not known, so many new numerical methods have been presented to find approximate solutions for FDIEs, such as the Galerkin method [24], fractional differential transformation method [39], homotopy analysis method [26], variational iteration method [50], Jacobi operational matrix method [19], wavelet method [51], cubic B-spline collocation method [37], shifted Jacobi collocation method [12], hybrid Taylor and block-pulse operational matrix method [46] and Chebyshev method [5].

There are two important topics where fractional calculus plays an essential role: the fractional calculus of variations and the fractional optimal control theory. An FVP is a classical variational problem in which the performance index depends on fractional derivatives. The FVPs can be found in several physical applications [9, 40]. One of the earliest papers that were interested in discussing these problems and searching for their optimal solutions was [1]. It presented a formulation of Euler–Lagrange equations for two types of FVPs with fractional derivatives described in the Riemann–Liouville sense. A discrete method based on discretizing the performance index by using appropriate approximations for derivatives has been given in [44]. Another numerical method, based on Jacobi polynomials has been presented in [10]. The authors have considered a problem for a performance index, where the integration interval is a proper subset of the whole domain of the admissible functions. The shifted Legendre polynomials, together with Gauss–Legendre quadrature formula are used in [25] to reduce a class of FVPs into a system of algebraic equations. Recently, El-Kalaawy et al. [23] have proposed a computational method based on Gegenbauer functions to solve FVPs.

Optimal control theory is an expanded mathematical branch that has been used for mathematical modeling in science, engineering, and operations research [13, 32]. An optimal control problem concerns finding a control variable for a given system of differential equations such that a performance index is optimized. If the given system is a system of fractional differential equations, then the problem is the FOCP. A general formulation for a class of FOCPs, which extends the classical optimal control theory to the fractional dynamical system has been presented in [2, 4] where the fractional derivatives are considered in the Riemann–Liouville sense. While the formulation for FOCPs using fractional derivatives described in the Caputo sense is presented in [3]. These methods derive the fractional Euler–Lagrange equations and use them to develop a numerical scheme for FOCPs. In [49], the FOCP has been converted into an integer-order optimal control problem by using Oustaloup's recursive approximation to model the fractional dynamical system in terms of a state-space realization. The neural networks are used to approximate a solution of FOCPs in [45], while Ritz's direct method is applied to give a numerical method for FOCPs in [31]. A combination of Bernstein polynomials and block-pulse functions has been used in [36] to transform FOCP into an optimization problem that can be handled easily by optimization techniques. A discretization technique for FOCPs based on a second-order numerical integration scheme for the fractional system has been proposed in [33]. By using the control parameterization method, the FOCPs has been approximated by a sequence of finite-dimensional optimization problems in [41].

Many operational matrix techniques have been introduced to provide numerical solutions of FOCPs [14, 17, 27–29, 34]. These methods are based on various types of orthogonal polynomials that are used to approximate the state and control variables. The main advantage of using these functions is to simplify the treatment of FDIEs by converting their solution to the solution of a system of algebraic equations. The Chelyshkov polynomials are a class of orthogonal polynomials that were presented in [16]. Utilizing these functions, solutions have been obtained of a class of nonlinear fractional differential equations in [35], linear weakly singular Volterra integral equations in [48] and multi-order fractional differential equations in [11].

Basis functions of integer-order give good convergence when they are used to solve integer-order differential equations, while their usage for solving fractional differential equations may produce some problems [27]. The main drawback occurs when the solutions of FDIEs contain terms with fractional powers which leads to a poor rate of convergence and a high number of basis functions is required to obtain good results. To overcome this disadvantage, it will be useful to use orthogonal functions of fractional order to approximate the solutions of FDIEs to increase efficiency.

Based on the above considerations, the motivations of this paper are as follows:

- We consider FVPs and FOCPs due to their importance in applications where their numerical solutions are crucial and the presented method can be useful in treating these problems.
- The operational matrix of fractional integration is not new but to the best knowledge of the authors, there is no article about solving FVPs and FOCPs using FCHFs.

Therefore, we adopt the FCHFs for solving FVPs and FOCPs. To achieve these aims, the operational matrix of fractional integral for FCHFs is used together with the Lagrange multiplier method to reduce the fractional optimization problem into a system of algebraic equations. The key feature of the presented method is that a small number of FCHFs is needed to obtain satisfactory results. Our method possesses some beneficial properties as follows:

- The present method uses fractional-order orthogonal functions, which overcome the disadvantages of the slow rate of convergence and the needed high number of basis functions
- It transforms the fractional optimization problem into a system of algebraic equations, which simplify the solution procedure.
- The obtained numerical results show that the introduced method is more efficient than conventional methods.

The paper is arranged as follows. Section 2 presents some definitions and preliminaries of fractional calculus, together with the FCHFs and their properties. In Section 3, a new direct method is presented to solve a class of FVPs and FOCPs via FCHFs as basis functions. The convergence of the proposed method is studied in Section 4. Section 5 demonstrates the performance of the introduced algorithm by considering some illustrative examples. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

2.1. Fractional calculus

This subsection presents some basic concepts and primary results about fractional calculus. Here, the definitions of Riemann–Liouville fractional integral and Caputo fractional derivative are considered.

Definition 2.1. *The fractional integral of order $\eta \geq 0$ of a given function $\phi(t)$ according to Riemann–Liouville is defined as [43]*

$$I^{\eta}\phi(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}\phi(\tau)d\tau, \quad \eta > 0, t > 0, \tag{2.1}$$
$$I^0\phi(t) = \phi(t).$$

The Riemann–Liouville fractional integral satisfies the properties

$$I^{\eta}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\eta+1)}t^{\gamma+\eta}, \quad \gamma > -1, \tag{2.2}$$

$$I^{\eta_1}I^{\eta_2}\phi(t) = I^{\eta_1+\eta_2}\phi(t) = I^{\eta_2}I^{\eta_1}\phi(t), \quad \eta_1,\ \eta_2 \geq 0, \tag{2.3}$$

$$I^{\eta}\left(\alpha_1\phi_1(t) + \alpha_2\phi_2(t)\right) = \alpha_1 I^{\eta}\phi_1(t) + \alpha_2 I^{\eta}\phi_2(t), \tag{2.4}$$

where α_1 and α_2 are constants.

Definition 2.2. *The Caputo fractional derivative of order $\eta > 0$ of a given function $\phi(t)$ is given by [43]*

$$D^{\eta}\phi(t) = I^{\lceil\eta\rceil-\eta}D^{\lceil\eta\rceil}\phi(t) = \frac{1}{\Gamma(\lceil\eta\rceil-\eta)}\int_0^t (t-\tau)^{\lceil\eta\rceil-\eta-1}\phi^{(\lceil\eta\rceil)}(\tau)d\tau, \quad \lceil\eta\rceil - 1 < \eta \leq \lceil\eta\rceil, t > 0, \tag{2.5}$$

where $\lceil\eta\rceil$ denotes the smallest integer greater than or equal to η.

For Caputo fractional derivative we have

$$D^{\eta}K = 0, \quad \text{K is constant}, \tag{2.6}$$

$$D^{\eta}t^{\gamma} = \begin{cases} 0, & \gamma \in N_0 \ and \ \gamma < \lceil\eta\rceil, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\eta+1)}t^{\gamma-\eta}, & \gamma \in N_0 \ and \ \gamma \geq \lceil\eta\rceil \quad \text{or} \quad \gamma \notin N \ and \ \gamma > \lfloor\eta\rfloor, \end{cases} \tag{2.7}$$

$$D^{\eta}I^{\eta}\phi(t) = \phi(t), \tag{2.8}$$

$$I^{\eta}D^{\eta}\phi(t) = \phi(t) - \sum_{k=0}^{\lceil\eta\rceil-1}\frac{\phi^{(k)}(0^+)}{k!}t^k, \quad t > 0, \tag{2.9}$$

$$D^{\eta}\left(\alpha_1\phi_1(t) + \alpha_2\phi_2(t)\right) = \alpha_1 D^{\eta}\phi_1(t) + \alpha_2 D^{\eta}\phi_2(t), \tag{2.10}$$

where $\lfloor\eta\rfloor$ denotes the largest integer less than or equal to η, $N_0 = \{0, 1, 2, \ldots\}$ and $N = \{1, 2, \ldots\}$.

Definition 2.3. *(Generalized Taylor's formula). Suppose that $D^{l\gamma}\phi(t) \in C(0,1]$ for $l = 0, 1, \ldots, m+1$ and $0 < \gamma \leq 1$, then [42]*

$$\phi(t) = \sum_{k=0}^{m}\frac{t^{k\gamma}}{\Gamma(k\gamma+1)}D^{k\gamma}\phi(0^+) + \frac{t^{(m+1)\gamma}}{\Gamma\left((m+1)\gamma+1\right)}D^{(m+1)\gamma}\phi(\hbar), \quad 0 < \hbar \leq t, \forall t \in (0,1], \tag{2.11}$$

where $D^{k\gamma} = \underbrace{D^{\gamma}D^{\gamma}\ldots D^{\gamma}}_{k\ times}$.

If $\gamma = 1$, then the generalized Taylor's formula converts to the classical Taylor's formula.

