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*Research article*

# Variational approach to non-instantaneous impulsive differential equations with $p$-Laplacian operator

**Wangjin Yao***

Key Laboratory of Applied Mathematics of Fujian Province University, School of Mathematics and Finance, Putian University, Putian 351100, China

* **Correspondence:** Email: 13635262963@163.com.

**Abstract:** In this paper, we consider the existence, multiplicity and nonexistence of solutions for a class of $p$-Laplacian differential equations with non-instantaneous impulses. By using variational methods and critical point theory, we obtain that the impulsive problem has at least one nontrivial solution, at least two nontrivial solutions and no nontrivial solution.

**Keywords:** Dirichlet boundary value problem; $p$-Laplacian differential equations; non-instantaneous impulse; variational methods; critical point theory
**Mathematics Subject Classification:** 34B37, 34K45, 58E30

## 1. Introduction

In this paper, we consider the following problem

$$\begin{cases} -(\Phi_p(u'(t)))' = D_xF_i(t, u(t) - u(t_{i+1})), \qquad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., n, \\ \Phi_p(u'(t)) = \beta_i, \qquad t \in (t_i, s_i],\ i = 1, 2, ..., n, \\ \Phi_p(u'(s_i^+)) = \Phi_p(u'(s_i^-)), \qquad i = 1, 2, ..., n, \\ \Phi_p(u'(0)) = \beta_0, \quad u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n < t_{n+1} = T$, $\beta_i$ are given constants, and impulsive jumps starts abruptly at the points $t_i$ and keep the derivative constant on a finite time interval $(t_i, s_i]$. Here, $\Phi_p(u'(s_i^\pm)) = \lim_{s\to s^\pm} \Phi_p(u'(s))$, and the nonlinear functions $D_xF_i(t, x)$ are the derivatives of $F_i(t, x)$ with respect to $x$ for every $i = 0, 1, 2, ..., n$.

In recent years, variational methods and critical point theory have been widely used to study the existence and multiplicity of solutions for impulsive differential equations which possess variational structures under certain boundary conditions. In this field of research, the pioneering studies were initiated by Tian-Ge [29] and Nieto-O'Regan [21]. Since this time, many scholars investigated different types of impulsive differential equations, such as second-order, fourth-order and fractional order impulsive differential equations, by means of variational approach and critical point theory. For some general and recent works, we refer the interested reader to [11, 12, 15, 20, 22, 23, 28, 32, 33, 35] and some of references for additional details.

On the other hand, the $p$-Laplacian operator appears in non-Newtonian fluid flows, turbulent filtration in porous media, some reaction-diffusion and many other application areas, so it has deep background in physics [3, 10, 17]. In 1983, Leibenson [17] considered the one-dimensional turbulent flow of a polytropic gas in a porous medium and introduced the $p$-Laplacian equation as follows:

$$(\Phi_p(u'(t)))' = f(t, u(t), u'(t)), \tag{1.2}$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$. Obviously, when $p = 2$, the $p$-Laplacian equations reduce to the second-order differential equations. Many important results relative to (1.2) have been achieved over the past few decades, see for instance [4,6,9,31]. In 2008, Tian-Ge [29] first investigated a class of $p$-Laplacian boundary value problem with impulsive effects via variational methods and obtained it has at least two positive solutions. Since then, the impulsive differential equations with $p$-Laplacian operators have received extensive attention. For example, in [5,13,14,19,25,26,30], authors studied different types of $p$-Laplacian impulsive differential equations by applying variational methods and critical point theory.

Although the above works about impulsive differential equations have achieved many important research results, they focused primarily on differential equations with instantaneous impulses, which can't describe all the phenomena in real life, such as earthquakes and tsunamis. Thus, the study of non-instantaneous impulsive differential equations has attracted widespread attention in recent years. In 2013, Hernández-O'Regan [16] were inspired by a simplified situation concerning the hemodynamical equilibrium of a person and introduced the non-instantaneous impulses, which start abruptly at some points and remain active on a finite time interval. Obviously, non-instantaneous impulsive differential equations are a natural generalization of impulsive differential equations. From then on, the existence of solutions for non-instantaneous impulsive differential equations have been investigated by some approaches, such as fixed point theory, theory of analytic semigroup and variational methods, see for instance [1, 2, 7, 16, 24, 36]. In [1], Bai-Nieto studied the following linear differential equations with non-instantaneous impulses and obtained the existence and uniqueness of weak solutions.

$$\begin{cases} -u''(t) = \sigma_i(t), \qquad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., N, \\ u'(t) = \alpha_i, \qquad t \in (t_i, s_i],\ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \qquad i = 1, 2, ..., N, \\ u(0) = u(T) = 0, \quad u'(0) = \alpha_0, \end{cases}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_N < s_N < t_{N+1} = T$, the impulses start abruptly at the points $t_i$ and keeps the derivative constants on finite interval $(t_i, s_i]$, $\sigma_i \in L^2((s_i, t_{i+1}), \mathbb{R})$ and $\alpha_i$ are given constants. Here, $u'(s_i^\pm) = \lim_{s\to s^\pm} u'(s)$.

On the basis of [1], Bai-Nieto-Wang [2] considered the following nonlinear differential equations with non-instantaneous impulses via variational methods and critical point theory. They obtained the problem has at least two distinct nontrivial weak solutions.

$$\begin{cases} -u''(t) = D_xF_i(t, u(t) - u(t_{i+1})), \ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ u'(t) = \alpha_i, \qquad t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \qquad i = 1, 2, ..., N, \\ u(0) = u(T) = 0, \quad u'(0) = \alpha_0, \end{cases}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_N < s_N < t_{N+1} = T$, the impulses start abruptly at the points $t_i$ and keeps the derivative constants on finite interval $(t_i, s_i]$. Here, $u'(s_i^{\pm}) = \lim_{s\to s^{\pm}} u'(s)$ and $\alpha_i$ are given constants.

