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*Research article*

# On Geraghty $\perp$-contractions in $O$-metric spaces and an application to an ordinary type differential equation

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**Abstract:** By combining the concept of orthogonality and the Geraghty type contraction, we give some fixed point results in the class of $O$-metric spaces. Our obtained results extend the existing results in the literature. We also resolve an ordinary type differential equation.

**Keywords:** $O$-metric space; Geraghty contraction mapping; orthogonal set; ordinary differential equation
**Mathematics Subject Classification:** 39B55, 47H10

## 1. Introduction

Fixed point theory is one of very important tools for proving the existence and uniqueness of the solutions to various mathematical models like integral and partial differential equations, variational inequalities, optimization and approximation theories, etc. It has gained a considerable importance because of the Banach contraction mapping principle [5]. Since then, there have been many results related to the mapping satisfying various types of contractive inequalities; we refer the reader to [7, 16, 19, 21] and references therein. In recent years, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with a partial order;

see [6, 14, 17, 18]. Recently, Gordji et al. [13] coined an exciting notion of the orthogonal sets and after that, they introduced orthogonal metric spaces. The concepts of sequence, continuity and completeness were redefined for these spaces. Further, they gave an extension of Banach contraction principle (BCP) on this newly described shape and also applied their theorem to show the existence of a solution for a differential equation that cannot be attained using BCP. Many authors generalized BCP by using some control functions, see [15, 22].

First, we review some facts of $O$-metric spaces that we need in the sequel. The references [1–4, 8–11, 20] are useful.

**Definition 1.1.** *Let $\mathcal{E} \neq \phi$ and $\perp \subseteq \mathcal{E} \times \mathcal{E}$ be a binary relation. If there is $\Omega_0 \in \mathcal{E}$ such that $\omega \perp \Omega_0$ for all $\omega \in \mathcal{E}$ or $\Omega_0 \perp \omega$ for each $\omega \in \mathcal{E}$, then $\mathcal{E}$ is called an O-set. We show this O-set by $(\mathcal{E}, \perp)$.*

The following example gives a view on $O$-sets.

**Example 1.1.** *Let $\mathcal{E} = [0, 1)$. Define: $\Omega \perp \omega$ iff $\Omega\omega < 1$. Note that $\frac{1}{2} \perp \Omega$ for each $\Omega \in [0, 1)$. Hence $(\mathcal{E}, \perp)$ is an O-set.*

**Definition 1.2.** *Let $(\mathcal{E}, \perp)$ be an O-set. A sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ is called an O-sequence if for each n, $\Omega_n \perp \Omega_{n+1}$ or $\Omega_{n+1} \perp \Omega_n$.*

The concept of continuity in such spaces is defined as follows.

**Definition 1.3.** *Let $(\mathcal{E}, \perp, d_o)$ be an O-metric space. Then $\Upsilon : \mathcal{E} \to \mathcal{E}$ is O-continuous (or, $\perp$-continuous) at $a \in \mathcal{E}$ provided that for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $a_n \to a$, then $\Upsilon(a_n) \to \Upsilon(a)$. Furthermore, $\Upsilon$ is $\perp$-continuous on $\mathcal{E}$ if $\Upsilon$ is $\perp$-continuous at each $a \in \mathcal{E}$.*

**Remark 1.1.** *Note that every continuous mapping is $\perp$-continuous. In the following, the converse is not true.*

**Example 1.2.** *Take $\mathcal{E} = \mathbb{R}$. Consider that $\Omega \perp \omega$ iff $\Omega = 0$ or $0 \neq \omega \in \mathbb{Z}$. Clearly, $(\mathcal{E}, \perp)$ is an O-set. Given $\Upsilon : \mathcal{E} \to \mathcal{E}$,*

$$\Upsilon(\Omega) = \begin{cases} 1, & \text{if } \Omega \in \mathbb{Z}, \\ 0, & \text{if } \Omega \in \mathcal{E} \setminus \mathbb{Z}. \end{cases}$$

*Such $\Upsilon$ is $\perp$-continuous, but it is not continuous on $\mathbb{Z}$. On the other hand, if $\{\Omega_n\}$ is an arbitrary O-sequence in $\mathcal{E}$ such that $\{\Omega_n\}$ converges to $\Omega \in \mathcal{E}$, then we have the following cases:*

