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Research article

Solving the time-fractional inverse Burger equation involving fractional Heydari-Hosseininia derivative

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Abstract: In this paper, we present a powerful numerical scheme based on energy boundary functions to get the approximate solutions of the time-fractional inverse Burger equation containing HH-derivative.This problem has never been investigated earlier so, this is our motivation to work on this important problem. Some numerical examples are presented to verify the efficiency of the presented technique. Graphs of the exact and numerical solutions along with the plot of absolute error are provided for each example. Tables are given to see and compare the results point by point for each example.

Keywords: Optimization; fractional inverse Burger equation; Heydari-Hosseininia derivative
Mathematics Subject Classification: 34K37, 35R11, 90C32

1. Introduction

This study is devoted to investigate the numerical solutions of the time-fractional Burger equation. The time-fractional Burger equation can be regarded as the generic convection model. To see more details see [1]. Also, this model has a huge interest, physically. In fact, researchers consider this problem as a proper reduction of the Navier–Stokes system. Indeed, the flow model which consists of some important phenomena such as heat's diffusion and turbulence can be described by the Burger model. In recent years different methods have been worked on this important problem numerically and analytically. For example, an application of the fictitious time integration method for obtaining the numerical solutions of this equation can be seen in [2]. Also, to get the approximate solutions of generalized time-fractional Burgers equation a linear finite difference scheme has been worked in [3]. To get the analytical solutions of this problem, an application of the (G′/G)-expansion method can be found in [4]. Moreover, in [5] approximate analytical solution of the nonlinear fractional KdV–Burgers equation has been provided. Also, an efficient method for time-fractional damped Burger and time-sharma-tasso-Olver equations employing the FRDTM was presented in [6]. A reliable method for solving the space–time fractional Burgers and time-fractional Cahn-Allen equations can be seen in [7]. Approximate solution of the fractional Burgers model using cubic B-spline finite was obtained in [8]. Analytical solution of the time and space fractional Burgers' problem can be seen in [9]. Also, approximate solutions of the fractional Burger equation using differential transform approach can be read in [10]. Furthermore, the applications of other method on this problem can be observed in [11–16].

In recent years, many researchers have paid attention to non-integer differential and integral operators as generalizations of differential and integral operators of integer orders. These non-integer operators are able to consider any arbitrary value for their orders. There are various definitions for fractional operators. The Caputo sense is one of the most famous fractional derivatives. In spite of the beneficial properties of this operator, the chief drawback of this derivative is the singularity of its kernel. To dominate this issue, new definitions of fractional derivatives such as Caputo–Fabrizio and Atangana–Baleanu senses have been introduced. In the Caputo–Fabrizio definition, the exponential function has been employed instead of the singular kernel in the definition of the Caputo derivative, while in the Atangana–Baleanu definition, the Mittag–Leffler function has been applied. Albeit the mentioned definitions solve the issue of the singularity of the Caputo fractional derivatives, they themselves have a number of limitations. For instance, in the Caputo–Fabrizio fractional derivative, due to the structure of the kernel function it is difficult to extract a closed-form for the fractional derivative of functions. Moreover, the expansion of numerical schemes for solving differential equations involving this type of fractional derivative can be hard. On the other hand, although it is easy to work with the definition provided by Atangana and Baleanu, the fractional derivatives of analytic functions in this sense will be non-analytic functions. In the sequel, to overcome the mentioned limitations, we use a new non-singular fractional differentiation called Heydari-Hosseininia fractional derivative [17] Motivated by the mentioned arguments in the current study we investigate the time-fractional inverse Burger problem involving a new generation of fractional derivative: Heydari-Hosseininia (HH) derivative. This is the first time that this problem is under investigation. We consider the following problem:

$$^{HH}\mathcal{D}_{\tau}^{\alpha}h(z,\tau) - h(z,\tau)h_z(z,\tau) = h_{zz} + H(z,\tau), \quad 0 < z < 1, 0 < \tau < \tau_f, \tag{1.1}$$

$$h(0,\tau) = F_0(\tau), \quad h(l,\tau) = F_l(\tau). \tag{1.2}$$

where HH derivative introduced in [15] as:

$$^{HH}\mathcal{D}_{\tau}^{\alpha}f(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau} f'(c)T_1\Big(\frac{-\alpha}{1-\alpha}(\tau-c)\Big)\mathrm{d}c, \quad 0 < \alpha < 1, \tag{1.3}$$

where $T_{\alpha}(z)$ is Mittag-Leffler function represented via:

$$T_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}. \tag{1.4}$$

We have

$$h(0,\tau) = F_0(\tau), \quad h(l,\tau) = F_l(\tau), \quad h(0,\tau) = Q_0(\tau), \quad h_z(l,\tau) = Q_l(\tau), \tag{1.5}$$

$$H(0,\tau) = H_0(\tau), \quad H(l,\tau) = H_l(\tau), \quad H_z(0,\tau) = H_0^{'}(\tau), \quad h(l,\tau) = H_l^{'}(\tau). \tag{1.6}$$

We order our manuscript as follows: We explain a new notion of energy border functional equation in Section 2. We provide the method to construct $H(x,t)$ in Section 3. Section 4 solves two numerical examples. Lastly, the conclusion section is given.