In [36], Zhao-Luo-Chen considered the following fractional differential equations with non-instantaneous impulses. They proved that the problem has at least one nontrivial weak solution, at least two nontrivial weak solutions and does not admit any nontrivial solution by using critical point theory and variational methods.

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) = D_xF_i(t, u(t) - u(t_{i+1})), \qquad t \in (s_i, t_{i+1}], \ i = 0, 1, 2, ..., N, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t) = \beta_i, \qquad t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(s_i^-) = {}_t D_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(s_i^+), \qquad i = 1, 2, ..., N, \\ {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(0) = \beta_0, \quad u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{2} < \alpha \le 1$, ${}_0D_t^{\gamma}$ and ${}_tD_T^{\gamma}$ denote the left and right Riemann–Liouville fractional derivatives of order $\gamma$, respectively. ${}_0^cD_t^{\alpha}$ is the left Caputo fractional derivative of order $\alpha$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < s_N < t_{N+1} = T$, $\beta_i$ are given constants, and the impulsive jump starts abruptly at the fixed points $t_i$ and keeps the derivative constants on finite interval $(t_i, s_i]$. Here, ${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(s_i^{\pm}) = \lim_{s\to s^{\pm}} {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(s)$, and the nonlinear functions $D_xF_i(t, x)$ are the derivatives of $F_i(t, x)$ with respect to $x$ for every $i = 0, 1, 2, ..., N$.

Motivated by the above research, our aim of this paper is to study a class of non-instantaneous impulsive differential equations with $p$-Laplacian operator. As far as we know, there is no paper considered the problem (1.1) and the existence, multiplicity and nonexistence of solutions for it are obtained via variational methods and critical point theory. So we generalize the existing results in [1, 2, 36].

Throughout this paper, we need the following assumptions.

$(H1)$ $F_i(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}$ and continuously differentiable in $x$ for a.e. $t \in (s_i, t_{i+1}]$, and there exist functions $m_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $m_2 \in L^1((s_i, t_{i+1}); \mathbb{R}^+)$ such that

$$|F_i(t, x)| \le m_1(|x|)m_2(t), \ |D_xF_i(t, x)| \le m_1(|x|)m_2(t)$$

for each $x \in \mathbb{R}$ and a.e. in $t \in (s_i, t_{i+1}]$.

$(H2)$ There exist constant $\gamma \in [0, p)$ and the functions $b_0, b_1 \in L^1(s_i, t_{i+1})$ such that

$$|F_i(t, x)| \le b_0(t)|x|^{\gamma} + b_1(t)$$

for *a.e.* $t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$.

$(H3)$ $F_i(t, 0) = 0$ for *a.e.* $t \in (s_i, t_{i+1}]$ and there exist constants $\mu_i > p$ and $M > 0$ such that $0 < \mu_iF_i(t, x) \le xD_xF_i(t, x)$ for *a.e.* $t \in (s_i, t_{i+1}]$, $x \in \mathbb{R}$ with $|x| > M$, $i = 0, 1, 2, ..., n$.

$(H4)$ $\limsup\limits_{|x|\to 0} \frac{F_i(t,x)}{|x|^{\mu_i}} < A^*$ uniformly for *a.e.* $t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$, where

$$A^* := \frac{1 - 2^{p-1} C^p p \sum_{i=1}^{n} |\beta_{i-1} - \beta_i|}{2^p C^p p \sum_{i=0}^{n} (t_{i+1} - s_i)} > 0,$$

and $C$ is defined in Lemma 2.10.

$(H5)$ $F_i(t,x) = o(|x|^{\mu_i})$ as $|x| \to 0$ uniformly for *a.e.* $t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$.

$(H6)$ $m_1(x)$ is nondecreasing function for any $x \geq 0$.

$(H7)$ $m_1(2Cx) < \frac{x^{p-1} - \sum_{i=1}^{n} C|\beta_{i-1} - \beta_i|}{2C \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} m_2(t)dt}$ for all $x > 0$, where $C$ is defined in Lemma 2.10.

Our main results are as follows.

**Theorem 1.1.** *Assume that (H1) and (H2) hold. Then problem* (1.1) *has at least one nontrivial weak solution.*

**Theorem 1.2.** *Assume that (H1), (H3) and (H4) hold. Then problem* (1.1) *has at least two nontrivial weak solutions.*

**Theorem 1.3.** *Assume that (H1), (H3) and (H5) hold. Then problem* (1.1) *has at least two nontrivial weak solutions.*

**Theorem 1.4.** *Assume that (H1), (H6) and (H7) hold. Then problem* (1.1) *has no nontrivial solution.*

**Remark 1.5.** **i** *The condition (H1) ensures the existence and continuity of Gateaux derivative $\phi_i'(u)$, so we can obtain $\varphi'(u)$ exists and is continuous. Moreover, the condition (H2) ensures the functional $\varphi(u)$ is coercive. Hence, we can use minimization methods to prove the Theorem 1.1 holds.*

**ii** *The conditions (H3) and (H4) ensure $\varphi(u)$ has mountain pass geometric structure. So we can obtain the problem* (1.1) *has at least one nontrivial weak solution by using mountain pass theorem. Furthermore, we can get another nontrivial weak solution of the problem* (1.1) *by applying Lemma 2.7. Therefore, the Theorem 1.2 is proved. Similarly, we can prove Theorem 1.3 by mean of the same methods.*

**iii** *In the case u is a nontrivial solution of problem* (1.1), $\langle \varphi'(u), u \rangle = 0$. *The conditions (H6) and (H7) ensure $\langle \varphi'(u), u \rangle > 0$. Hence, we can use inequalities to prove the Theorem 1.4 holds.*

**Remark 1.6.** *If $p = 2$, problem* (1.1) *reduces to [2]. Furthermore, if $p = 2$ and $D_x F_i(t, u(t) - u(t_{i+1})) = \sigma_i(t)$, problem* (1.1) *reduces to [1]. So our problem* (1.1) *generalizes the works of [1, 2].*

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we will prove our main results.