2.2. *Fractional-order Chelyshkov functions and their properties*

The fractional-order Chelyshkov functions (FCHFs) are a class of orthogonal polynomials in the interval $\sigma = [0, 1]$ with respect to the weight function $w_\gamma(t) = t^{\gamma-1}$. These functions, denoted by $\rho^\gamma_{mn}(t)$, take the form [6, 11]:

$$\rho^\gamma_{mn}(t) = \sum_{j=n}^{m} C_{jn} t^{j\gamma}, \quad n = 0, 1, \ldots m. \tag{2.12}$$

where

$$C_{jn} = (-1)^{j-n} \binom{m-n}{j-n} \binom{m+j+1}{m-n}, \quad j = n, n+1, ..., m. \tag{2.13}$$

The orthogonality relationship for the FCHFs is

$$\int_0^1 \rho^\gamma_{mr}(t) \rho^\gamma_{mi}(t) w_\gamma(t) dt = \frac{\delta_{ri}}{(r+i+1)\gamma}, \quad \delta_{ri} = \begin{cases} 1, & r = i, \\ 0, & r \neq i, \end{cases}, \quad r, i = 0, 1, \ldots, m. \tag{2.14}$$

From Eqs (2.12) and (2.13), it can be seen that all the functions $\rho^\gamma_{mn}(t)$, $n = 0, 1, ..., m$ are of degree m. This is the fundamental difference between the Chelyshkov polynomials and the other sets of orthogonal polynomials. Suppose the weighted space $L^2_{w_\gamma}(\sigma)$ is defined by

$$L^2_{w_\gamma}(\sigma) = \left\{ \nu : \sigma \longrightarrow \mathbb{R};\ \nu \text{ is measurable on } \sigma \ \& \ \int_0^1 |\nu(t)|^2 w_\gamma(t) dt < \infty \right\}. \tag{2.15}$$

The inner product and the norm in this space are given by

$$\langle \nu(t), \upsilon(t) \rangle_{w_\gamma} = \int_0^1 \nu(t) \upsilon(t) w_\gamma(t) dt, \tag{2.16}$$
$$\|\nu(t)\|_{w_\gamma} = \langle \nu(t), \nu(t) \rangle^{\frac{1}{2}}_{w_\gamma}.$$

Assume that $\mathbb{P}_m = \text{span}\left\{\rho^\gamma_{m0}(t), \rho^\gamma_{m1}(t), \ldots, \rho^\gamma_{mm}(t)\right\}$. Since $\mathbb{P}_m$ is a finite dimensional and closed subspace of $L^2_{w_\gamma}(\sigma)$, then for every function $\phi(t) \in L^2_{w_\gamma}(\sigma)$ there exists a unique best estimation $\phi_m(t) \in \mathbb{P}_m$ such that

$$\forall \xi(t) \in \mathbb{P}_m, \quad \|\phi(t) - \phi_m(t)\|_{w_\gamma} \leq \|\phi(t) - \xi(t)\|_{w_\gamma}. \tag{2.17}$$

Since $\phi_m(t) \in \mathbb{P}_m$, then there are unique coefficients $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that

$$\phi(t) \simeq \phi_m(t) = \sum_{k=0}^{m} \lambda_k \rho^\gamma_{mk}(t) = \Lambda^T \rho_\gamma(t), \tag{2.18}$$

where

$$\Lambda = \left(\lambda_0, \lambda_1, \ldots, \lambda_m\right)^T, \quad \lambda_k = \gamma(2k+1) \int_0^1 \phi(t) \rho^\gamma_{mk}(t) w_\gamma(t) dt, \quad k = 0, 1, \ldots, m, \tag{2.19}$$

and $\rho_\gamma(t)$ denotes the FCHFs vector which takes the form

$$\rho_\gamma(t) = \left(\rho^\gamma_{m0}(t), \rho^\gamma_{m1}(t), \ldots, \rho^\gamma_{mm}(t)\right)^T. \tag{2.20}$$

The Riemann–Liouville fractional integral of order $\eta > 0$ of the vector $\rho_\gamma(t)$ can be given by

$$I^\eta \rho_\gamma(t) \simeq H^{(\eta)} \rho_\gamma(t), \tag{2.21}$$

where $H^{(\eta)} = (h_{nr})_{n,r=0}^m$ is the $(m+1) \times (m+1)$ operational matrix of fractional integral of order $\eta > 0$ in the Riemann–Liouville sense and its elements take the form [6, 11]:

$$h_{nr} = \sum_{j=n}^{m} \sum_{l=r}^{m} \frac{\gamma(2r+1) C_{lr} C_{jn}}{\gamma(j+l+1)+\eta} \frac{\Gamma(j\gamma+1)}{\Gamma(j\gamma+\eta+1)}. \tag{2.22}$$

3. Method of the solution

The current section aims to use the FCHFs set and its operational matrix of fractional integral to introduce a new numerical method for solving some types of FVPs and FOCPs accurately. A description of the suggested methodology is given below.

3.1. Fractional variational problems

Consider the following FVP:

$$\text{Minimize} \quad J(x) = \int_0^1 L\left(t, x(t), D^{\eta_1} x(t), D^{\eta_2} x(t), \ldots, D^{\eta_q} x(t), D^{\eta} x(t)\right) dt, \tag{3.1}$$

subject to the initial conditions

$$x^{(k)}(0) = p_k, \; k = 0, 1, \ldots, \lceil \eta \rceil - 1, \tag{3.2}$$

where $0 < \eta_1 < \eta_2 < \ldots < \eta_q < \eta$, D^η denotes the Caputo fractional derivative of order η, J is called the performance index, $x(t)$ is the unknown function and L is a smooth function. To find an approximate solution of the above problem, we expand $D^\eta x(t)$ in terms of the FCHFs as follows

$$D^\eta x(t) \simeq \sum_{l=0}^{m} \lambda_l \rho_{ml}^{\gamma}(t) = \Lambda^T \rho_\gamma(t). \tag{3.3}$$

where Λ is an unknown coefficient vector. By using Eq (2.9) and the initial conditions given in (3.2), we get

$$I^\eta D^\eta x(t) = x(t) - \sum_{k=0}^{\lceil \eta \rceil - 1} \frac{p_k}{k!} t^k, \tag{3.4}$$

from Eqs (2.21) and (3.3), we rewrite Eq (3.4) as

$$\begin{aligned} x(t) &\simeq \Lambda^T I^\eta \rho_\gamma(t) + \sum_{k=0}^{\lceil \eta \rceil - 1} \frac{p_k}{k!} t^k \\ &\simeq \Lambda^T H^{(\eta)} \rho_\gamma(t) + \sum_{k=0}^{\lceil \eta \rceil - 1} \frac{p_k}{k!} t^k. \end{aligned} \tag{3.5}$$

Then, for $r = 1, 2, \ldots, q$, we have

$$D^{\eta_r} x(t) \simeq \Lambda^T H^{(\eta-\eta_r)} \rho_\gamma(t) + \sum_{k=0}^{\lceil\eta\rceil-1} \frac{p_k}{k!} D^{\eta_r} t^k, \tag{3.6}$$

and from Eq (2.7), one can write

$$D^{\eta_r} x(t) \simeq \Lambda^T H^{(\eta-\eta_r)} \rho_\gamma(t) + \sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_r+1)} t^{k-\eta_r}, \quad \lceil\eta_r\rceil \leq \lceil\eta\rceil - 1, \tag{3.7}$$

we can see that the second term of the previous equation will vanish if $\lceil\eta_r\rceil \geq \lceil\eta\rceil$. Let $\eta_0 = 0$, then Eq (3.5) together with Eq (3.7) yield

$$D^{\eta_r} x(t) \simeq \Lambda^T H^{(\eta-\eta_r)} \rho_\gamma(t) + \sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_r+1)} t^{k-\eta_r}, \quad r = 0, 1, \ldots, q. \tag{3.8}$$

Also, the second term in the right hand side of the above equation can be expanded in terms of the FCHFs as follows

$$\sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_r+1)} t^{k-\eta_r} \simeq \sum_{s=0}^{m} y_{rs} \rho_{ms}^{\gamma}(t) = Y_r^T \rho_\gamma(t), \quad r = 0, 1, \ldots, q, \tag{3.9}$$

where the known vectors $Y_r = \left(y_{r0}, y_{r1}, \ldots, y_{rm}\right)^T$, $r = 0, 1, \ldots, q$ can be obtained by

$$\begin{aligned} y_{rs} &= \gamma(2s+1) \sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_r+1)} \int_0^1 t^{k-\eta_r} \rho_{ms}^{\gamma}(t) w_\gamma(t) dt \\ &= \gamma(2s+1) \sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \sum_{j=s}^{m} \frac{C_{js} p_k}{\Gamma(k-\eta_r+1)} \int_0^1 t^{\gamma(j+1)+k-\eta_r-1} dt \\ &= \gamma(2s+1) \sum_{k=\lceil\eta_r\rceil}^{\lceil\eta\rceil-1} \sum_{j=s}^{m} \frac{C_{js} p_k}{(\gamma(j+1)+k-\eta_r)\Gamma(k-\eta_r+1)}, \quad r = 0, 1, \ldots, q, \ s = 0, 1, \ldots, m. \end{aligned} \tag{3.10}$$

Therefore, Eq (3.8) becomes

$$D^{\eta_r} x(t) \simeq \left(\Lambda^T H^{(\eta-\eta_r)} + Y_r^T\right) \rho_\gamma(t) = X_r^T \rho_\gamma(t), \quad r = 0, 1, \ldots, q. \tag{3.11}$$

By substituting Eqs (3.3) and (3.11) in Eq (3.1), the performance index (3.1) can be approximated as

$$J(x) \simeq J(\Lambda) = \int_0^1 L\left(t, X_0^T \rho_\gamma(t), X_1^T \rho_\gamma(t), \ldots, X_q^T \rho_\gamma(t), \Lambda^T \rho_\gamma(t)\right) dt. \tag{3.12}$$

The necessary conditions for optimality of the FVP (3.1)–(3.2) are

$$\frac{\partial J}{\partial \lambda_l} = 0, \quad l = 0, 1, \ldots, m. \tag{3.13}$$

The above equations are a system of $m+1$ algebraic equations that can be solved for the unknown vector Λ. Consequently, an approximate solution of $x(t)$ can be obtained from Eq (3.11).