2. Energy functional of boundary functions

Some implementations of energy functional boundary functions can be found in [18–26]. The structure of energy functional boundary functions is given as follows. By multiplying Eq (1.1) by $h(z,\tau)$ we obtain:

$$h(z,\tau)^{HH}\mathcal{D}_\tau^\alpha h(z,\tau) - h^2(z,\tau)h_z(z,\tau) = h(z,\tau)h_{zz}(z,\tau) + h(z,\tau)H(z,\tau). \tag{2.1}$$

Using Eq (1.5) and integration by parts from $x = 0$ to $x = l$ yields:

$$\begin{aligned} &\int_0^\ell h(z,\tau)^{HH}\mathcal{D}_\tau^\alpha h(z,\tau)dz + \int_0^\ell h_z^2(z,\tau)dz - \int_0^\ell H(z,\tau)h(z,\tau)dz \\ &= Q_\ell(\tau)F_\ell(\tau) - Q_0(\tau)F_0(\tau) + \frac{1}{3}[h^3(l,\tau) - h^3(o,\tau)] = \\ &Q_\ell(\tau)F_\ell(\tau) - Q_0(\tau)F_0(\tau) + \frac{1}{3}[F^3(l,\tau) - F^3(o,\tau)] = F(\tau). \end{aligned} \tag{2.2}$$

The above problem is an energy equation and we use it to discover $H(z,\tau)$. Thus, we describe the following relation:

$$v(z,\tau) = h(z,\tau) - G_0(z,\tau), \tag{2.3}$$

where $G_0(z,\tau)$ is the homogenization function

$$\begin{aligned} G_0(z,\tau) &= \frac{1}{\ell^3}\left[2F_0(\tau) - 2F_\ell(\tau) + Q_0(\tau)\ell + Q_\ell(\tau)\ell\right]z^3 \\ &-\frac{1}{\ell^2}\left[3F_0(\tau) - 3F_\ell(\tau) + 2Q_0(\tau)\ell + Q_\ell(\tau)\ell\right]z^2 + Q_0(\tau)z + F_0(\tau). \end{aligned} \tag{2.4}$$

Regarded to homogeneous boundary conditions:

$$\int_0^\ell [v(z,\tau)) + G_0(z,\tau)]\left[{}^{HH}\mathcal{D}_\tau^\alpha v(z,\tau) +^{HH} \mathcal{D}_\tau^\alpha G_0(z,\tau)\right]dz + \int_0^\ell \left[v_z(z,\tau) + G_0'(z,\tau)\right]^2 dz$$

$$-\int_0^\ell [v(z,\tau)+G_0(z,\tau)]\, H(z,\tau)dz = F(\tau) \tag{2.5}$$

$$v(0,\tau)=0, v(\ell,\tau)=0, \quad v_z(0,\tau)=0, \quad v_z(\ell,\tau)=0. \tag{2.6}$$

Since we are not aware of the exact value of $v(z,\tau)$, so some functions can be considered to proximate this function. We can gather the boundary function which spontaneously gladdens the boundary conditions in Eq (2.9):

$$G_j(z) = \left(x^4 - 2\ell x^3 + \ell^2 x^2\right) x^{j-1}, j \geq 1. \tag{2.7}$$

with

$$G_j(0)=0, \quad G_j(\ell)=0, \quad G_j'(0)=0, \quad G_j'(\ell)=0. \tag{2.8}$$

According to Eqs (2.8) and (2.9), it is inevitable that $\beta G_j(z), \beta \in \mathbb{R}$, is a function with boundary, if $G_j(z)$ is a function with boundary, and if $G_j(z)$ and $G_k(z)$ are functions with boundary, $G_j(y)+G_k(z)$ is said to be a boundary function. The boundary functions are closed under a scalar addition and multiplication; thus, the group of

$$\left\{G_j(z)\right\}, \quad j \geq 1. \tag{2.9}$$

and the zero component set up a linear space of homogeneous functions with boundary, represented as $\mathcal{G}$.

For identifying $H(z,\tau)$, an approximate functional equation can be obtained in the following way.