## 2. Preliminaries

In this section, we first introduce some definitions, lemmas and theorems, which are used further in this paper.

**Definition 2.1.** *( [18], Minimizing sequence) Let X be a Banach space. A minimizing sequence of a functional* $\varphi : X \to \mathbb{R}$ *is a sequence* $\{u_k\} \subset X$*, such that*

$$\lim_{k\to\infty} \varphi(u_k) = \inf_{u\in X} \varphi(u).$$

**Definition 2.2.** *( [18], Weakly lower semi-continuous) Let X be a Banach space. A functional* $\varphi : X \to \mathbb{R}$*, is weakly lower semi-continuous, if*

$$u_k \rightharpoonup u \Rightarrow \liminf_{k\to\infty} \varphi(u_k) \geq \varphi(u).$$

**Definition 2.3.** *( [18], Coercive) Let X be a Banach space. A functional* $\varphi : X \to \mathbb{R}$*, is called coercive if, for every* $u \in X$*,*

$$\varphi(u) \to +\infty, \quad \text{if} \quad \|u\|_X \to \infty.$$

**Theorem 2.4.** *[18] If* $\varphi : X \to (-\infty, +\infty]$ *is coercive, then* $\varphi$ *has a bounded minimizing sequence.*

**Theorem 2.5.** *[18] Let X be a reflexive Banach space and let* $\varphi : X \to (-\infty, +\infty]$ *be weakly lower semi-continuous on X. If* $\varphi$ *has a bounded minimizing sequence, then* $\varphi$ *has a minimum on X.*

**Definition 2.6.** *( [18], (PS) condition) Let X be a real reflexive Banach space. For any sequence* $\{u_k\} \subset X$*, if* $\{\varphi(u_k)\}$ *is bounded and* $\varphi'(u_k) \to 0$ *as* $k \to \infty$ *possesses a convergent subsequence, then we say that* $\varphi$ *satisfies the Palais-Smale condition.*

**Lemma 2.7.** *[34, Theorem 38.A] For the functional* $\varphi : M \subset X \to [-\infty, +\infty]$ *with* $M \neq \emptyset$*;* $\min_{u\in M} \varphi(u) = \alpha$ *has a solution in case the following hold:*

**(i)** *X is a real reflexive Banach space;*

**(ii)** *M is bounded and weak sequentially closed;*

**(iii)** $\varphi$ *is sequentially weakly lower semi-continuous on M.*

**Lemma 2.8.** *[18, Theorem 4.10] Let X be a Banach space and let* $\varphi \in C^1(X, \mathbb{R})$*. Assume that there exist* $u_0 \in X$*,* $u_1 \in X$*, and a bounded open neighborhood* $\Omega$ *of* $u_0$ *such that* $u_1 \in X\backslash\Omega$ *and* $\inf_{\partial\Omega} \varphi > \max\{\varphi(u_0), \varphi(u_1)\}$*. Let*

$$\Gamma = \{g \in C([0, 1], X) | g(0) = u_0, g(1) = u_1\} \quad \text{and} \quad c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)).$$

*If* $\varphi$ *satisfies the* $(PS)_c$*-condition, then c is a critical value of* $\varphi$ *and* $c > \max\{\varphi(u_0), \varphi(u_1)\}$*.*

Next, we recall the well-known Poincaré inequality

$$\int_0^T |u'(t)|^p dt \geq \lambda_1 \int_0^T |u(t)|^p dt \quad \text{for all} \quad u \in W_0^{1,p}([0, T]),$$

where $\lambda_1$ is the first eigenvalues of the following problem

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + \lambda|u(t)|^{p-2}u(t) = 0, & t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{2.1}$$

Moreover, it has been shown in [8] that all eigenvalues of the problem (2.1) are given by the sequence of positive numbers

$$\lambda_k = (p-1)\left(\frac{k\pi_p}{T}\right)^p \quad \text{for} \quad k = 1, 2, ...$$

where $\pi_p = \frac{2\pi}{p\sin\frac{\pi}{p}}$.

Let $X := W_0^{1,p}([0,T])$ equipped with the norm

$$\|u\|_X = \left(\int_0^T |u'(t)|^p\right)^{\frac{1}{p}}.$$

Obviously, $X$ is a reflexive Banach space. We also define the norms in $L^p([0,T])$, $C[0,T]$ as $\|u\|_{L^p} = \left(\int_0^T |u|^p dt\right)^{\frac{1}{p}}$ and $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$, respectively.