3.2. Fractional optimal control problems

Consider the following FOCP:

$$\text{Minimize} \quad J(x,u) = \int_0^1 L\left(t, x(t), u(t)\right) dt, \tag{3.14}$$

subject to the fractional dynamical system

$$f\left(t, x(t), D^{\eta_1}x(t), D^{\eta_2}x(t), \ldots, D^{\eta_q}x(t), D^{\eta}x(t), u(t)\right) = 0, \tag{3.15}$$

and the initial conditions

$$x^{(k)}(0) = p_k, \; k = 0, 1, \ldots, \lceil \eta \rceil - 1, \tag{3.16}$$

where $x(t)$ is the state function, $u(t)$ is the control function, and L and F are smooth functions. Following the same method illustrated in the previous subsection, we approximate $D^{\eta}x(t)$ and $D^{\eta_r}x(t),\; r = 0, 1, \ldots, q$ by Eqs (3.3) and (3.11) respectively. Also, $u(t)$ can be expanded in terms of the FCHFs as

$$u(t) \simeq \sum_{i=0}^{m} u_i \rho_{mi}^{\gamma}(t) = U^T \rho_{\gamma}(t). \tag{3.17}$$

By substituting these approximations into the performance index (3.14) and the fractional dynamic constraint (3.15), we obtain

$$J(x,u) \simeq J(\Lambda, U) = \int_0^1 L\Big(t, X_0^T \rho_{\gamma}(t), U^T \rho_{\gamma}(t)\Big) dt, \tag{3.18}$$

$$f(t, \Lambda, U) = f\Big(t, X_0^T \rho_{\gamma}(t), X_1^T \rho_{\gamma}(t), \ldots, X_q^T \rho_{\gamma}(t), \Lambda^T \rho_{\gamma}(t), U^T \rho_{\gamma}(t)\Big) \simeq 0. \tag{3.19}$$

Collocating Eq (3.19) at the nodes $t_j = \dfrac{j}{m},\; j = 0, 1, \ldots, m$, we obtain a system of $(m+1)$ algebraic equations as

$$f(t_j, \Lambda, U) \simeq 0, \quad j = 0, 1, \ldots, m. \tag{3.20}$$

Finally, we define the Lagrange function $J^*(\Lambda, U, \mu)$ by introducing one Lagrange multiplier μ_j for every constraint $f(t_j, \Lambda, U)$ in the form

$$J^*(\Lambda, U, \mu) = J(\Lambda, U) + \mu_0 f(t_0, \Lambda, U) + \cdots + \mu_m f(t_m, \Lambda, U), \tag{3.21}$$

where $\mu = \Big(\mu_0, \mu_1, \ldots, \mu_m\Big)^T$. The necessary conditions for the optimality of J^*, which also corespond to the optimum solution of the FOCP (3.14)–(3.16) are

$$\frac{\partial J^*}{\partial \lambda_l} = 0, \quad \frac{\partial J^*}{\partial u_l} = 0, \quad \frac{\partial J^*}{\partial \mu_l} = 0, \quad l = 0, 1, \ldots, m. \tag{3.22}$$

Equations (3.22) are a system of $3(m+1)$ algebraic equations in terms of the unknown coefficients $\lambda_l,\; u_l,\; \mu_l,\;\; l = 0, 1, \ldots, m$. By determining the unknowns, an approximate solutions of $x(t)$ and $u(t)$ can be obtained from Eqs (3.11) and (3.17) respectively.

4. Convergence analysis

In this section, we discuss the convergence of the proposed method. First, let us recall a theorem from [6].

Theorem 4.1. *Suppose that $D^{l\gamma}\phi(t) \in C(0,1]$ for $l = 0, 1, \ldots, m+1$. If $\phi_m(t)$ is the best estimation to $\phi(t)$ from $\mathbb{P}_m$, then the following inequality holds*

$$\left\|\phi(t) - \phi_m(t)\right\|_{w_\gamma} \leq \frac{K_\gamma}{\Gamma\left((m+1)\gamma+1\right)} \frac{1}{\sqrt{(2m+3)\gamma}}, \tag{4.1}$$

where

$$K_\gamma = sup\left\{|D^{(m+1)\gamma}\phi(t)|\right\}, \quad t \in (0,1].$$

According to Theorem 4.1, the error bound given in inequality (4.1) depends on both m and γ, so for a fixed γ, $\dfrac{K_\gamma}{\Gamma\left((m+1)\gamma+1\right)} \dfrac{1}{\sqrt{(2m+3)\gamma}} \longrightarrow 0$ as $m \longrightarrow \infty$. Then $\lim_{m\longrightarrow\infty} \left\|\phi(t) - \phi_m(t)\right\|_{w_\gamma} = 0$, which means that the approximation $\phi_m(t)$ converges to the function $\phi(t)$.

Let E_η be the error vector of the operational matrix of fractional integration $H^{(\eta)}$ and is given by

$$E_\eta = I^\eta \rho_\gamma(t) - H^{(\eta)}\rho_\gamma(t), \quad E_\eta = \left(e_0, e_1, \ldots, e_m\right)^T. \tag{4.2}$$

This vector satisfies the following [6]:

$$\|e_n\|_{w_\gamma} \leq \sum_{j=n}^{m} \left| C_{jn} \frac{\Gamma(j\gamma+1)}{\Gamma\left(j\gamma+\eta+1\right)} \right| \frac{\bar{K}_\gamma}{\Gamma((m+1)\gamma+1)} \frac{1}{\sqrt{(2m+3)\gamma}}, \quad n = 0, 1, \ldots, m, \tag{4.3}$$

where

$$\bar{K}_\gamma = \sup\left\{|D^{(m+1)\gamma} t^{j\gamma+\eta}|\right\}, \quad t \in (0,1].$$

From inequality (4.3), it can be seen that by increasing the number of FCHFs, the error vector E_η tends to zero.

By using (2.9) and considering $z(t) = D^\eta x(t)$, it is easy to write the FVP (3.1)–(3.2) in the following equivalent form,

Minimize

$$J(z) = \int_0^1 L\left(t, I^\eta z(t) + \sum_{k=0}^{\lceil\eta\rceil-1} \frac{p_k}{k!} t^k, I^{\eta-\eta_1} z(t) + \sum_{k=\lceil\eta_1\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_1+1)} t^{k-\eta_1}, \ldots, I^{\eta-\eta_q} z(t) + \sum_{k=\lceil\eta_q\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_q+1)} t^{k-\eta_q}, z(t)\right) dt.$$

Also, in the same way the FOCP (3.14)–(3.16) is equivalent to the following problem

$$\text{Minimize} \quad J(z,u) = \int_0^1 L\left(t, I^\eta z(t) + \sum_{k=0}^{\lceil\eta\rceil-1} \frac{p_k}{k!} t^k, u(t)\right) dt, \tag{4.4}$$

subject to the fractional dynamical system

$$f\left(t, I^\eta z(t) + \sum_{k=0}^{\lceil\eta\rceil-1} \frac{p_k}{k!} t^k, I^{\eta-\eta_1} z(t) + \sum_{k=\lceil\eta_1\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_1+1)} t^{k-\eta_1}, \ldots, I^{\eta-\eta_q} z(t) + \sum_{k=\lceil\eta_q\rceil}^{\lceil\eta\rceil-1} \frac{p_k}{\Gamma(k-\eta_q+1)} t^{k-\eta_q}, z(t), u(t)\right) = 0. \tag{4.5}$$

Theorem 4.2. *The approximate solutions $z_n(t) = \Lambda^T \rho_\gamma(t)$ and $u_n(t) = U^T \rho_\gamma(t)$ converge to the exact solutions of the FOCP (4.4)–(4.5), when the number of FCHFs tends to infinity.*

Proof. Let $\mathbb{S}$ be the space of all functions $(z(t), u(t))$ that satisfy the constraint (4.5). Consider $\mathbb{S}_m$ as m-dimensional subspace of $\mathbb{S}$ consisting of all functions $(\Lambda^T \rho_\gamma(t), U^T \rho_\gamma(t))$. According to Theorem 4.1, for every $(\widehat{\Lambda}^T \rho_\gamma(t), \widehat{U}^T \rho_\gamma(t)) \in \mathbb{S}_m$ there exists a unique $(\widehat{z}(t), \widehat{u}(t)) \in \mathbb{S}$ that satisfy

$$(\widehat{\Lambda}^T \rho_\gamma(t), \widehat{U}^T \rho_\gamma(t)) \longrightarrow (\widehat{z}(t), \widehat{u}(t)) \qquad \text{as} \qquad m \longrightarrow \infty. \tag{4.6}$$