*(i) If $\Omega_n = 0$ for each n, then $\Omega = 0$ and $\Upsilon(\Omega_n) = 1 \to 1 = \Upsilon(\Omega)$;*

*(ii) If $\Omega_{n_0} \neq 0$ for some $n_0$, then there is $n \in \mathbb{Z}$ such that $\Omega_n \in \mathbb{Z}$ for each $n \geq n_0$. Thus, $\Omega \in \mathbb{Z}$ and $\Upsilon(\Omega_n) = 1 \to 1 = \Upsilon(\Omega)$.*

*It follows that $\Upsilon$ is $\perp$-continuous on $\mathbb{Z}$, but it is not continuous on $\mathbb{Z}$.*

In the sequel, the following definition will be needed.

**Definition 1.4.** *Let $(\mathcal{E}, \perp, d_o)$ be an O-set with a metric $d_o$. We say that $\mathcal{E}$ is O-complete, if every Cauchy O-sequence is convergent.*

**Remark 1.2.** *Note that every complete metric space is O-complete, but the converse is not true.*

In the following, the concepts of $\perp$-preserving and weakly $\perp$-preserving in $O$-metric spaces are equivalent to the concepts of decreasing and increasing in metric spaces.

**Definition 1.5.** *Let $(\mathcal{E}, \perp)$ be an O-set. We say that the mapping $\Upsilon : \mathcal{E} \to \mathcal{E}$ is $\perp$-preserving if $\Upsilon(\Omega) \perp \Upsilon(\omega)$, when $\Omega \perp \omega$. Also, we say that $\Upsilon : \mathcal{E} \to \mathcal{E}$ is weakly $\perp$-preserving if $\Upsilon(\Omega) \perp \Upsilon(\omega)$ or $\Upsilon(\omega) \perp \Upsilon(\Omega)$ when $\Omega \perp \omega$.*

**Remark 1.3.** *Every $\perp$-preserving is weakly $\perp$-preserving. But, the converse is not true.*

Gordji et al. [13] considered a real extension of BCP.

**Theorem 1.1.** *Let $(\mathcal{E}, \perp, d_o)$ be an O-complete metric space (not necessarily a complete metric space). Suppose that $\Upsilon : \mathcal{E} \to \mathcal{E}$ is $\perp$-continuous, $\perp$-preserving and an $\perp$-contraction, i.e.,*

$$d_o(\Upsilon\Omega, \Upsilon\omega) \leq \lambda d_o(\Omega, \omega) \;\; for\ all \;\; \Omega \perp \omega,$$

*where $0 < \lambda < 1$. Then $\Upsilon$ has a unique fixed point in $\mathcal{E}$, say $\Omega^*$. Also, $\Upsilon$ is a Picard operator, that is, $\lim_{n\to\infty} \Upsilon^n(\Omega) = \Omega^*$ for each $\Omega \in \mathcal{E}$.*

## 2. Main results

First, let $\mathcal{S}$ denote the class of functions $\beta : [0, \infty) \to [0, 1)$ such that $\beta(v_n) \to 1$ implies $v_n \to 0$ as $n \to \infty$. We begin with the following definition, which is useful to prove our main theorem. In fact, we extend the results in [1] by orthogonality and obtain some fixed point results in $O$-metric spaces.

**Definition 2.1.** *Let $(\mathcal{E}, \perp, d_o)$ be an O-metric space. A mapping $\Upsilon : \mathcal{E} \to \mathcal{E}$ is said to be a Geraghty $\perp$-contraction if, for $\Omega, \omega \in \mathcal{E}$,*

$$\Omega \perp \omega \,\text{implies}\, d_o(\Upsilon(\Omega), \Upsilon(\omega)) \leq \beta(d_o(\Omega, \omega))d_o(\Omega, \omega), \tag{2.1}$$

*where $\beta \in \mathcal{S}$.*

Now, we give an example showing that every Geraghty contraction (for more details and results using this concept, see [2–4]) is a Geraghty $\perp$-contraction, but the converse is not true.