Theorem 1:

We have

$$T_j(z) = \gamma_j G_j(z), \quad j \geq 1. \tag{2.10}$$

Where $H(z,\tau)$ is solution of the following equation in terms of $T_j(z)$.

$$\begin{aligned} &\int_0^\ell \left[T_j(z)+G_0(z,\tau)\right]\left[{}^{HH}\mathcal{D}_\tau^\alpha G_0(z,\tau) - H(z,\tau)\right] dz \\ &+ \int_0^\ell \left[T_j'(z)+G_0'(z,\tau)\right]^2 dz = F(\tau), \end{aligned} \tag{2.11}$$

where

$$a_2 = \int_0^\ell G_j'(z)^2 dz, \quad a_1 = \int_0^\ell \left\{2G_0'(z,\tau)G_j'(z) + \left[{}^{HH}\mathcal{D}_\tau^\alpha G_0(z,\tau) - H(z,\tau)\right] G_j(z)\right\} dz$$

$$a_0 = \int_0^\ell \left\{G_0'(z,\tau)^2 + G_0(z,\tau)\left[{}^{HH}\mathcal{D}_\tau^\alpha G_0(z,\tau) - H(z,\tau)\right]\right\} dz - F(\tau), \tag{2.12}$$

$$\gamma_j = \frac{-a_1 - \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}. \tag{2.13}$$

Proof. $G_j(z) \in \mathcal{G}$ is an element in the linear space $\mathcal{G}$. Thus, the scalar multiplication in Eq (2.11) renders $T_j(z) \in \mathcal{G}$, which satisfies the following homogeneous boundary conditions:

$$T_j(0) = 0, \quad T_j(\ell) = 0, \quad T_j'(0) = 0, \quad T_j'(\ell) = 0, \tag{2.14}$$

with regard to Eq (2.9). The function $T_j(z)$ already satisfies the conditions in Eq (2.15) as those of Eq (2.7) for $v(z,\tau)$. The power identity Eq (2.6) is introduced to $T_j(z)$, from which we can get approximately $v(z,\tau)$ by $T_j(z)$ and obtain Eq (2.12) which is a functional energy equation of $T_j(z)$ in linear space $\mathcal{G}$. Using Eq (2.11) for $T_j(z)$ and

$$T_j'(z) = \gamma_j G_j'(z), \tag{2.15}$$

for $T_j(x)$ into Eq (2.12) we obtain:

$$a_2\gamma_j^2 + a_1\gamma_j + a_0 = 0, \tag{2.16}$$

The coefficients a_0, a_1, a_2 are described in Eq (2.13). Therefore, the solution of γ_j is obtained in Eq (2.14). So, this completes the proof. □

3. Numerical algorithm

We approximate $H(z,\tau)$ as follows:

$$H(z,\tau) = D(z,\tau) + \sum_{i=1}^{m} c_i T_i(z), \tag{3.1}$$

where

$$\begin{aligned} D(z,\tau) = &\frac{1}{\ell^3}\left[2H_0(\tau) - 2H_\ell(\tau) + H_0'(\tau)\ell + H_\ell'(\tau)\ell\right] z^3 \\ -&\frac{1}{\ell^2}\left[3H_0(\tau) - 3H_\ell(\tau) + 2H_0'(\tau)\ell + H_\ell'(\tau)\ell\right] z^2 + H_0'(t)z + H_0(\tau), \end{aligned} \tag{3.2}$$

Then, we get

$$\begin{aligned} &c_i \int_0^\ell \left[T_j(z) + G_0(z,\tau)\right] T_i(z)dz = \\ &\int_0^\ell \left[T_j(z) + G_0(z,\tau))\right]\left[{}^{HH}\mathcal{D}_\tau^\alpha h(z,\tau) G_0(z,\tau) - D(z,\tau)\right] dz + \\ &\int_0^\ell \left[T_j'(z) + G_0'(z,\tau)\right]^2 dz - F(\tau). \end{aligned} \tag{3.3}$$

Then, we obtain:

$$H(0,\tau) = D(0,\tau) + \sum_{i=1}^{m} c_i T_i(0) = H_0(\tau), D(0,\tau) = H_0(\tau), T_i(0) = 0,$$
$$H(\ell,\tau) = D(\ell,\tau) + \sum_{i=1}^{m} c_i T_i(\ell) = H_\ell(\tau), D(\ell,\tau) = H_\ell(\tau), T_i(\ell) = 0,$$
$$H_z(0,t) = D_z(0,\tau) + \sum_{i=1}^{m} c_i T_i'(0) = H_0'(\tau), \text{ due to } D_z(0,\tau) = H_0'(\tau), T_i'(0) = 0,$$
$$H_z(\ell,\tau) = D_z(\ell,\tau) + \sum_{i=1}^{m} c_i T_i'(\ell) = H_\ell'(\tau), \quad \text{ due to } D_z(\ell,\tau) = H_\ell'(\tau), T_i'(\ell) = 0,$$

where we have applied Eq (3.1) and Eq (3.3). Secondly, the coefficient matrix $\int_0^\ell T_j(z)T_i(z)dz$ in Eq (3.5) is symmetric.