For each $v \in X$, we have

$$\begin{aligned}\int_0^T (\Phi_p(u'(t)))'v(t)dt &= \int_0^{t_1} (\Phi_p(u'(t)))'v(t)dt + \sum_{i=1}^n \int_{t_i}^{s_i} (\Phi_p(u'(t)))'v(t)dt \\ &\quad + \sum_{i=1}^{n-1} \int_{s_i}^{t_{i+1}} (\Phi_p(u'(t)))'v(t)dt + \int_{s_n}^T (\Phi_p(u'(t)))'v(t)dt \\ &= -\int_0^T |u'(t)|^{p-2}u'(t)v'(t)dt + \sum_{i=1}^n (|u'(t_i^-)|^{p-2}u'(t_i^-) \\ &\quad - |u'(t_i^+)|^{p-2}u'(t_i^+))v(t_i) + \sum_{i=1}^n (|u'(s_i^-)|^{p-2}u'(s_i^-) \\ &\quad - |u'(s_i^+)|^{p-2}u'(s_i^+))v(s_i),\end{aligned}$$

which combined with (1.1) yields that

$$\begin{aligned}\int_0^T (\Phi_p(u'(t)))'v(t)dt = &-\int_0^T |u'(t)|^{p-2}u'(t)v'(t)dt + \sum_{i=1}^n (\beta_{i-1} - \beta_i)v(t_i) \\ &- \sum_{i=0}^{n-1} \left(\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))dt\right) v(t_{i+1}).\end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned}\int_0^T (\Phi_p(u'(t)))'v(t)dt &= -\sum_{i=0}^n \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t)dt \\ &\quad + \sum_{i=1}^n \int_{t_i}^{s_i} \frac{d}{dt}(\beta_i)v(t)dt \\ &= -\sum_{i=0}^n \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t)dt.\end{aligned} \tag{2.3}$$

Thus, according to $v(t_{n+1}) = v(T) = 0$, (2.2), and (2.3), we have

$$\begin{aligned}&-\int_0^T |u'(t)|^{p-2}u'(t)v'(t)dt + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)v(t_i)\\ =&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} D_xF_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt.\end{aligned} \tag{2.4}$$

**Definition 2.9.** *A function $u \in X$ is said to be a weak solution of problem* (1.1) *if* (2.4) *holds for any $v \in X$.*

We define the following functional on $X$

$$\varphi(u) = \frac{1}{p}\|u\|_X^p - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)u(t_i) - \sum_{i=0}^{n}\phi_i(u), \tag{2.5}$$

where

$$\phi_i(u) = \int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1}))dt.$$

For $u$ and $v$ fixed in $X$ and $\lambda \in [-1, 1]$, by Lemma 2.10, we have

$$|u(t) - u(t_{i+1})| \le 2\|u\|_\infty \le 2C\|u\|_X. \tag{2.6}$$

Hence,

$$|u(t) - u(t_{i+1}) + \lambda\theta(v(t) - v(t_{i+1}))| \le 2C(\|u\|_X + \|v\|_X), \text{ for } \theta \in (0, 1), \tag{2.7}$$

and

$$\begin{aligned}&\lim_{\lambda\to 0}\frac{1}{\lambda}\left[F_i(t, u(t) - u(t_{i+1}) + \lambda(v(t) - v(t_{i+1}))) - F_i(t, u(t) - u(t_{i+1}))\right]\\ =&D_xF_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1})), \text{ for } a.e.\ t \in (s_i, t_{i+1}].\end{aligned}$$

It follows from condition (H1), (2.7) and mean value theorem that

$$\begin{aligned}&\left|\frac{1}{\lambda}\left[F_i(t, u(t) - u(t_{i+1}) + \lambda(v(t) - v(t_{i+1}))) - F_i(t, u(t) - u(t_{i+1}))\right]\right|\\ =&|D_xF_i(t, u(t) - u(t_{i+1}) + \lambda\theta(v(t) - v(t_{i+1}))(v(t) - v(t_{i+1}))|\\ \le&2C\|v\|_X \max_{z\in[0,2C(\|u\|_X+\|v\|_X)]} m_1(z)m_2(t) \in L^1((s_i, t_{i+1}]; \mathbb{R}^+).\end{aligned}$$

for some $\theta \in (0, 1)$. By Lebesgue's dominated convergence theorem, we obtain that $\phi_i$ has at every point $u$ a directional derivative

$$\langle\phi_i'(u), v\rangle = \int_{s_i}^{t_{i+1}} D_xF_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt. \tag{2.8}$$

Moreover,

$$|\langle\phi_i'(u), v\rangle| \le \int_{s_i}^{t_{i+1}} |D_xF_i(t, u(t) - u(t_{i+1}))||(v(t) - v(t_{i+1}))|dt$$

$$\leq 2C\int_{s_i}^{t_{i+1}} m_2(t)dt \max_{z\in[0,2C(\|u\|_X+\|v\|_X)]} m_1(z)\|v\|_X.$$

Thus, $\phi_i' \in X^*$ (The space $X^*$ is the topological dual of $X$). Suppose $u_k \rightharpoonup u$ in $X$, then $u_k \to u$ on $[0, T]$ [18, Proposition 1.2]. Furthermore, by (2.8), we obtain

$$\|\phi_i'(u_k) - \phi_i'(u)\| \leq 2C\int_{s_i}^{t_{i+1}} |D_x F_i(t, u_k(t) - u_k(t_{i+1})) - D_x F_i(t, u(t) - u(t_{i+1}))|dt.$$

Hence, $\phi_i'(u)$ is continuous from $X$ to $X^*$, $\varphi \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle \varphi'(u), v\rangle =& \int_0^T |u'|^{p-2}u'v'dt - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)v(t_i)\\ &- \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt.\end{aligned} \tag{2.9}$$

Then, it is clear that the critical points of $\varphi$ correspond to weak solutions of problem (1.1).