**Example 2.1.** *Take $\mathcal{E} = [0, 3)$ endowed with the Euclidean metric. Consider that $\Omega \perp \omega$ iff $\Omega\omega \leq \Omega$. Define $\Upsilon : \mathcal{E} \to \mathcal{E}$ by*

$$\Upsilon(\Omega) = \begin{cases} \dfrac{\Omega}{4}, & if\, \Omega \leq 2, \\ 0, & if\, \Omega > 2. \end{cases}$$

*The following cases hold:*

*(i) If $\Omega = 0, \;\omega \leq 2,$ then $\Upsilon(\Omega) = 0, \;\Upsilon(\omega) = \frac{\omega}{4}$;*

*(ii) If $\Omega = 0, \;\omega > 2,$ then $\Upsilon(\Omega) = 0 = \Upsilon(\omega)$;*

*(iii) If $\omega \leq 1, \;\Omega \leq 2,$ then $\Upsilon(\Omega) = \frac{\Omega}{4}, \;\Upsilon(\omega) = \frac{\omega}{4}$;*

*(iv) If* $\omega \leq 1,\ \Omega > 2,$ then $\Upsilon(\Omega) = 0,\ \Upsilon(\omega) = \frac{\omega}{4}$.

*Given* $\beta : [0,\infty) \to [0,1)$ *as* $\beta(d_o(\Omega,\omega)) = \frac{1}{1+d_o(\Omega,\omega)}$, *note that* $\beta(d_o(\Omega,\omega)) \to 1$ *implies* $d_o(\Omega,\omega) \to 0$. *Also,* $|\Upsilon(\Omega) - \Upsilon(\omega)| \leq \frac{|\Omega-\omega|}{1+|\Omega-\omega|}$. *Hence,* $\Upsilon$ *is a Geraghty* $\perp$*-contraction. But,* $\Upsilon$ *is not a Geraghty contraction. Indeed, for* $\Omega = 2$ *and* $\omega = \frac{5}{2}$*, we have*

$$|\Upsilon(2) - \Upsilon(\frac{5}{2})| = \frac{1}{2} \not< \frac{|2-\frac{5}{2}|}{1+|2-\frac{5}{2}|} = \frac{1}{3}.$$

An additional importance for applications of orthogonal spaces is represented by the richness of the structure of metric spaces that we may see in the following result.

**Theorem 2.1.** *Let* $(\mathcal{E}, \perp, d_o)$ *be an O-complete metric space and let* $\Upsilon : \mathcal{E} \to \mathcal{E}$ *be* $\perp$*-continuous,* $\perp$*-preserving and a Geraghty* $\perp$*-contraction. Then* $\Upsilon$ *has a unique fixed point in* $\mathcal{E}$*. Also,* $\Upsilon$ *is a Picard operator.*

*Proof.* We first show that $\Upsilon$ has a fixed point. Since $\mathcal{E}$ is an $O$-set, there is $\Omega_0 \in \mathcal{E}$ such that for every $\omega \in \mathcal{E}$, $\Omega_0 \perp \omega$ or $\omega \perp \Omega_0$. It follows that $\Omega_0 \perp \Upsilon(\Omega_0)$ or $\Upsilon(\Omega_0) \perp \Omega_0$. Since $\Upsilon$ is $\perp$-preserving, by induction we obtain that

$$\Omega_0 \perp \Upsilon(\Omega_0) \perp \Upsilon^2(\Omega_0) \perp \Upsilon^3(\Omega_0) \perp \ldots \perp \Upsilon^n(\Omega_0) \perp \Upsilon^{n+1}(\Omega_0) \perp \ldots$$

Set $\Omega_n = \Upsilon^n(\Omega_0)$ for $n = 1, 2, \ldots$. Since $\Omega_n \perp \Omega_{n+1}$ for each $n \in \mathbb{N}$, by (2.1),

$$\begin{aligned} d_o(\Omega_{n+1}, \Omega_{n+2}) &= d_o(\Upsilon^{n+1}(\Omega_0), \Upsilon^{n+2}(\Omega_0)) \\ &\leq \beta(d_o(\Omega_n, \Omega_{n+1}))d_o(\Omega_n, \Omega_{n+1}) \\ &\leq d_o(\Omega_n, \Omega_{n+1}). \end{aligned}$$