This means that every element in $\mathbb{S}_m$ converges to an element in $\mathbb{S}$ when the number of FCHFs tends to infinity. Also, from (4.6) we can get

$$J(\widehat{\Lambda}^T \rho_\gamma(t), \widehat{U}^T \rho_\gamma(t)) \longrightarrow J(\widehat{z}(t), \widehat{u}(t)) \qquad \text{as} \qquad m \longrightarrow \infty. \tag{4.7}$$

Assume that $\mu_m = \inf_{\mathbb{S}_m} J$ and $\mu = \inf_{\mathbb{S}} J$. Since $\mathbb{S}_m \subseteq \mathbb{S}_{m+1}$, then $\mu_m \geq \mu_{m+1}$. Now we will show that $\lim_{m\longrightarrow\infty} \mu_m = \mu$. Given $\varepsilon > 0$ then by the definition of Infimum there exists $(z^*(t), u^*(t)) \in \mathbb{S}$ such that

$$J(z^*(t), u^*(t)) < \mu + \varepsilon. \tag{4.8}$$

Since $J(z(t), u(t))$ is continuous on $\mathbb{S}$, then for this ε, there is $\delta(\varepsilon)$ such that

$$\left| J(z(t), u(t)) - J(z^*(t), u^*(t)) \right| < \varepsilon. \tag{4.9}$$

Now suppose that there exists an element $(\widetilde{\Lambda}^T \rho_\gamma(t), \widetilde{U}^T \rho_\gamma(t)) \in \mathbb{S}_m$ such that

$$\left| J(\widetilde{\Lambda}^T \rho_\gamma(t), \widetilde{U}^T \rho_\gamma(t)) - J(z^*(t), u^*(t)) \right| < \varepsilon. \tag{4.10}$$

It is clear that for sufficiently large m the element $(\widetilde{\Lambda}^T \rho_\gamma(t), \widetilde{U}^T \rho_\gamma(t))$ exists. From Eq (4.10), we get

$$J(\widetilde{\Lambda}^T \rho_\gamma(t), \widetilde{U}^T \rho_\gamma(t)) < J(z^*(t), u^*(t)) + \varepsilon < \mu + 2\varepsilon. \tag{4.11}$$

Also,

$$J(\widetilde{\Lambda}^T \rho_\gamma(t), \widetilde{U}^T \rho_\gamma(t)) \geq \mu_m \geq \mu. \tag{4.12}$$

Then from (4.11) and (4.12), we get

$$\mu \leq \mu_m < \mu + 2\varepsilon. \tag{4.13}$$

Since ε is chosen arbitrary, then we have

$$\lim_{m\longrightarrow\infty} \mu_m = \mu. \tag{4.14}$$

□

5. Numerical examples

To illustrate the performance of the presented method, we apply it to solve some examples and we compare the obtained numerical results by our method with those in the literature. All numerical results have been obtained using Mathematica 11 software and a laptop with Intel(R) Core(TM) i7-4600M 2.90 GHz CPU.

Example 1. Consider the following FVP [23]:

$$\text{Minimize} \quad J = \int_0^1 \left(\frac{1}{2} \left(D^\eta x(t) \right)^2 - x(t) \right) dt, \tag{5.1}$$

subject to the initial condition

$$x(0) = 0. \tag{5.2}$$

The exact solution to this problem at $\eta = 1$ is $x(t) = t - \frac{1}{2}t^2$. By using the technique described in Section 3, we get

$$\begin{aligned} D^\eta x(t) &= \Lambda^T \rho_\gamma(t), \\ x(t) &= X_0^T \rho_\gamma(t), \end{aligned} \tag{5.3}$$

where X_0^T can be calculated from Eq (3.11). Equation (5.1) can be written after using Eqs (5.3) in the form

$$\text{Minimize} \quad J = \int_0^1 \left(\frac{1}{2} \left(\Lambda^T \rho_\gamma(t) \right)^2 - X_0^T \rho_\gamma(t) \right) dt. \tag{5.4}$$

In case $\eta = 1$ and $m = 2$, $\gamma = 1$, we get

$$X_0 = \begin{pmatrix} \frac{1}{18}\lambda_0 - \frac{1}{36}\lambda_1 + \frac{1}{180}\lambda_2 \\ \frac{5}{12}\lambda_0 + \frac{1}{6}\lambda_1 - \frac{1}{30}\lambda_2 \\ \frac{19}{36}\lambda_0 + \frac{11}{18}\lambda_1 + \frac{5}{18}\lambda_2 \end{pmatrix}. \tag{5.5}$$

By substituting Eq (5.5) in (5.4) and applying the necessary conditions for optimality to the resulting equation, we obtain

$$\lambda_0 - \frac{1}{3} = 0, \quad \frac{1}{3}\lambda_1 - \frac{1}{4} = 0, \quad \frac{1}{5}\lambda_2 - \frac{1}{12} = 0. \tag{5.6}$$

Solving Eqs (5.6) gives the solution $\Lambda = \left(\frac{1}{3}, \frac{3}{4}, \frac{5}{12} \right)^T$, and hence

$$x(t) = X_0^T \rho_1(t) = \left(0, \frac{1}{4}, \frac{3}{4} \right) \begin{pmatrix} 3 - 12t + 10t^2 \\ 4t - 5t^2 \\ t^2 \end{pmatrix} = t - \frac{1}{2}t^2, \tag{5.7}$$

which is the exact solution. Also, for $m = 4$ and $\gamma = \frac{1}{2}$ we can again obtain the exact solution, where

$$X_0 = \begin{pmatrix} \frac{1}{495}\lambda_0 - \frac{2}{275}\lambda_1 + \frac{142}{17325}\lambda_2 - \frac{137}{34650}\lambda_3 + \frac{1}{1386}\lambda_4 \\ \frac{1}{55}\lambda_0 + \frac{4}{275}\lambda_1 - \frac{8}{275}\lambda_2 + \frac{61}{3850}\lambda_3 - \frac{1}{330}\lambda_4 \\ -\frac{31}{693}\lambda_0 + \frac{14}{165}\lambda_1 + \frac{26}{495}\lambda_2 - \frac{23}{495}\lambda_3 + \frac{1}{99}\lambda_4 \\ \frac{16}{495}\lambda_0 + \frac{1}{275}\lambda_1 + \frac{532}{2475}\lambda_2 + \frac{329}{2475}\lambda_3 - \frac{7}{165}\lambda_4 \\ -\frac{3}{385}\lambda_0 + \frac{39}{550}\lambda_1 + \frac{87}{550}\lambda_2 + \frac{117}{275}\lambda_3 + \frac{3}{11}\lambda_4 \end{pmatrix}. \tag{5.8}$$

Similarly, by using Eq (5.8) and applying the necessary conditions for optimality, we get the solution $\Lambda = \left(\frac{1}{5}, \frac{3}{5}, \frac{20}{21}, \frac{16}{15}, \frac{18}{35}\right)^T$, which yields

$$x(t) = X_0^T \rho_{\frac{1}{2}}(t) = \left(0, 0, \frac{1}{21}, \frac{1}{3}, \frac{11}{14}\right) \begin{pmatrix} 5 - 60t^{\frac{1}{2}} + 210t - 280t^{\frac{3}{2}} + 126t^2 \\ 20t^{\frac{1}{2}} - 105t + 168t^{\frac{3}{2}} - 84t^2 \\ 21t - 56t^{\frac{3}{2}} + 36t^2 \\ 8t^{\frac{3}{2}} - 9t^2 \\ t^2 \end{pmatrix} = t - \frac{1}{2}t^2. \tag{5.9}$$

Figure 1 displays the approximate solution $x(t)$ for various values of η, while Figure 2 shows the approximate solution $x(t)$ for several values of m. From Figure 1, it can be observed that as η approaches to 1, the approximate solution $x(t)$ converges to the integer-order solution and from Figure 2, it is clear that the exact solution can be obtained by using a few terms of FCHFs. Table 1 shows a comparison of the computational results obtained by the present method and those in [23] in terms of the optimal values of the performance index J. It can be seen that the introduced method outperforms the method in [23] since we achieve lower values on 2 and the same values on 2 out of all 4 tests. Furthermore, the authors of [23] used nine terms of the fractional-order Gegenbauer functions to get an approximate solution for $\eta = 1$, while we used only three terms of FCHFs to achieve the exact solution. This means that the presented method is more coincidental with the exact solution than [23] for this problem and can find solutions with high accuracy within acceptable computational costs.