(i) We give $s \in \left(0, \tau_f\right]$, and $m, \varepsilon, \gamma_j = 0,\ j = 0$, and $\mathbf{c} = (c_1, \ldots, c_{\text{rn}})^{\text{T}}$, $\mathbf{c}^0 = \mathbf{0}$.

(ii) For $\tau = 0, 1, \ldots,$

$$T_j(z) = \gamma_j G_j(z),$$
$$H(z,\tau) = D(z,\tau) + \sum_{j=1}^{m} c_j^k T_j(z),$$

and we determine

$$a_2 = \int_0^\ell G_j'(z)^2 dy,$$
$$a_1 = \int_0^\ell \left\{2G_0'(z,\tau)G_j'(z) + \left[{}^{HH}\mathcal{D}_\tau^\alpha G_0(z,\tau) - H(z,\tau)\right] G_j(z)\right\} dy,$$
$$a_0 = \int_0^\ell \left\{G_0'(y,s)^2 + \left[{}^{HH}\mathcal{D}_s^\alpha G_0(z,\tau) - H(z,\tau)\right] G_0(z,\tau)\right\} dz - F(\tau).$$

(iii) We compute

$$\gamma_j = \frac{-a_1 - \sqrt{\left|a_1^2 - 4a_0a_2\right|}}{2a_2},$$
$$T_j(z) = \gamma_j G_j(z).$$
$$T_j'(z) = \gamma_j G_j'(z),$$

(iv) We insert the above $T_j(z)$ and $T_j'(z)$ in

$$\begin{aligned} &c_i \int_0^\ell \left[T_j(z) + G_0(z,\tau)\right] T_i(z) dz \\ &= \int_0^\ell \left[T_j(z) + G_0(z,\tau)\right]\left[{}^{HH}\mathcal{D}_\tau^\alpha G_0(z,\tau) - D(z,\tau)\right] dz + \int_0^\ell \left[T_j'(z) + G_0'(z,\tau)\right]^2 dz - F(\tau). \end{aligned} \tag{3.4}$$

By solving these equations to reach c_1^{k+1} and if the convergence criteria satisfies for the norm of c^k

$$\left\|\mathbf{c}^{k+1} - \mathbf{c}^k\right\| \leq \varepsilon. \tag{3.5}$$

then we stop iterations. Otherwise, we check on (ii) for the next step. For the first iteration, $a_1^2 - 4a_0a_2$ is likely to be negative, and we employ $\left|a_1^2 - 4a_0a_2\right|$ to prevent from program interuption. In relation Eq (3.5), $\left\|\mathbf{c}^{k+1} - \mathbf{c}^k\right\|$ is the Euclidean norm for $\mathbf{c}^{k+1} - \mathbf{c}^k$.

4. Numerical results

In this section we examine the performance of the presented method by solving two examples. The computations are generated by using Matlab 2021. We take into consideration:

$$\begin{aligned}
&\hat{u}\left(0,\tau_i\right) = F_0\left(\tau_i\right)\left[1 + sR(i)\right], \quad \hat{u}\left(\ell,\tau_i\right) = F_\ell(t)[1 + sR(i)],\\
&\hat{u}_z\left(0,\tau_i\right) = Q_0\left(\tau_i\right)\left[1 + sR(i)\right], \quad \hat{u}_z(\ell,\tau) = Q_\ell\left(\tau_i\right)\left[1 + sR(i)\right],\\
&\hat{H}\left(0,\tau_i\right) = H_0\left(\tau_i\right)\left[1 + sR(i)\right], \quad \hat{H}\left(\ell,\tau_i\right) = H_\ell\left(\tau_i\right)\left[1 + sR(i)\right],\\
&\hat{H}_z\left(0,\tau_i\right) = H_0'\left(\tau_i\right)\left[1 + sR(i)\right], \quad \hat{H}_z\left(\ell,\tau_i\right) = H_\ell'\left(\tau_i\right)\left[1 + sR(i)\right].
\end{aligned} \tag{4.1}$$

Example 1:

We take into consideration

$$h(z,\tau) = z\tau(z-1)(\tau-2), \tag{4.2}$$

and the function

$$H(z,\tau) = z\tau(z-1)(\tau-2)(z\tau(\tau-2) + \tau(\tau-2)(z-1)) - 2\tau(\tau-2). \tag{4.3}$$

We apply the current method to solve this problem using different parameters. Figure 1 is devoted to show the approximate and exact solutions along with errors obtained by selecting $m = 3$, $s = 0.001$ and the fractional order $\alpha = 0.45$. Satisfactory results are gained which are displayed point by point through Table 1. Also we show the results by choosing $m = 7$, $s = 0.001$ and $\alpha = 0.55$ in Figure 2. Table 2 is responsible to display the values of results containing errors, exact and numerical solutions.