Then $\{d_o(\Omega_n, \Omega_{n+1})\}$ is an $\perp$-preserving sequence and is bounded below, so $\lim_{n\to\infty} d_o(\Omega_n, \Omega_{n+1}) = r \geq 0$. Assume $r > 0$, then, from (2.1), we have

$$\frac{d_o(\Omega_{n+1}, \Omega_{n+2})}{d_o(\Omega_n, \Omega_{n+1})} \leq \beta(d_o(\Omega_n, \Omega_{n+1})), \qquad n = 1, 2, \ldots$$

The above inequality yields $\lim_{n\to\infty} \beta(d_o(\Omega_n, \Omega_{n+1})) = 1$. Since $\beta \in \mathcal{S}$, we get that $r = 0$. Then $\lim_{n\to\infty} d_o(\Omega_n, \Omega_{n+1}) = 0$.

Now, we show that $\{\Omega_n\}$ is a Cauchy $O$-sequence. On the contrary, assume that

$$\limsup_{m,n\to\infty} d_o(\Omega_n, \Omega_m) > 0. \tag{2.2}$$

By the triangle inequality,

$$d_o(\Omega_n, \Omega_m) \leq d_o(\Omega_n, \Omega_{n+1}) + d_o(\Omega_{n+1}, \Omega_{m+1}) + d_o(\Omega_{m+1}, \Omega_m).$$

Hence from (2.1),

$$d_o(\Omega_n, \Omega_m) \leq (1 - \beta(d_o(\Omega_n, \Omega_m)))^{-1}[d_o(\Omega_n, \Omega_{n+1}) + d_o(\Omega_{m+1}, \Omega_m)].$$

Since $\limsup_{m,n\to\infty} d_o(\Omega_n, \Omega_m) > 0$ and $\lim_{n\to\infty} d_o(\Omega_n, \Omega_{n+1}) = 0$, we have

$$\limsup_{m,n\to\infty}(1 - \beta(d_o(\Omega_n, \Omega_m)))^{-1} = \infty.$$

We obtain $\limsup_{m,n\to\infty} \beta(d_o(\Omega_n, \Omega_m)) = 1$. But since $\beta \in \mathcal{S}$, we get

$$\limsup_{m,n\to\infty} d_o(\Omega_n, \Omega_m) = 0.$$

This contradicts (2.2), so $\{\Omega_n\}$ is a Cauchy $O$-sequence in $\mathcal{E}$. Since $(\mathcal{E}, d_o)$ is an $O$-complete metric space, there is $t \in \mathcal{E}$ such that $\lim_{n\to\infty} \Omega_n = t$. To prove that $t$ is a fixed point of $\Upsilon$, in the case that $\Upsilon$ is $\perp$-continuous, one writes

$$t = \lim_{n\to\infty} \Omega_n = \lim_{n\to\infty} \Upsilon^n(\Omega_0) = \lim_{n\to\infty} \Upsilon^{n+1}(\Omega_0) = \Upsilon(\lim_{n\to\infty} \Upsilon^n(\Omega_0)) = \Upsilon(t).$$

Hence $\Upsilon(t) = t$, i.e., $t$ is a fixed point.

Let $\Omega$ be another fixed point of $\Upsilon$. So, we have

$$\Upsilon^n(\Omega) = \Omega \ , \ \ \Upsilon^n(\omega) = \omega, \quad \text{for each } \ n \in \mathbb{N}.$$

By choosing $\Omega_0$ in the first part of proof, we get

$$(\Omega_0 \perp \Omega \ , \ \Omega_0 \perp \omega) \text{ or } (\Omega \perp \Omega_0 \ , \ \omega \perp \Omega_0).$$

Since $\Upsilon$ is $\perp$-preserving, we infer that

$$(\Upsilon^n(\Omega_0) \perp \Upsilon^n(\Omega) \ , \ \Upsilon^n(\Omega_0) \perp \Upsilon^n(\omega)) \quad \text{for each } \ n \in \mathbb{N}$$

or

$$(\Upsilon^n(\Omega) \perp \Upsilon^n(\Omega_0) \ , \ \Upsilon^n(\omega) \perp \Upsilon^n(\Omega_0)) \quad \text{for each } \ n \in \mathbb{N}.$$