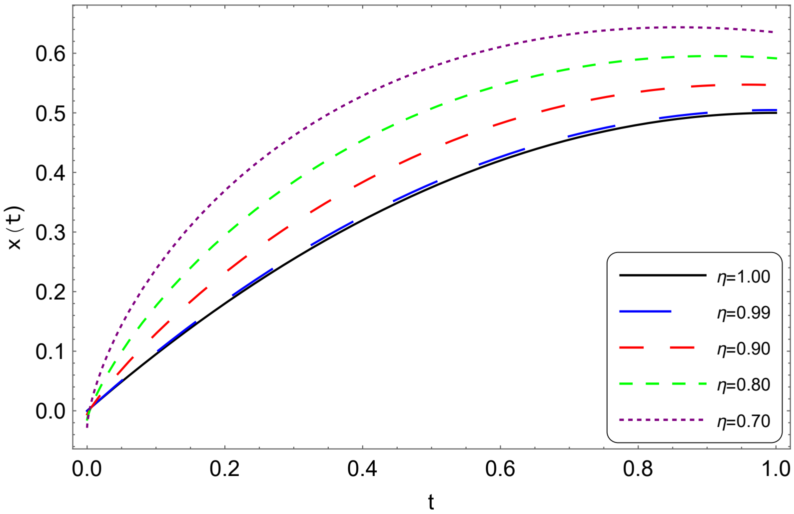

Figure 1. The behavior of the approximate solution $x(t)$ for various values of η at $m = 2$ and $\gamma = \eta$ for Example 1.

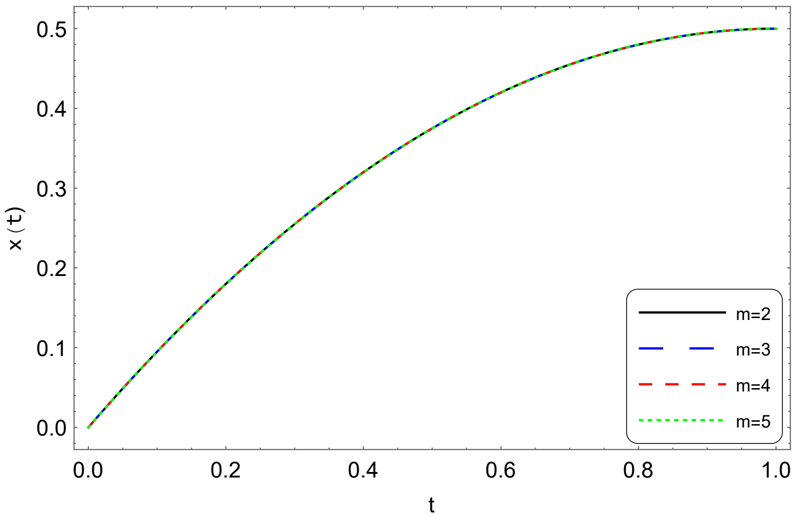

Figure 2. The behavior of the approximate solution $x(t)$ for various values of m at $\eta = \gamma = 1$ for Example 1.

Table 1. Comparison of the optimal values of J for several choices of η and $\gamma = \eta$ for Example 1.

	Our method	El-Kalaawy's [23]
η	$m = 2$	$m = 8$
1.00	−0.166667	−0.166667
0.99	−0.169199	−0.169299
0.90	−0.193088	−0.193051
0.80	−0.221813	−0.221687

Example 2. Consider the following FVP [23, 30]:

$$\text{Minimize} \quad J = \int_0^1 \left((D^\eta x(t))^2 + t\,(D^\eta x(t)) \right) dt, \tag{5.10}$$

subject to the initial condition

$$x(0) = 0. \tag{5.11}$$

The exact solution of this problem for $\eta = 1$ is $x(t) = -\frac{1}{4}t^2$. Using the proposed method yields

$$\begin{aligned} D^\eta x(t) &= \Lambda^T \rho_\gamma(t), \\ x(t) &= X_0^T \rho_\gamma(t), \end{aligned} \tag{5.12}$$

where X_0^T can be calculated from Eq (3.11). Equation (5.12) transforms Eq (5.10) to

$$\text{Minimize} \quad J = \int_0^1 \left(\left(\Lambda^T \rho_\gamma(t)\right)^2 + t\left(\Lambda^T \rho_\gamma(t)\right) \right) dt. \tag{5.13}$$

When $\eta = 1$ and $m = 2$, $\gamma = 1$, we get

$$X_0 = \begin{pmatrix} \frac{1}{18}\lambda_0 - \frac{1}{36}\lambda_1 + \frac{1}{180}\lambda_2 \\ \frac{5}{12}\lambda_0 + \frac{1}{6}\lambda_1 - \frac{1}{30}\lambda_2 \\ \frac{19}{36}\lambda_0 + \frac{11}{18}\lambda_1 + \frac{5}{18}\lambda_2 \end{pmatrix}, \tag{5.14}$$

and the generated set of linear algebraic equations is

$$2\lambda_0 = 0, \quad \frac{2}{3}\lambda_1 + \frac{1}{12} = 0, \quad \frac{2}{5}\lambda_2 + \frac{1}{4} = 0. \tag{5.15}$$

The solution of Eqs (5.15) is $\Lambda = \left(0, -\frac{1}{8}, -\frac{5}{8}\right)^T$. Hence,

$$x(t) = X_0^T \rho_1(t) = \left(0, 0, -\frac{1}{4}\right) \begin{pmatrix} 3 - 12t + 10t^2 \\ 4t - 5t^2 \\ t^2 \end{pmatrix} = -\frac{1}{4}t^2, \tag{5.16}$$

which is the exact solution. Figures 3 and 4 display the approximate solution $x(t)$ for various values of η and m, respectively. It can be observed that as η tends to 1, the approximate solution $x(t)$ converges to the integer-order solution and the exact solution can be achieved with a few terms of FCHFs. The optimal value of the performance index J for various values of η is represented in Table 2. Problem (5.10)–(5.11) has been solved in [23, 30]. By comparing our solution with those in [23, 30], it can be seen that the presented method is more accurate since we get the exact solution for $\eta = 1$ with three terms of FCHFs, while they used nine terms of the fractional-order Gegenbauer functions and eight terms of Haar wavelet functions in [23] and [30], respectively to get an approximate solution.

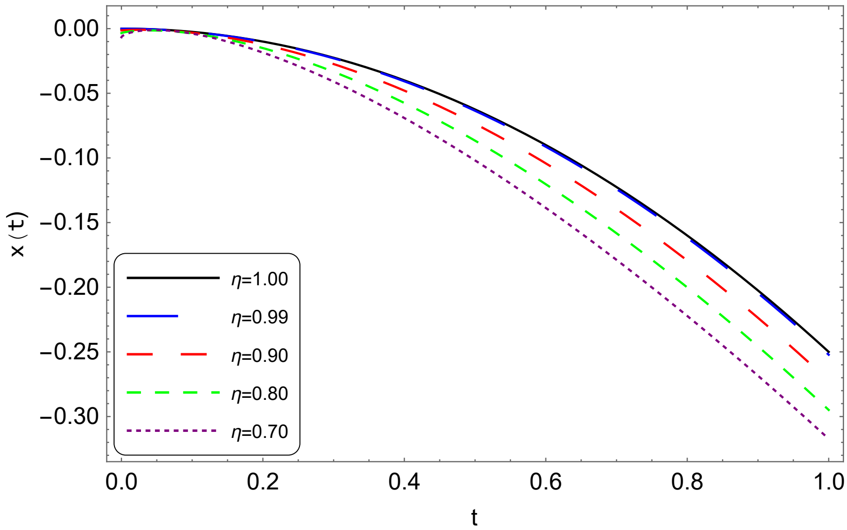

Figure 3. The behavior of the approximate solution $x(t)$ for various values of η at $m = 2$ and $\gamma = \eta$ for Example 2.

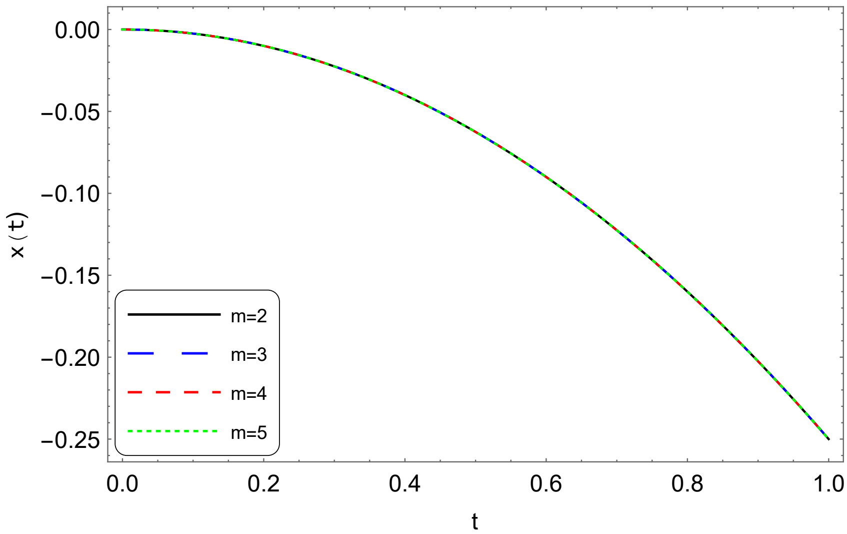

Figure 4. The behavior of the approximate solution $x(t)$ for various values of m at $\eta = \gamma = 1$ for Example 2.

Table 2. The optimal value of J for various choices of η at $m = 2$ and $\gamma = \eta$ for Example 2.