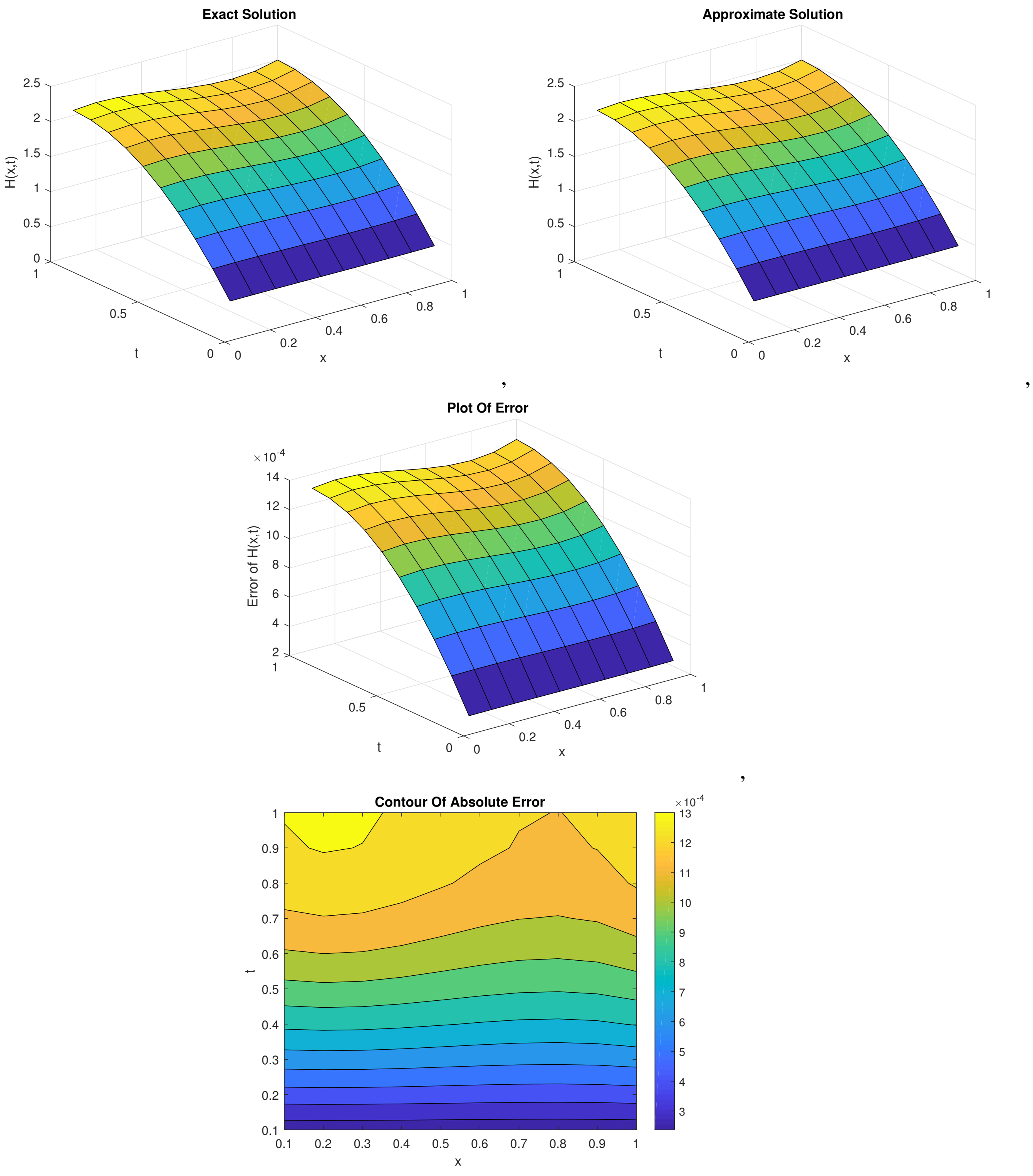

Figure 1. Simulations for $m = 3$, $s = 0.001$ and $\alpha = 0.45$ of Example 1.