**Lemma 2.10.** *For any $u \in X$, there exist a constant $C > 0$ such that $\|u\|_\infty \leq C\|u\|_X$.*

*Proof.* For any $u \in X$, it follows from mean value theorem that

$$u(\tau) = \frac{1}{T}\int_0^T u(s)ds$$

for some $\tau \in [0, T]$. Thus, for $t \in [0, T]$, by Hölder and Poincaré inequality,

$$\begin{aligned}|u(t)| =& \left|u(\tau) + \int_\tau^t u'(s)ds\right| \leq |u(\tau)| + \int_0^T |u'(s)|ds\\ \leq& \frac{1}{T}\left|\int_0^T u(s)ds\right| + T^{\frac{1}{q}}\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}}\\ \leq& T^{-\frac{1}{p}}\left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}} + T^{\frac{1}{q}}\left(\int_0^T |u'(s)|^p ds\right)^{\frac{1}{p}}\\ =& T^{-\frac{1}{p}}\|u\|_{L^p} + T^{\frac{1}{q}}\|u\|_X^{\frac{1}{p}} \leq \left((\lambda_1 T)^{-\frac{1}{p}} + T^{\frac{1}{q}}\right)\|u\|_X,\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, there exist a constant $C = (\lambda_1 T)^{-\frac{1}{p}} + T^{\frac{1}{q}} > 0$ such that $\|u\|_\infty \leq C\|u\|_X$. □

## 3. Main results

**Lemma 3.1.** *The functional $\varphi : X \to \mathbb{R}$ is weakly lower semi-continuous.*

*Proof.* Let $u_k \rightharpoonup u$ in $X$, then $u_k \to u$ on $[0, T]$. Furthermore, it follows from the continuity and convexity of $\frac{\|u\|_X^p}{p}$ that $\liminf\limits_{k\to+\infty}\frac{\|u_k\|_X^p}{p}\geq\frac{\|u\|_X^p}{p}$. Thus, we have

$$\begin{aligned}\liminf_{k\to+\infty}\varphi(u_k)=&\liminf_{k\to+\infty}\Bigg(\frac{\|u_k\|_X^p}{p}-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)u_k(t_i)\\&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}F_i(t,u_k(t)-u_k(t_{i+1}))dt\Bigg)\\ \geq&\frac{\|u\|_X^p}{p}-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)u(t_i)-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}F_i(t,u(t)-u(t_{i+1}))dt\\=&\varphi(u).\end{aligned}\tag{3.1}$$

Therefore, $\varphi$ is weakly lower semi-continuous. □

*Proof of Theorem 1.1.* By condition (H2), Lemma 2.10 and (2.5), we have

$$\begin{aligned}\varphi(u)=&\frac{1}{p}\|u\|_X^p-\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)u(t_i)-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}F_i(t,u(t)-u(t_{i+1}))dt\\ \geq&\frac{1}{p}\|u\|_X^p-\sum_{i=1}^{n}|\beta_{i-1}-\beta_i|\|u\|_\infty-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}(b_0(t)|u(t)-u(t_{i+1})|^\gamma+b_1(t))dt\\ \geq&\frac{1}{p}\|u\|_X^p-\sum_{i=1}^{n}C|\beta_{i-1}-\beta_i|\|u\|_X-2^\gamma C^\gamma\|u\|_X^\gamma\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}b_0(t)dt\\&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}b_1(t)dt.\end{aligned}\tag{3.2}$$

Since $\gamma\in[0,p)$, (3.2) implies that $\lim\limits_{\|u\|\to\infty}\varphi(u)=+\infty$, i.e., $\varphi$ is coercive. By using Lemma 3.1 and Theorem 2.4, we obtain that $\varphi$ satisfies all the conditions of Theorem 2.5. Thus $\varphi$ has a minimum on $X$, which is a critical point of $\varphi$. Hence, problem (1.1) has at least one nontrivial weak solution. □

*Proof of Theorem 1.2.* We need four steps to complete the proof.

**Step 1.** $\varphi(u)$ satisfies (PS) condition on $X$.

Let $\{u_k\}\subset X$ such that $\varphi(u_k)$ is a bounded sequence and $\lim\limits_{k\to\infty}\varphi'(u_k)=0$. Obviously,

$$\left|\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)u(t_i)\right|\leq\sum_{i=1}^{n}|\beta_{i-1}-\beta_i|\|u\|_\infty\leq\sum_{i=1}^{n}|\beta_{i-1}-\beta_i|C\|u\|_X.\tag{3.3}$$

Since $\mu_iF_i(t,x)-xD_xF_i(t,x)$ is continuous for $t\in(s_i,t_{i+1}]$ and $|x|\leq M$, there is constant $C_0>0$ such that

$$\mu_iF_i(t,x)\leq xD_xF_i(t,x)+C_0,\quad t\in(s_i,t_{i+1}],\ \ |x|\leq M.$$

From (H3), we have

$$\mu_iF_i(t,x)\leq xD_xF_i(t,x)+C_0,\quad t\in(s_i,t_i+1],\ \ x\in\mathbb{R}.\tag{3.4}$$

Let $\mu := \min\{\mu_i, i = 0, 1, 2, ..., n\}$, then $\mu > p$. By (3.3) and (3.4), we have

$$\begin{aligned}
&\mu\varphi(u_k) - \langle \varphi'(u_k), u_k\rangle \\
=&\frac{\mu-1}{p}\|u_k\|_X^p - (\mu-1)\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)u(t_i) \\
&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} \mu F_i(t, u_k(t)-u_k(t_i)) - F_i(t, u_k(t)-u_k(t_i))(u_k(t)-u_k(t_i))dt \\
\geq&\frac{\mu-1}{p}\|u_k\|_X^p - (\mu-1)\sum_{i=1}^{n}|\beta_{i-1}-\beta_i|C\|u_k\|_X - C_0\sum_{i=0}^{n}(t_{i+1}-s_i),
\end{aligned} \tag{3.5}$$

which implies that $\{u_k\}$ is bounded in $X$. Since $X$ is a reflexive Banach space, going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ in $X$, then $u_k \to u$ in $L^p[0, T]$ and $u_k \to u$ $a.e.$ $t \in [0, T]$.