η	1.00	0.99	0.90	0.80	0.7
J	−0.0833333	−0.0833333	−0.0833322	−0.0833304	−0.0833298

Example 3. Consider the following FOCP [21, 38]:

$$\text{Minimize} \quad J = \int_0^1 \left((u(t) - t)^2 + \left(x(t) - \frac{t^{\eta+1}}{\Gamma(\eta+2)} - \frac{t^{\eta}}{\Gamma(\eta+1)} \right)^2 \right) dt, \tag{5.17}$$

subject to the fractional dynamical system

$$D^{\eta} x(t) - u(t) - 1 = 0, \quad 0 \le \eta \le 1, \tag{5.18}$$

and the initial condition

$$x(0) = 0. \tag{5.19}$$

The exact solution for $0 \le \eta \le 1$ is

$$\begin{aligned} x(t) &= \frac{t^{\eta+1}}{\Gamma(\eta+2)} + \frac{t^{\eta}}{\Gamma(\eta+1)}, \\ u(t) &= t. \end{aligned} \tag{5.20}$$

By applying the method described in Section 3, we obtain

$$\begin{aligned} D^{\eta} x(t) &= \Lambda^T \rho_{\gamma}(t), \\ x(t) &= X_0^T \rho_{\gamma}(t), \\ u(t) &= U^T \rho_{\gamma}(t). \end{aligned} \tag{5.21}$$

where X_0^T can be calculated from Eq (3.11). Hence, Eqs (5.17) and (5.18) take the form

$$\text{Minimize} \quad J = \int_0^1 \left((U^T \rho_{\gamma}(t) - t)^2 + \left(X_0^T \rho_{\gamma}(t) - \frac{t^{\eta+1}}{\Gamma(\eta+2)} - \frac{t^{\eta}}{\Gamma(\eta+1)} \right)^2 \right) dt, \tag{5.22}$$

subject to

$$\Lambda^T \rho_{\gamma}(t) - U^T \rho_{\gamma}(t) - 1 = 0, \quad 0 \le \eta \le 1. \tag{5.23}$$

Collocating Eq (5.23) at the nodes $t_j = \frac{j}{m}$, $j = 0, 1, \ldots, m$ and define the Lagrange function J^* by introducing one Lagrange multiplier μ_j for every constraint, we get

$$\begin{aligned} J^* = &\int_0^1 \left((U^T \rho_{\gamma}(t) - t)^2 + \left(X_0^T \rho_{\gamma}(t) - \frac{t^{\eta+1}}{\Gamma(\eta+2)} - \frac{t^{\eta}}{\Gamma(\eta+1)} \right)^2 \right) dt + \mu_0 \left(\Lambda^T \rho_{\gamma}(t_0) - U^T \rho_{\gamma}(t_0) - 1 \right) \\ &+ \ldots + \mu_m \left(\Lambda^T \rho_{\gamma}(t_m) - U^T \rho_{\gamma}(t_m) - 1 \right). \end{aligned} \tag{5.24}$$

With $\eta = 1$ and $m = 2$, $\gamma = 1$, we get

$$X_0 = \begin{pmatrix} \frac{1}{18}\lambda_0 - \frac{1}{36}\lambda_1 + \frac{1}{180}\lambda_2 \\ \frac{5}{12}\lambda_0 + \frac{1}{6}\lambda_1 - \frac{1}{30}\lambda_2 \\ \frac{19}{36}\lambda_0 + \frac{11}{18}\lambda_1 + \frac{5}{18}\lambda_2 \end{pmatrix}, \tag{5.25}$$

and the generated system of linear algebraic equations is

$$
\begin{aligned}
\frac{7}{30}\lambda_0 + \frac{31}{180}\lambda_1 + \frac{1}{20}\lambda_2 + 3\mu_0 - \frac{1}{2}\mu_1 + \mu_2 - \frac{79}{180} &= 0,\\
\frac{31}{180}\lambda_0 + \frac{61}{360}\lambda_1 + \frac{23}{360}\lambda_2 + \frac{3}{4}\mu_1 - \mu_2 - \frac{41}{90} &= 0,\\
\frac{1}{20}\lambda_0 + \frac{23}{360}\lambda_1 + \frac{19}{600}\lambda_2 + \frac{1}{4}\mu_1 + \mu_2 - \frac{17}{90} &= 0,\\
-\frac{1}{2}\lambda_0 + \frac{3}{4}\lambda_1 + \frac{1}{4}\lambda_2 + \frac{1}{2}u_0 - \frac{3}{4}u_1 - \frac{1}{4}u_2 - 1 &= 0,\\
\lambda_0 - \lambda_1 + \lambda_2 - u_0 + u_1 - u_2 - 1 &= 0,\\
2u_0 - 3\mu_0 + \frac{1}{2}\mu_1 - \mu_2 &= 0,\\
\frac{2}{3}u_1 - \frac{3}{4}\mu_1 + \mu_2 - \frac{1}{6} &= 0,\\
\frac{2}{5}u_2 - \frac{1}{4}\mu_1 - \mu_2 - \frac{1}{2} &= 0,\\
3\lambda_0 - 3u_0 - 1 &= 0.
\end{aligned}
\tag{5.26}
$$

The solution of system (5.26) is $\Lambda = \left(\frac{1}{3}, \frac{5}{4}, \frac{35}{12}\right)^T$, $U = \left(0, \frac{1}{4}, \frac{5}{4}\right)^T$ and $\mu = \left(0, 0, 0\right)^T$, which leads to the exact solution for $\eta = 1$. The approximate solutions $x(t)$ and $u(t)$ for various values of η and m are plotted in Figures 5 and 6, respectively. It can be seen that as η approaches to 1, the approximate solutions $x(t)$ and $u(t)$ converge to the integer-order solutions and the exact solutions can be achieved with a few terms of FCHFs. Table 3 shows the absolute errors of $x(t)$ and $u(t)$ for various values of η. Table 4 provides a comparison between the results obtained by the proposed method at $m = 6$, $\gamma = \eta$ and those in [38] in terms of the optimal values of the performance index J. It can be observed that the optimal values achieved by the presented method are less than those in [38] for almost all values of η. This means that the introduced method is in more agreement with the exact solution than [38] for this problem and can find solutions with high accuracy within acceptable computational costs.

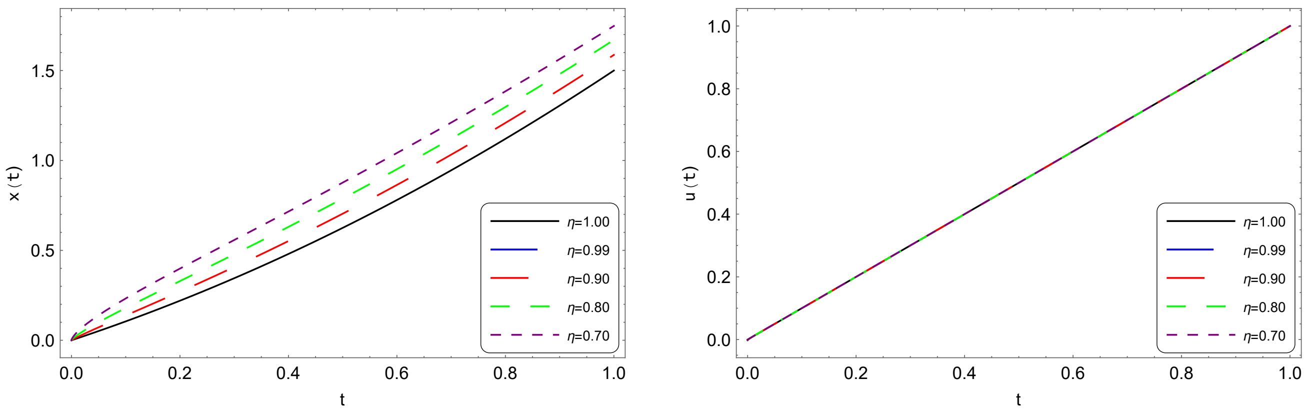

Figure 5. The behavior of the approximate solutions, $x(t)$ (left side) and $u(t)$ (right side), for various choices of η at $m = 6$ and $\gamma = \eta$ for Example 3.

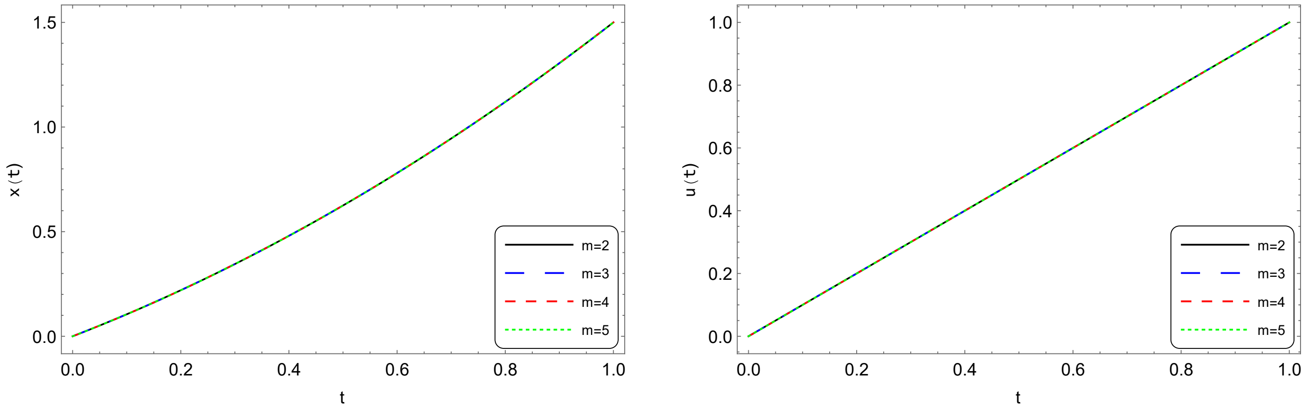

Figure 6. The behavior of the approximate solutions, $x(t)$ (left side) and $u(t)$ (right side), for various choices of m at $\eta = \gamma = 1$ for Example 3.