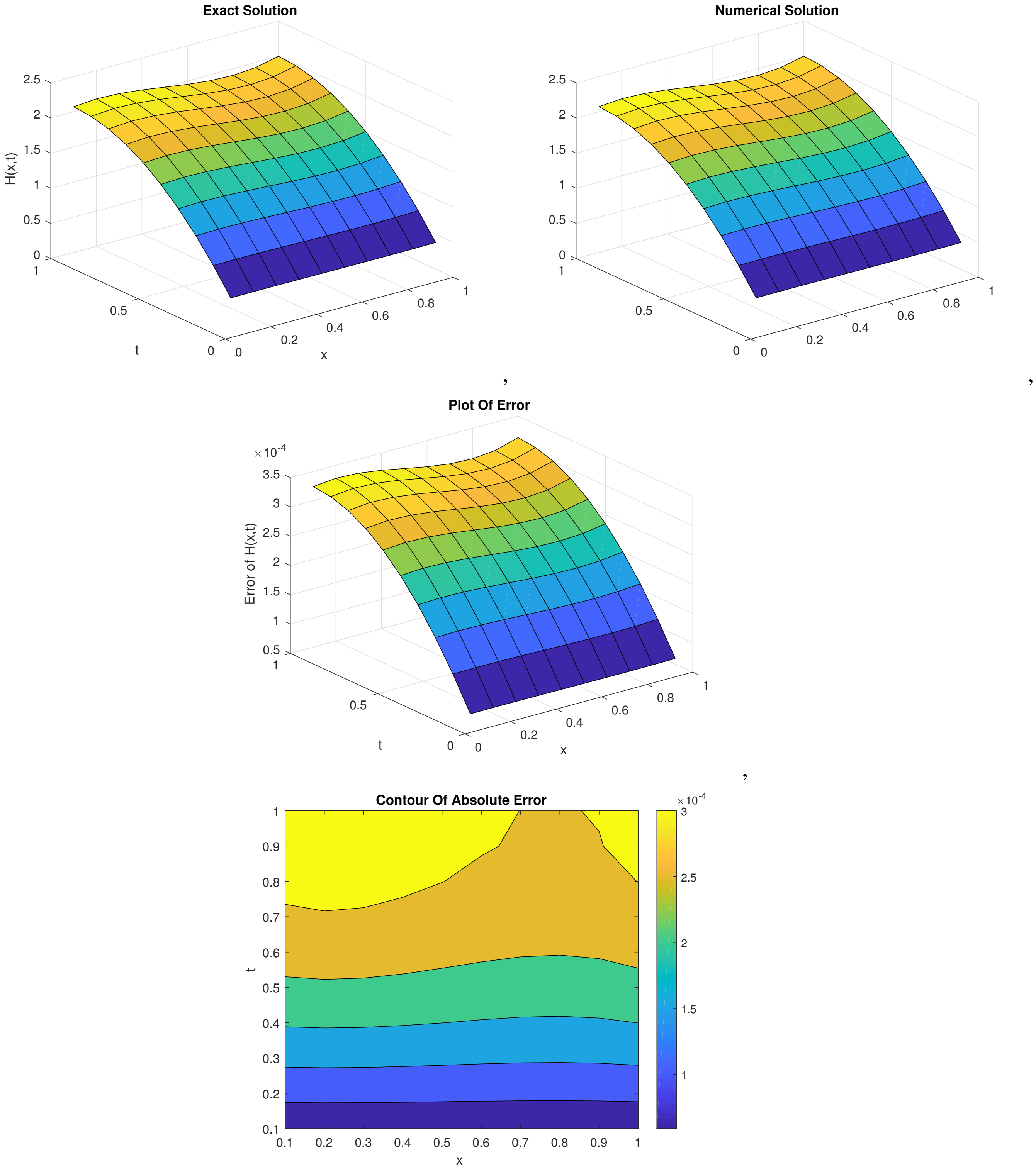

Figure 2. Simulations for $m = 7$, $s = 0.001$ and $\alpha = 0.55$ of Example 1.

Table 1. Approximate solution (AS), exact solution (ES) and absolute error (AE) for $m = 3$, $s = 0.001$ and $\alpha = 0.45$ for Example 1.

(y, s)	AS	ES	AS
(0.1,0.1)	0.3828	0.3826	2.2977e-04
(0.3,0.3)	1.0425	1.0418	6.2569e-04
(0.5,0.5)	1.5009	1.5000	9.0084e-04
(0.7,0.7)	1.7515	1.7504	0.0011
(0.9,0.9)	1.9106	1.9094	0.0011

Table 2. Approximate solution (AS), exact solution (ES) and absolute error (AE) for $m = 7$, $s = 0.001$ and $\alpha = 0.55$ for Example 1.

(y, s)	AS	ES	AE
(0.1,0.1)	0.3825	0.3826	5.9868e-05
(0.3,0.3)	1.0418	1.0417	1.6303e-04
(0.5,0.5)	1.4998	1.5000	2.3472e-04
(0.7,0.7)	1.7502	1.7504	2.7390e-04
(0.9,0.9)	1.9091	1.9094	2.9878e-04

Example 2:

Consider the following variable

$$h(z, \tau) = sin(z)exp(-\tau), \tag{4.4}$$

and the function

$$H(z, \tau) = sin(z)exp(-\tau) + cos(z)sin(z)exp(-2\tau). \tag{4.5}$$

We solve the second example via using various parameters. Figure 3 is dedicated to illustrate the numerical, analytical solutions and errors obtained by selecting $m = 2$, $s = 0.00001$ and the fractional order of $\alpha = 0.15$. Considerable results are derived which are shown point by point by Table 3. Indeed, we depict the results by selecting $m = 3$, $s = 0.001$ and $\alpha = 0.9$ in Figure 4. The values of solutions and errors can be seen in Table 4.