Since $\lim\limits_{k\to\infty} \varphi'(u_k) = 0$ and $\{u_k\}$ converges weakly to some $u$, one has

$$\langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle \to 0, \quad \text{as } k \to +\infty. \tag{3.6}$$

From (2.9), we have

$$\begin{aligned}
&\langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle \\
=&\int_0^T (|u_k'(t)|^{p-2}u_k'(t) - |u'(t)|^{p-2}u'(t))(u_k'(t) - u'(t))dt \\
&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} (D_xF_i(t, u_k(t)-u_k(t_{i+1})) - D_xF_i(t, u(t)-u(t_{i+1}))) \\
&\times (u_k(t) - u_k(t_{i+1}) - u(t) + u(t_{i+1})) \\
\geq&\int_0^T (|u_k'|^{p-2} - |u'|^{p-2})(u_k' - u')dt \\
&-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} |D_xF_i(t, u_k(t)-u_k(t_{i+1})) - D_xF_i(t, u(t)-u(t_{i+1}))| \\
&\times |u_k(t) - u_k(t_{i+1}) - u(t) + u(t_{i+1})|.
\end{aligned} \tag{3.7}$$

In view of assumption (H1) and

$$\begin{aligned}
|u_k(t) - u_k(t_{i+1}) - u(t) + u(t_{i+1})| \leq& |u_k(t) - u(t)| + |u_k(t_{i+1}) - u(t_{i+1})| \\
\leq& 2\|u_k - u\|_\infty \to 0, \quad \text{as } k \to \infty,
\end{aligned}$$

we obtain that the second term on the right hand of (3.7) converges to 0 as $k \to \infty$.

By [27, Eq (2.2)], there exist $c_p, d_p > 0$ such that

$$\begin{aligned}
&\int_0^T (|u_k'(t)|^{p-2}u_k'(t) - |u'(t)|^{p-2}u'(t))(u_k'(t) - u'(t))dt \\
\geq&\begin{cases} c_p\int_0^T |u_k'(t) - u'(t)|^p dt, & \text{if } p \geq 2. \\ d_p\int_0^T \frac{|u_k'(t)-u'(t)|^2}{(|u_k'(t)|+|u'(t)|)^{2-p}}dt, & \text{if } 1 < p < 2. \end{cases}
\end{aligned}$$

If $p \geq 2$, it follows from (3.6) and (3.7) that $\|u_k - u\|_X \to 0$ as $k \to \infty$.

If $1 < p < 2$, by Hölder's inequality, one has

$$\begin{aligned}
&\int_0^T |u_k'(t) - u'(t)|^p dt \\
\leq&\left(\int_0^T \frac{|u_k'(t) - u'(t)|^2}{(|u_k'(t) + u'(t)|)^{2-p}} dt\right)^{\frac{p}{2}} \left(\int_0^T (|u_k'(t)| + |u'(t)|)^p dt\right)^{\frac{2-p}{2}} \\
\leq&\left(\int_0^T \frac{|u_k'(t) - u'(t)|^2}{(|u_k'(t) + u'(t)|)^{2-p}} dt\right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^T (|u_k'(t)|^p + |u'(t)|^p) dt\right)^{\frac{2-p}{2}} \\
\leq&2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^T \frac{|u_k'(t) - u'(t)|^2}{(|u_k'(t) + u'(t)|)^{2-p}} dt\right)^{\frac{p}{2}} (\|u_k\|_X + \|u\|_X)^{\frac{(2-p)p}{2}}.
\end{aligned} \tag{3.8}$$

It follows from $1 < p < 2$ and (3.8) that

$$\begin{aligned}
&\int_0^T (|u_k'(t)|^{p-2} - |u'(t)|^{p-2})(u_k(t) - u(t)) dt \\
\geq&\frac{2^{\frac{(p-1)(p-2)}{p}} d_p}{(\|u_k\|_X + \|u\|_X)^{2-p}} \left(\int_0^T |u_k'(t) - u'(t)|^p dt\right)^{\frac{2}{p}} = \frac{2^{\frac{(p-1)(p-2)}{p}} d_p \|u_k - u\|_X^2}{(\|u_k\|_X + \|u\|_X)^{2-p}}.
\end{aligned} \tag{3.9}$$

In view of (3.6)–(3.9), one has $\|u_k - u\|_X \to 0$ as $k \to \infty$.

Thus, $\{u_k\}$ converges strongly to $u$ in $X$.