Table 3. The absolute errors of $x(t)$ and $u(t)$ for various choices of η at $m = 6$ and $\gamma = \eta$ for Example 3.

	$x(t)$			$u(t)$		
t	$\eta = 0.70$	$\eta = 0.80$	$\eta = 0.90$	$\eta = 0.70$	$\eta = 0.80$	$\eta = 0.90$
0.0	5.16165×10^{-5}	4.90724×10^{-5}	2.86898×10^{-5}	2.05227×10^{-3}	2.30118×10^{-3}	1.62647×10^{-3}
0.1	1.09465×10^{-5}	1.40953×10^{-5}	7.24095×10^{-6}	1.31687×10^{-4}	1.63971×10^{-4}	7.90082×10^{-5}
0.2	1.26782×10^{-5}	7.65696×10^{-6}	3.63482×10^{-7}	5.14652×10^{-5}	5.10128×10^{-7}	6.22997×10^{-5}
0.3	2.81197×10^{-6}	9.14049×10^{-6}	7.55975×10^{-6}	5.91737×10^{-5}	1.11909×10^{-4}	9.24534×10^{-5}
0.4	1.15633×10^{-5}	7.07895×10^{-6}	6.55911×10^{-7}	3.55884×10^{-5}	3.19948×10^{-6}	3.79693×10^{-5}
0.5	7.81750×10^{-6}	1.14683×10^{-5}	7.68199×10^{-6}	5.83855×10^{-5}	8.90662×10^{-5}	6.51814×10^{-5}
0.6	7.41367×10^{-6}	1.88089×10^{-6}	2.30957×10^{-6}	5.15458×10^{-6}	2.90472×10^{-5}	5.06001×10^{-5}
0.7	1.30771×10^{-5}	1.35627×10^{-5}	7.52784×10^{-6}	5.77482×10^{-5}	7.29243×10^{-5}	4.69784×10^{-5}
0.8	2.65467×10^{-6}	2.32250×10^{-6}	4.25030×10^{-6}	1.34246×10^{-5}	4.34349×10^{-5}	5.36115×10^{-5}
0.9	1.77025×10^{-5}	1.79426×10^{-5}	1.06507×10^{-5}	6.65696×10^{-5}	9.16151×10^{-5}	7.27357×10^{-5}
1.0	4.73750×10^{-5}	4.65554×10^{-5}	2.80269×10^{-5}	1.57216×10^{-4}	2.25944×10^{-4}	1.96024×10^{-4}

Table 4. Comparison of the optimal values of J for several choices of η for Example 3.

η	Our method	Mohammadi's [38]
1.00	0.00000	1.5910×10^{-29}
0.90	1.27366×10^{-8}	8.5111×10^{-9}
0.80	1.68360×10^{-8}	7.7437×10^{-8}
0.70	8.07703×10^{-9}	3.7701×10^{-7}
0.60	1.23408×10^{-9}	1.3712×10^{-6}
0.50	-1.95295×10^{-10}	4.0965×10^{-6}

Example 4. Consider the following FOCP [8, 20, 23]:

$$\text{Minimize} \quad J = \frac{1}{2}\int_0^1 \left(x^2(t) + u^2(t)\right) dt, \tag{5.27}$$

subject to the fractional dynamical system

$$D^{\eta}x(t) + x(t) - u(t) = 0, \tag{5.28}$$

and the initial condition

$$x(0) = 1. \tag{5.29}$$

The exact solution of this problem for $\eta = 1$, is given as

$$\begin{aligned} x(t) &= \cosh\left(\sqrt{2}t\right) + c\sinh\left(\sqrt{2}t\right), \\ u(t) &= \left(\sqrt{2}c + 1\right)\cosh\left(\sqrt{2}t\right) + \left(c + \sqrt{2}\right)\sinh\left(\sqrt{2}t\right), \end{aligned} \tag{5.30}$$

where

$$c = -\frac{\cosh\left(\sqrt{2}\right) + \sqrt{2}\sinh\left(\sqrt{2}\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right) + \sinh\left(\sqrt{2}\right)}.$$

Figures 7 and 8 represent the approximate solutions $x(t)$ and $u(t)$ for various values of η and m. It can be noted that as η approaches to 1, the approximate solutions $x(t)$ and $u(t)$ tend to the exact solutions for $\eta = 1$. Table 5 displays the absolute errors of $x(t)$ and $u(t)$ for various values of m. It can be seen that when m increases, the errors are reduced and the approximate solutions approach to the exact solutions. Tables 6, 7 and 8 show a comparison between the results obtained by the introduced method and the collected results in [23] in terms of the maximum absolute errors (MAEs) of $x(t)$, MAEs of $u(t)$ and the optimal values of the performance index J. The symbol M that has used in Tables 6 and 7 denotes the total number of iterations used in [8] for solving problem (5.27)–(5.29). From Tables 6 and 7, it can be observed that MAEs obtained by the presented method are less than those in [8, 23] for all the listed values in these tables, while our method achieves better results than [20] for MAEs of $u(t)$ only. However, the proposed method is also better than [20] since we use a number of the basis functions smaller than the used number of the basis functions in [20]. From Table 8, it can be seen that our results are in good agreement with the results in [23] and slightly different from those in [7, 8]. Therefore, we can say that the presented method is efficient and superior to the methods [8, 20, 23] for this problem.

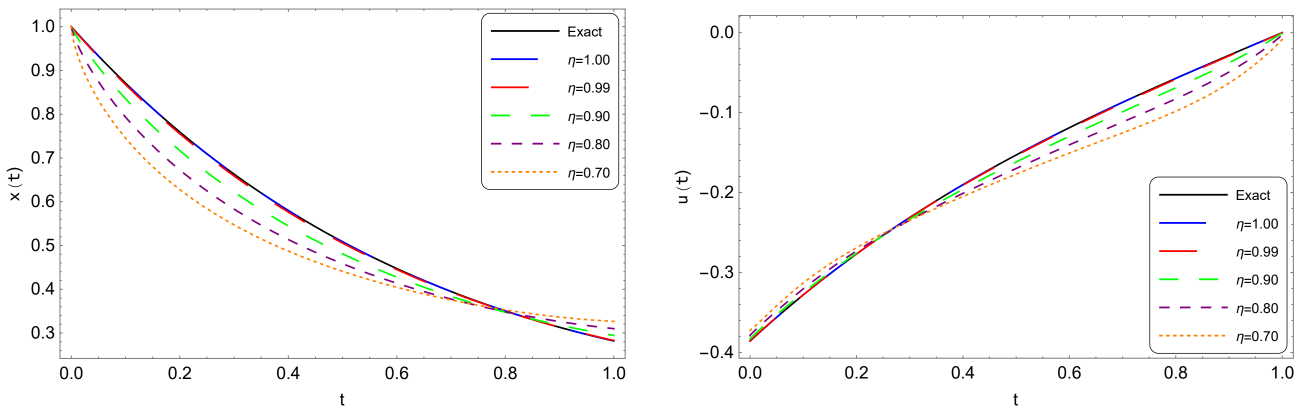

Figure 7. The behavior of the approximate solutions, $x(t)$ (left side) and $u(t)$ (right side), for various values of η at $m = 6$ and $\gamma = \eta$ with the exact solutions for Example 4.

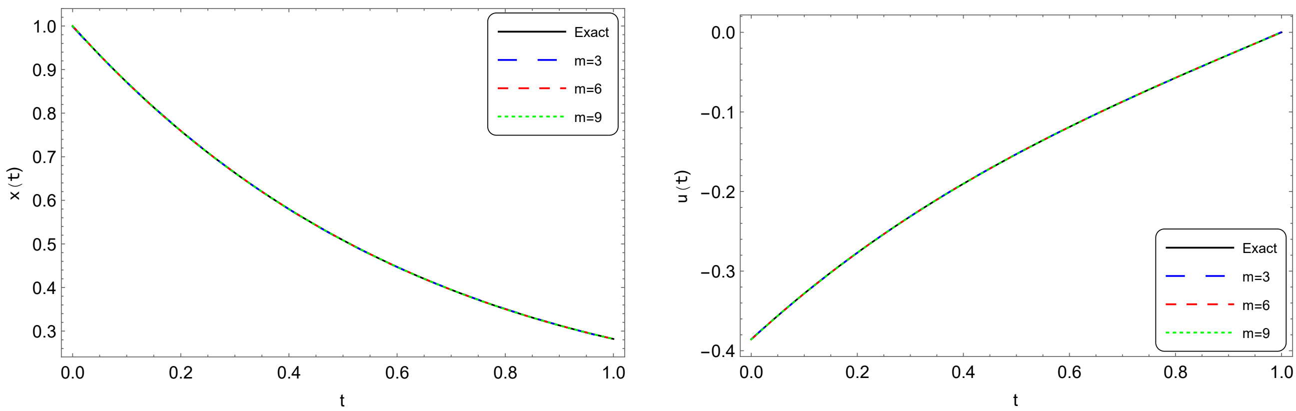

Figure 8. The behavior of the approximate solutions, $x(t)$ (left side) and $u(t)$ (right side), for various choices of m at $\eta = \gamma = 1$ with the exact solutions for Example 4.