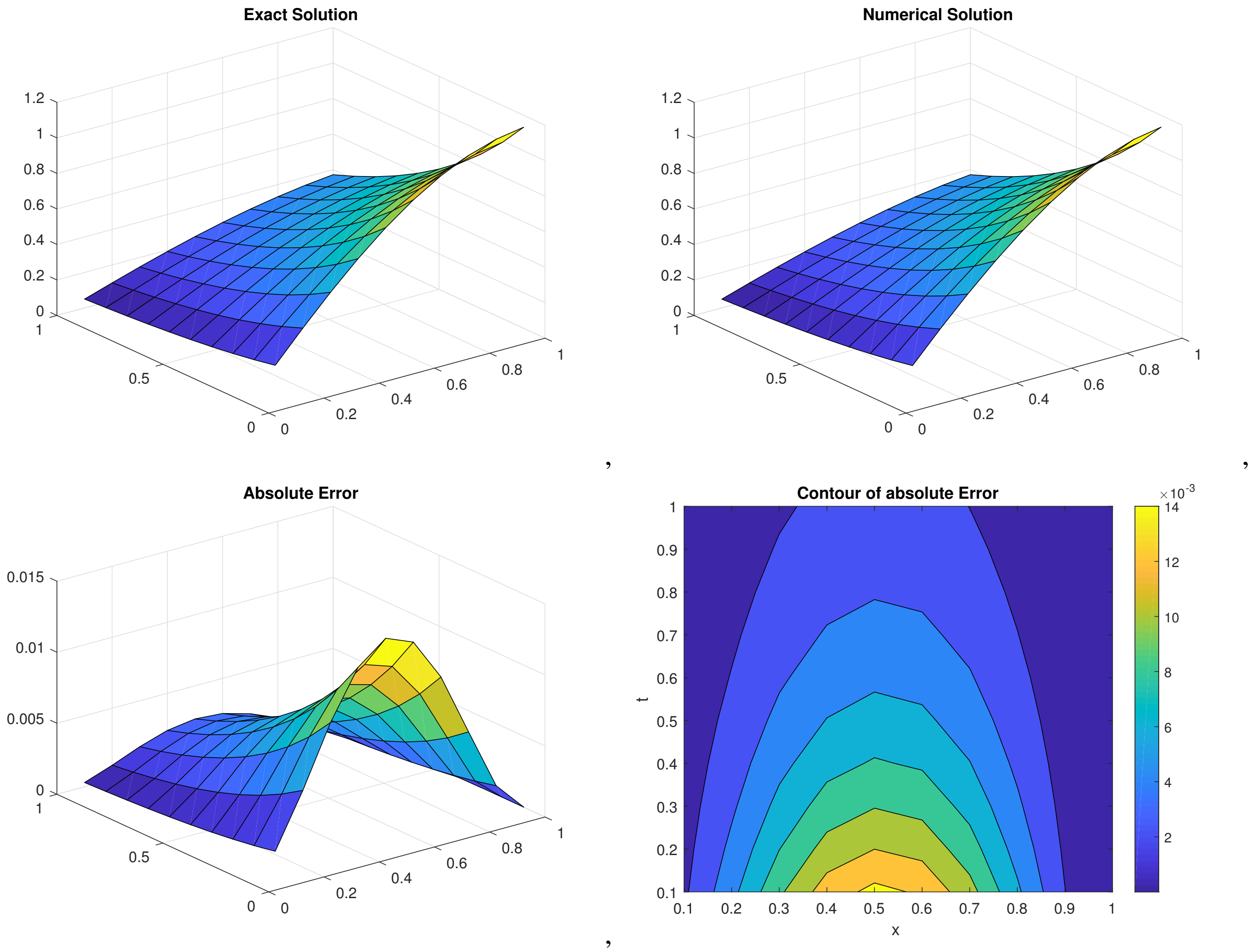

Figure 3. Simulations for $m = 2$, $s = 0.00001$ and $\alpha = 0.15$ of Example 2.