**Step 2.** $\varphi(u)$ has mountain pass geometric structure.

By condition (H4), there exists $\epsilon \in (0, 1)$ such that

$$F_i(t, x) \leq (1 - \epsilon) A^* |x|^{\mu_i}$$

uniformly for *a.e.* $t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$ with $|x| \leq 1$. Taking $\rho = \frac{1}{2C}$, where $C$ is listed in Lemma 2.10, then

$$\|u\|_\infty \leq \frac{1}{2} \quad \text{and} \quad |u(t) - u(t_i)| \leq 1$$

for each $u \in X$ with $\|u\|_X = \rho$. Thus, for every $u \in X$, we have

$$\begin{aligned}
\varphi(u) =&\frac{1}{p}\|u\|_X^p - \sum_{i=1}^n (\beta_{i-1} - \beta_i) u(t_i) - \sum_{i=0}^n \int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1})) dt \\
\geq&\frac{1}{2^p C^p p} - \frac{1}{2}\sum_{i=1}^n |\beta_{i-1} - \beta_i| - (1 - \epsilon) A^* \sum_{i=0}^n \int_{s_i}^{t_{i+1}} |u(t) - u(t_{i+1})|^{\mu_i} dt \\
\geq&\frac{1}{2^p C^p p} - \frac{1}{2}\sum_{i=1}^n |\beta_{i-1} - \beta_i| - (1 - \epsilon) A^* \sum_{i=0}^n (t_{i+1} - s_i) \\
\geq&\frac{1}{2^p C^p p} - \frac{1}{2}\sum_{i=1}^n |\beta_{i-1} - \beta_i| - A^* \sum_{i=0}^n (t_{i+1} - s_i) + \epsilon A^* \sum_{i=0}^n (t_{i+1} - s_i) \\
=&\epsilon A^* \sum_{i=0}^n (t_{i+1} - s_i) = \eta > 0
\end{aligned} \tag{3.10}$$

In view of (3.10), we have $\varphi(u) \geq \eta > 0 = \varphi(0)$ for any $u \in \partial\Omega_\rho$. Thus,

$$\inf_{u\in\partial\Omega_\rho} \varphi(u) > \varphi(0).$$

It follows from condition (H3) that there exist constants $c_i,\ d_i > 0$ such that

$$F_i(t,x) \geq c_i|x|^{\mu_i} - d_i, \quad x \in \mathbb{R},\ a.e.\ t \in (s_i, t_{i+1}],\ i = 1,2,...,n.$$

Let $\xi > 0$ and $w \in X$ with $\|w\|_X = 1$, and $w(t)$ is not constant for a.e. $t \in [0, t_1]$. Thus, we have

$$\begin{aligned}\varphi(\xi w) =& \frac{1}{p}\|\xi w\|_X^p - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\xi w(t_i) - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t, \xi w(t) - \xi w(t_{i+1}))dt \\ \leq& \frac{\xi^p}{p} + \xi\sum_{i=1}^{n}|\beta_{i-1} - \beta_i|C\|w\|_X - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} (c_i|\xi w(t) - \xi w(t_{i+1})|^{\mu_i} - d_i)dt \\ \leq& \frac{\xi^p}{p} + C\xi\sum_{i=1}^{n}|\beta_{i-1} - \beta_i| - \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} c_i\xi^{\mu_i}|w(t) - w(t_{i+1})|^{\mu_i}dt \\ &+ \sum_{i=0}^{n} d_i(t_{i+1} - s_i).\end{aligned} \tag{3.11}$$

Let $P_i = \int_{s_i}^{t_{i+1}} c_i|w(t) - w(t_{i+1})|^{\mu_i}dt$, then we have

$$0 \leq P_i \leq (2C)^{\mu_i}c_i(t_{i+1} - s_i), \quad i = 0,1,2,...,n.$$

Since $w(t)$ is not constant for a.e. $t \in [0, t_1]$, we have $P_0 = \int_0^{t_1} c_0|w(t) - w(t_1)|^{\mu_0}dt > 0$. From (3.11), we have

$$\varphi(\xi w) \leq \frac{\xi^p}{p} + C\xi\sum_{i=1}^{n}|\beta_{i-1} - \beta_i| - \sum_{i=0}^{n} P_i\xi^{\mu_i} + \sum_{i=0}^{n} d_i(t_{i+1} - s_i).$$

Since $\mu_i > p$, the above formula implies $\varphi(\xi w) \to -\infty$ as $\xi \to +\infty$. Hence, there exist $\xi_0$ with $\|\xi_0 w\|_X > \rho$ such that $\inf_{u\in\partial\Omega_\rho} \varphi(u) > \varphi(\xi_0 w)$. It follows from Steps 1–2 and Lemma 2.8 that there exists $u_* \in X$ such that

$$\varphi'(u_*) = 0 \quad \text{with} \quad \varphi(u_*) > \max\{\varphi(0), \varphi(\xi_0 w)\} \geq \varphi(0) = 0. \tag{3.12}$$

Hence, $u_*$ is a nontrival weak solution of problem (1.1).

**Step 3.** For some $i = 1,2,...,n$, $\beta_{i-1} \neq \beta_i$, we will prove that $\varphi(u)$ has a nonzero local minimum $u^*$ in $\overline{\Omega_\rho}$.

Since $\overline{\Omega_\rho}$ is closed-convex set, then $\overline{\Omega_\rho}$ is sequentially weakly closed. Moreover, $\varphi(u)$ is continuous differential and sequentially weakly lower semi-continuous on $X$ as the sum of a convex function and of a weakly continuous one. Therefore, it follows from Lemma 2.7 that there exists a $u^* \in \overline{\Omega_\rho}$ such that $\varphi(u^*) = \min_{\overline{\Omega_\rho}} \varphi(u)$.

By condition (H3), we have $F_i(t,0) = 0$ for $a.e.\ t \in (s_i, t_{i+1}]$ and $F_i(t,x) \geq 0$ for $a.e.\ t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$. Thus, $\phi_i(u) = \int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1}))dt \geq 0$. Let

$$\widehat{u(t)} = \begin{cases}\beta_{i-1} - \beta_i, & \text{if } t = t_i, \\ 0, & \text{if } t \in [0,T] \text{ and } t \neq t_i;\end{cases} \tag{3.13}$$

then $\widehat{u} \in \overline{\Omega_\rho}$. By (2.5) and (3.13), we have $\varphi(\widehat{u}) \le -(\beta_{i-1} - \beta_i)^2 < 0$. So

$$\varphi(u^*) \le \varphi(\widehat{u}) < 0 \tag{3.14}$$

which implies that $u^* \ne 0$.