Table 5. The absolute errors of $x(t)$ and $u(t)$ for several choices of m at $\eta = \gamma = 1$ for Example 4.

	$x(t)$			$u(t)$		
t	$m = 4$	$m = 7$	$m = 10$	$m = 4$	$m = 7$	$m = 10$
0.0	8.99562×10^{-5}	1.59413×10^{-8}	7.60751×10^{-13}	4.83358×10^{-5}	4.79765×10^{-9}	4.09006×10^{-13}
0.1	3.66886×10^{-5}	9.18310×10^{-12}	2.10942×10^{-13}	1.94735×10^{-5}	5.87345×10^{-11}	1.15075×10^{-13}
0.2	1.00765×10^{-5}	3.14893×10^{-9}	2.30149×10^{-13}	6.26409×10^{-6}	8.94861×10^{-10}	1.16074×10^{-13}
0.3	2.64695×10^{-5}	4.33599×10^{-9}	2.32148×10^{-13}	1.36257×10^{-5}	1.33000×10^{-9}	1.17600×10^{-13}
0.4	2.53003×10^{-5}	3.01195×10^{-10}	8.25229×10^{-13}	1.40474×10^{-5}	1.34353×10^{-11}	3.33511×10^{-13}
0.5	4.23504×10^{-6}	4.32806×10^{-9}	1.19904×10^{-14}	1.27483×10^{-6}	1.30256×10^{-9}	3.77476×10^{-15}
0.6	2.91334×10^{-5}	9.54414×10^{-10}	4.66849×10^{-13}	1.52015×10^{-5}	3.64447×10^{-10}	1.77192×10^{-13}
0.7	2.14187×10^{-5}	1.22907×10^{-10}	8.15603×10^{-12}	1.21062×10^{-5}	1.21618×10^{-9}	3.20079×10^{-12}
0.8	1.72645×10^{-5}	3.59572×10^{-9}	5.32663×10^{-12}	8.42679×10^{-6}	1.13498×10^{-9}	2.18819×10^{-12}
0.9	3.46536×10^{-5}	5.28987×10^{-10}	1.16771×10^{-10}	1.88604×10^{-5}	2.20655×10^{-10}	4.43365×10^{-11}
1.0	8.99549×10^{-5}	1.59413×10^{-8}	7.60947×10^{-13}	4.83361×10^{-5}	4.79765×10^{-9}	4.08396×10^{-13}

Table 6. Comparison of the MAEs of $x(t)$ for various values of m at $\eta = \gamma = 1$ for Example 4.

Our method		El-Kalaawy's [23]		Alizadeh's [8]		Doha's [20]	
m	MAE	m	MAE	M	MAE	m	MAE
2	1.15481×10^{-2}	2	1.15817×10^{-2}	10	2.18284×10^{-2}	3	1.00212×10^{-3}
4	3.76040×10^{-5}	4	8.99562×10^{-5}	30	5.05794×10^{-4}	5	5.05995×10^{-6}
6	9.75002×10^{-8}	6	3.11879×10^{-7}	50	4.19787×10^{-6}	7	1.29671×10^{-8}
8	2.79259×10^{-11}	8	6.08778×10^{-10}	80	1.63814×10^{-9}	9	2.01182×10^{-11}

Table 7. Comparison of the MAEs of $u(t)$ for various choices of m at $\eta = \gamma = 1$ for Example 4.

Our method		El-Kalaawy's [23]		Alizadeh's [8]		Doha's [20]	
m	MAE	m	MAE	M	MAE	m	MAE
2	6.22290×10^{-3}	2	6.22299×10^{-3}	10	6.76558×10^{-2}	3	5.99596×10^{-3}
4	2.01535×10^{-5}	4	4.83361×10^{-5}	30	2.95124×10^{-4}	5	3.31436×10^{-5}
6	5.29302×10^{-8}	6	1.67582×10^{-7}	50	1.50618×10^{-6}	7	1.06056×10^{-7}
8	1.91647×10^{-12}	8	3.27107×10^{-10}	80	6.18633×10^{-9}	9	1.94284×10^{-10}

Table 8. Comparison of the optimal values of J for various values of η and $\gamma = \eta$ for Example 4.

η	Our method	El-Kalaawy's [23]	Alizadeh's [8]	Akbarian's [7]
1.00	0.192909	0.192909	0.192909	0.192909
0.99	0.19153	0.19153	0.19153	0.19153
0.90	0.17953	0.17953	0.17953	0.17952
0.80	0.16708	0.16708	0.16711	0.16729

Example 5. Consider the following FOCP [8, 20, 23]:

$$\text{Minimize} \quad J = \frac{1}{2}\int_0^1 \left(x^2(t) + u^2(t)\right) dt, \tag{5.31}$$

subject to the fractional dynamical system

$$D^\eta x(t) - tx(t) - u(t) = 0, \tag{5.32}$$

and the initial condition

$$x(0) = 1. \tag{5.33}$$

Figure 9 presents the approximate solutions $x(t)$ and $u(t)$ for various choices of η. We can observe that as η approaches to 1, the approximate solutions $x(t)$ and $u(t)$ tend to the solutions for $\eta = 1$. Table 9 displays a comparison between the results obtained by the introduced method and the collected results in [23, 38] in terms of the optimal values of the performance index J. It can be seen that our results are in good agreement with the results in [23] and slightly different from those in [7, 8, 20, 38]. This means that the presented method is compatible with some other methods for this problem.

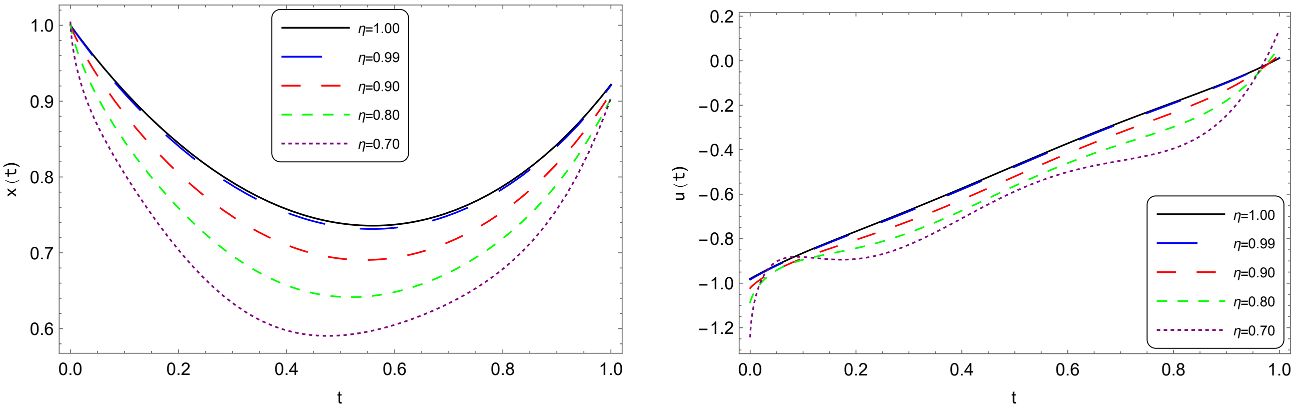

Figure 9. The behavior of the approximate solutions, $x(t)$ (left side) and $u(t)$ (right side), for various values of η at $m = 6$ and $\gamma = \eta$ for Example 5.

Table 9. Comparison of the optimal values of J for various choices of η and $\gamma = \eta$ for Example 5.

η	Our method	El-Kalaawy's [23]	Alizadeh's [8]	Doha's [20]	Akbarian's [7]	Mohammadi's [38]
1.00	0.48427	0.48427	0.48426	0.48426	0.48427	0.48426
0.99	0.48346	0.48346	0.48346	0.48346	0.48347	0.48346
0.90	0.47588	0.47588	0.47593	0.47588	0.47605	0.47588
0.80	0.46698	0.46698	0.46722	0.46697	0.46722	0.46697

6. Conclusions

The FCHFs and their properties have been used to develop an efficient numerical method to handle FVPs and FOCPs. The given fractional optimization problem is transformed to a system of algebraic equations in the unknown expansion coefficients. The presented method can be adapted to execute on computers. The error of the introduced method is evaluated. The applicability and the efficiency of the presented method are also tested on five problems. Comparisons with exact solutions and some recent numerical methods have been performed. From the obtained results, we can observe that the solutions are either coincidence with the exact solutions if we know the exact solutions or in good agreement with other methods in the literature if the exact solutions are unknown. Another advantage of our method is that there is no need for excessive computations since a small number of FCHFs is required to achieve high accuracy. Therefore, we can say that the new method is helpful, accurate and its performance is quite satisfactory which demonstrates the significance of the presented method for solving FVPs and FOCPs.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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