By the triangle inequality, we obtain

$$\begin{aligned} d_o(\Omega, \Upsilon^n(\Omega_0)) &= d_o(\Upsilon^n(\Omega), \Upsilon^n(\Omega_0)) \\ &\leq \beta(d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\Omega_0)))d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\Omega_0)) \\ &\leq d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\Omega_0)) \\ &= d_o(\Omega, \Upsilon^{n-1}(\Omega_0)). \end{aligned} \tag{2.3}$$

Since $\gamma_n = d_o(\Omega, \Upsilon^n(\Omega_0))$ is nonnegative and $\perp$-preserving,

$$\lim_{n\to\infty} d_o(\Omega, \Upsilon^n(\Omega_0)) = \gamma \geq 0.$$

We show that $\gamma = 0$. On the contrary, suppose that $\gamma > 0$. By passing to subsequences, if necessary, assume that $\lim_{n\to\infty} \beta(\gamma_n) = \lambda$ exists. From (2.3), we deduce that $\lambda\gamma = \gamma$, so $\lambda = 1$. Since $\beta \in \mathcal{S}$, we obtain that

$$\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} d_o(\Omega, \Upsilon^n(\Omega_0)) = 0.$$

This contradiction implies that $\gamma = 0$. Similarly, $\lim_{n\to\infty} d_o(\omega, \Upsilon^n(\Omega_0)) = 0$. Therefore,

$$d_o(\Omega, \omega) \leq d_o(\Omega, \Upsilon^n(\Omega_0)) + d_o(\Upsilon^n(\Omega_0), \omega) \longrightarrow 0 \ \text{(as } \ n \to \infty).$$

Consequently, $d_o(\Omega, \omega) = 0$, so $\Omega = \omega$.

Finally, suppose that $\omega$ is an arbitrary element in $\mathcal{E}$. Similarly,

$$(\Omega_0 \perp \Omega \ , \ \Omega_0 \perp \omega) \text{ or } (\Omega \perp \Omega_0 \ , \ \omega \perp \Omega_0)$$

and

$$(\Upsilon^n(\Omega_0) \perp \Upsilon^n(\Omega) \ , \ \Upsilon^n(\Omega_0) \perp \Upsilon^n(\omega))$$

or

$$(\Upsilon^n(\Omega) \perp \Upsilon^n(\Omega_0) \ , \ \Upsilon^n(\omega) \perp \Upsilon^n(\Omega_0))$$

for each $n \in \mathbb{N}$. By the triangle inequality, we get

$$\begin{aligned} d_o(\Omega, \Upsilon^n(\omega)) &= d_o(\Upsilon^n(\Omega), \Upsilon^n(\omega)) \\ &\leq \beta(d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\omega)))d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\omega)) \\ &\leq d_o(\Upsilon^{n-1}(\Omega), \Upsilon^{n-1}(\omega)) \\ &= d_o(\Omega, \Upsilon^{n-1}(\omega)). \end{aligned}$$

Similar to the above reasoning, $\lim_{n\to\infty} d_o(\Omega, \Upsilon^n(\omega)) = 0$. So $\Upsilon$ is a Picard operator.

In the following example, we show that Theorem 2.1 is a genuine generalization of Theorem 2.1 in [1].

**Example 2.2.** *According to Example 2.1 and by using Theorem 2.1, one can see that $\Upsilon$ has a unique fixed point. However, $\Upsilon$ is not a Geraghty contraction; so, by Theorem 2.1 in [1], we cannot find any fixed point for $\Upsilon$.*

We have the following corollary.