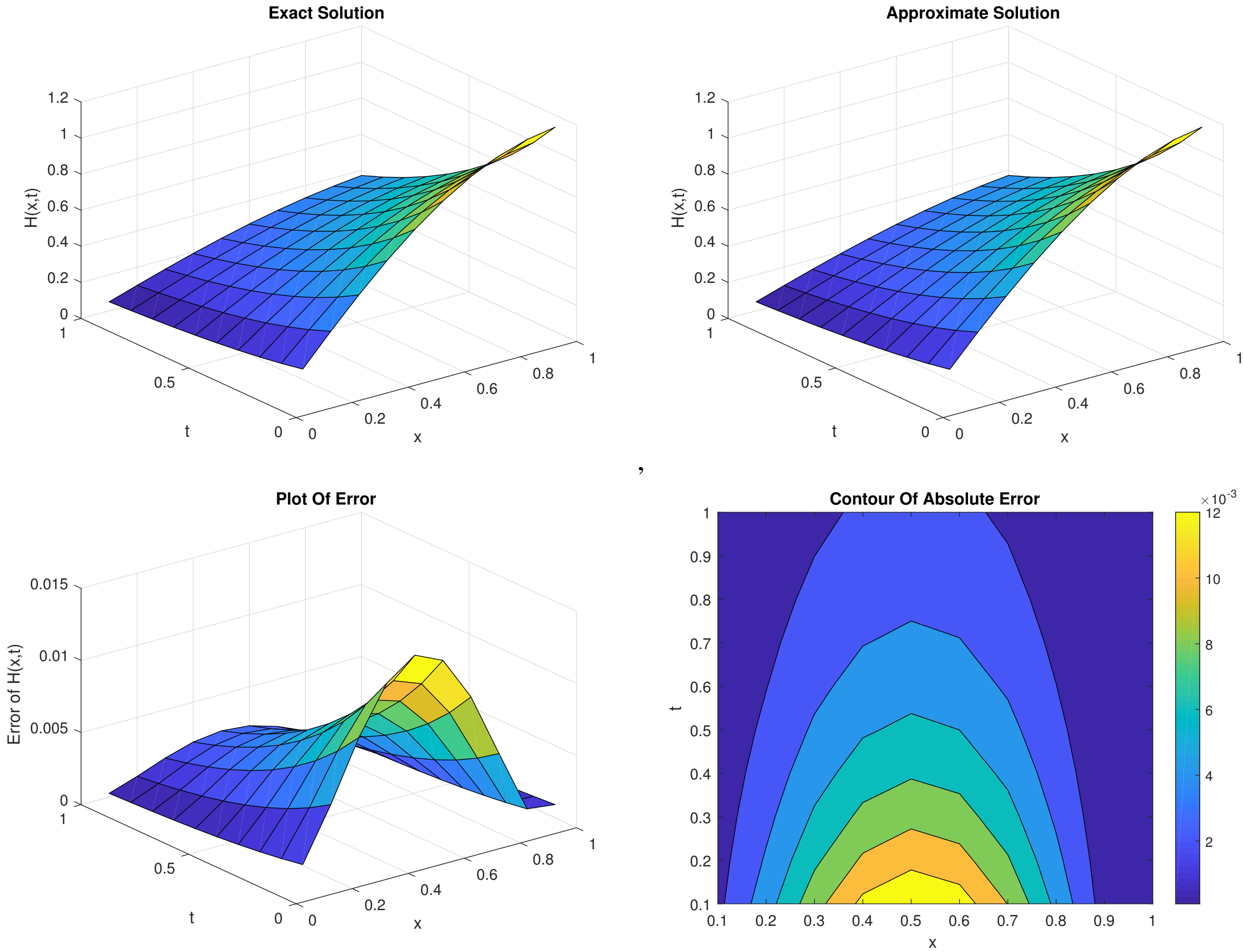

Figure 4. Simulations for $m = 3$, $s = 0.001$ and $\alpha = 0.9$ of Example 2.

Table 3. Approximate solution (AS), exact solution (ES) and absolute error (AE) for $m = 2$, $s = 0.00001$ and $\alpha = 0.15$ for Example 2.

(y, s)	AS	ES	AE
(0.1,0.1)	0.1700	0.1717	0.0016
(0.3,0.3)	0.3673	0.3739	0.0066
(0.5,0.5)	0.4388	0.4456	0.0068
(0.7,0.7)	0.4380	0.4414	0.0034
(0.9,0.9)	0.3985	0.3990	4.5253e-04

Table 4. Numerical results for $m = 3$, $s = 0.001$ and $\alpha = 0.9$ for Example 2.

(y, s)	AS	ES	AE
(0.1,0.1)	0.1717	0.1702	0.0015
(0.3,0.3)	0.3676	0.3739	0.0062
(0.5,0.5)	0.4456	0.4392	0.0064
(0.7,0.7)	0.4384	0.4414	0.0030
(0.9,0.9)	0.3989	0.3990	8.8942e-05

5. Conclusions

In this study, we presented an accurate numerical technique to approximate the source term of the time-fractional Burger problem containing HH derivative. The introduced method is relied on the energy boundary functions to obtain a linear system. Moreover, the suggested method is applied to solve the examples using different values of m and α to prove that our method is applicable for different parameters. In fact, two examples are solved to illustrate the accuracy and validity of the proposed technique. Furthermore, the figures of solutions and absolute errors are demonstrated, successfully. Also, tables demonstrating values of solutions and related errors are provided for each example.

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Conflict of interest

The authors declare no conflicts of interest.

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