**Corollary 2.1.** *The notions of O-completeness and O-continuity on O-metric spaces are weaker than the concepts of completeness and continuity on metric spaces. Therefore, Theorem 2.1 is a generalization of the corresponding results in [1].*

## 3. Application to ordinary differential equations

In this section, our purpose is to apply Theorem 2.1 to prove the existence of a solution for the following first-order problem:

$$\begin{cases} \Omega'(\ell) = \gamma(\ell, \Omega(\ell)), & \ell \in J = [0, L] \\ \\ \Omega(0) = \Omega(L). \end{cases} \tag{3.1}$$

Let $\mathcal{A}$ denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ such that

(i) $\psi$ is increasing;

(ii) for each $\xi > 0$, $\psi(\xi) < \xi$;

(iii) $\beta(\xi) = \frac{\psi(\xi)}{\xi} \in \mathcal{S}$.

Assume that $\gamma : J \times \mathbb{R} \to \mathbb{R}$ is an integrable function so that

(k1) $\gamma(\ell, \xi) \geq 0$ for all $\xi \geq 0$ and $\ell \in J$;

(k2) there are $\psi \in \mathcal{A}$ and $\lambda > 0$ such that for all $\xi, \mu \in \mathbb{R}$ with $\xi, \mu \geq 0$ and $\mu - \xi \geq 0$ and for $\ell \in J$, we have

$$0 \leq \gamma(\ell, \mu) + \lambda\mu - [\gamma(\ell, \xi) + \lambda\xi] \leq \lambda(\mu - \xi).$$

**Theorem 3.1.** *Under the above conditions, the ordinary differential equation* (3.1) *has a unique positive solution.*

*Proof.* Take $\mathcal{E} = \{\Omega \in C(J) : \Omega(\ell) \geq 0\,,\ \text{for all}\ \ell \in J\}$. Consider the following $O$-relation on $\mathcal{E}$:

$$x \perp y \text{ iff } y(\ell) - x(\ell) \geq 0 \text{ for each } \ell \in J.$$

Define $d_o(x, y) = \sup\{|x(\ell) - y(\ell)| : \ell \in J\}$ for all $x, y \in \mathcal{E}$. Clearly, $(\mathcal{E}, d_o)$ is a metric space. We show that $\mathcal{E}$ is $O$-complete (not necessarily complete). Take the Cauchy $O$-sequence $\{x_n\} \subseteq \mathcal{E}$. Now, we show that $\{x_n\}$ is converging to an element $x$ in $C(J)$. To ensure this, it suffices that $x \in \mathcal{E}$. Fix $\ell \in J$. Since $x_n(\ell) \geq 0$ for each $n \in \mathbb{N}$, the convergence of this real sequence to $x(\ell)$ implies that $x(\ell) \geq 0$. But, $\ell \in J$ is arbitrary; therefore, $x \geq 0$ and consequently, $x \in \mathcal{E}$.
Problem (3.1) can be expressed as

$$\begin{cases} \Omega'(\ell) + \lambda\Omega(\ell) = \gamma(\ell, \Omega(\ell)) + \lambda\Omega(\ell), & \ell \in J = [0, L] \\ \Omega(0) = \Omega(L). \end{cases} \tag{3.2}$$

This problem is equivalent to the integral equation

$$\Omega(\ell) = \int_0^L G(\ell, h)[\gamma(h, \Omega(h)) + \lambda\Omega(h)]dh,$$

where

$$G(\ell, h) = \begin{cases} \dfrac{e^{\lambda(L+h-\ell)}}{e^{\lambda L} - 1}, & 0 \leq h < \ell \leq L, \\ \dfrac{e^{\lambda(h-\ell)}}{e^{\lambda L} - 1}, & 0 \leq \ell < h \leq L. \end{cases}$$

Define $\Gamma : \mathcal{E} \to \mathcal{E}$ as

$$\Gamma_u(\ell) = \int_0^L G(\ell, h)[\gamma(h, u(h)) + \lambda u(h)]dh.$$

Note that a fixed point of $\Gamma$ is a solution of (3.2). We claim that the hypotheses of Theorem 2.1 hold.

Step 1) $\Gamma$ is $\perp$-preserving.

For $x \perp y$ and $\ell \in J$,

$$\begin{aligned}\Gamma_y(\ell) &= \int_0^L G(\ell, h)[\gamma(h, y(h)) + \lambda y(h)]dh \\ &\geq \int_0^L G(\ell, h)[\gamma(h, x(h)) + \lambda x(h)]dh = \Gamma_x(\ell)\end{aligned}$$

which yields that $\Gamma_y(\ell) - \Gamma_x(\ell) \geq 0$. Therefore, $\Gamma_x \perp \Gamma_y$.

Step 2) $\Gamma$ is a Geraghty $\perp$-contraction.

For $x \perp y$ and $\ell \in J$, condition $(k2)$ implies that

$$\begin{aligned}d_o(\Gamma_y, \Gamma_x) &= \sup_{\ell \in J} |\Gamma_y(\ell) - \Gamma_x(\ell)| \\ &\leq \sup_{\ell \in J} \int_0^L G(\ell, h)|\gamma(h, y(h)) + \lambda y(h) - \gamma(h, x(h)) - \lambda x(h)|dh \\ &\leq \sup_{\ell \in J} \int_0^L G(\ell, h).\lambda.\psi(y(h) - x(h))dh.\end{aligned}$$

As $\psi$ is increasing, $\psi(y(h) - x(h)) \leq \psi(d_o(x, y))$. Then

$$\begin{aligned}d_o(\Gamma_y, \Gamma_x) &\leq \sup_{\ell \in J} \int_0^L G(\ell, h).\lambda.\psi(y(h) - x(h))dh \\ &\leq \lambda.\psi(d_o(x, y)). \sup_{\ell \in J} \int_0^L G(\ell, h)dh \\ &= \lambda.\psi(d_o(x, y)). \sup_{\ell \in J} \frac{1}{e^{\lambda L} - 1}\left(\frac{1}{\lambda} e^{\lambda(L+h-\ell)}\Big|_0^\ell + \frac{1}{\lambda} e^{\lambda(h-\ell)}\Big|_\ell^L\right) \\ &= \lambda.\psi(d_o(x, y)). \frac{1}{\lambda(e^{\lambda L} - 1)}(e^{\lambda L} - 1) = \psi(d_o(x, y)) \\ &= \frac{\psi(d_o(x, y))}{d_o(x, y)} d_o(x, y) = \beta(d_o(x, y))d_o(x, y).\end{aligned}$$

For $x \perp y$, we get

$$d_o(\Gamma_x, \Gamma_y) \leq \beta(d_o(x, y))d_o(x, y).$$

Hence, $\Gamma$ is a Geraghty $\perp$-contraction.

Step 3) $\Gamma$ is $\perp$-continuous.

Let $\{x_n\} \subseteq \mathcal{E}$ be an $O$-sequence converging to $x \in \mathcal{E}$. Thus, $x_{n+1} \perp x_n$ or $x_n \perp x_{n+1}$ for each $n$. Hence, $x_n \geq x_{n+1}$ or $x_{n+1} \geq x_n$. Therefore, $x_n(\ell) - x(\ell) \geq 0$ or $x(\ell) - x_n(\ell) \geq 0$ for all $\ell \in J$ and $n \in \mathbb{N}$. In the case that $x_n(\ell) - x(\ell) \geq 0$ for each $n$, condition $(k2)$ implies that

$$\begin{aligned}d_o(\Gamma_{x_n}, \Gamma_x) &\leq \sup_{\ell \in J} \int_0^L G(\ell, h).\lambda.\psi(x_n(h) - x(h))dh \\ &\leq \beta(d_o(x_n, x))d_o(x_n, x).\end{aligned}$$

Thus,

$$d_o(\Gamma x_n, \Gamma x) \leq \beta(d_o(x_n, x))d_o(x_n, x),$$

for each $n \in \mathbb{N}$. This implies that $\Gamma x_n \to \Gamma x$. In the case that $x(\ell) - x_n(\ell) \geq 0$ for each $n$, the similar argument shows that $\Gamma x_n \to \Gamma x$. Consequently, $\Gamma$ is $\perp$-continuous.

By Theorem 2.1, we deduce the uniqueness of a solution of (3.1).

## 4. Conclusions

In this study, we have defined the notion of Geraghty $\perp$-contraction mappings and have established a fixed point theorem for such mappings in the setting of $O$-complete metric space which is not necessarily complete. Further, as an application, we solved an ordinary differential equation with the help of our main theorem.

## Conflict of interest

The authors declare to have no competing interests.

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