**Step 4.** $u_*$ and $u^*$ are different and both bounded.

From (3.12) and (3.14), we have

$$\varphi(u_*) > 0 > \varphi(u^*), \tag{3.15}$$

thus $u_*$ and $u^*$ are different. In view of the inf–max characterization of $u_*$ in Lemma 2.8 and (3.11), we have

$$\varphi(u_*) = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)) \le \max_{s\in[0,1]} \varphi(\xi_0 w s) \le \max_{s\in[0,1]} h(s), \tag{3.16}$$

where

$$h(s) = \frac{\xi_0^p s^p}{p} + \sum_{i=1}^n |\beta_{i-1} - \beta_i| C\xi_0 s - \sum_{i=0}^n P_i \xi_0^{\mu_i} s^{\mu_i} + \sum_{i=0}^n d_i(t_{i+1} - s_i).$$

Obviously, $h(s)$ is continuous on $[0, 1]$. (3.16) implies $\varphi(u_*)$ is bounded above and so is $\varphi(u^*)$ by (3.15). For $\widetilde{u} \in X$, similar as (3.5), we obtain

$$\mu\varphi(\widetilde{u}) - \langle \varphi'(\widetilde{u}), \widetilde{u}\rangle \ge \frac{\mu - 1}{p}\|\widetilde{u}\|_X^p - (\mu - 1)\sum_{i=1}^n |\beta_{i-1} - \beta_i| C\|\widetilde{u}\|_X - C_0 \sum_{i=0}^n (t_{i+1} - s_i). \tag{3.17}$$

For $u_*$ and $u^*$ are both critical points of $\varphi$, moreover $\varphi(u_*)$ and $\varphi(u^*)$ are both bounded above, (3.17) implies that $u_*$ and $u^*$ are both bounded in $X$. □

*Proof of Theorem 1.3.* According to condition (H5) and (3.10), for any $\epsilon > 0$, there exists a $\eta := \epsilon A^* \sum_{i=0}^n (t_{i+1} - s_i)$ such that the functional $\varphi(u) \ge \eta > 0$. By the similar proof of Theorem 1.2, we can obtain Theorem 1.3 holds, and we omit the rest of proof of it. □

*Proof of Theorem 1.4.* Let $u$ be a nontrivial solution of problem (1.1), then we obtain

$$\begin{aligned} 0 = \langle \varphi'(u), u\rangle =& \|u\|_X^p - \sum_{i=1}^n (\beta_{i-1} - \beta_i) u(t_i) \\ &- \sum_{i=0}^n \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))(u(t) - u(t_{i+1}))dt. \end{aligned}$$

By virtue of condition (H6), (2.6), we have

$$m_1(|u(t) - u(t_i)|) \le m_1(2C\|u\|_X).$$

Furthermore, in view of conditions (H1) and (H7), we obtain

$$\begin{aligned} 0 \ge& \|u\|_X^p - \sum_{i=1}^n |\beta_{i-1} - \beta_i||u(t_i)| \\ &- \sum_{i=0}^n \int_{s_i}^{t_{i+1}} |D_x F_i(t, u(t) - u(t_{i+1}))||(u(t) - u(t_{i+1}))|dt \end{aligned}$$

$$\geq \|u\|_X^p - \sum_{i=1}^{n} |\beta_{i-1} - \beta_i| C \|u\|_X$$
$$- \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} m_1(|u(t) - u(t_{i+1})|) m_2(t) |u(t) - u(t_{i+1})| dt$$
$$\geq \|u\|_X^p - \sum_{i=1}^{n} |\beta_{i-1} - \beta_i| C \|u\|_X - 2C\|u\|_X \cdot m_1(2C\|u\|_X) \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} m_2(t) dt > 0,$$

which is a contradiction. Hence, problem (1.1) has no nontrivial solution. □

## 4. Examples

*Example 4.1.* Let $T = 1$, consider the following problem

$$\begin{cases} -(|u'(t)|^2 u'(t))' = D_x F_i(t, u(t) - u(t_{i+1})), & t \in (s_i, t_{i+1}],\ i = 0, 1, \\ (|u'(t)|^2 u'(t)) = 0.1, & t \in (t_1, s_1], \\ |u'(s_1^+)|^2 u'(s_1^+) = |u'(s_1^-)|^2 u'(s_1^-), \\ |u'(0)|^2 u'(0) = 0.2, & u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $p = 4$, $n = 1$, $0 = s_0 < t_1 = \frac{1}{16} < s_1 = \frac{15}{16} < t_2 = 1$, and $D_x F_i(t, u(t) - u(t_{i+1})) = (1 + t)(u(t) - u(t_{i+1}))^2$. It is easy to verify that the conditions (H1) and (H2) are satisfied. Hence, the problem (4.1) has at least one nontrivial weak solution by Theorem 1.1. □

*Example 4.2.* Let $T = 1$, consider the following problem

$$\begin{cases} -(|u'(t)| u'(t))' = D_x F_i(t, u(t) - u(t_{i+1})), & t \in (s_i, t_{i+1}],\ i = 0, 1, \\ (|u'(t)| u'(t)) = 0.01, & t \in (t_1, s_1], \\ |u'(s_1^+)| u'(s_1^+) = |u'(s_1^-)| u'(s_1^-), \\ |u'(0)|^2 u'(0) = 0.02, & u(0) = u(1) = 0, \end{cases} \tag{4.2}$$

where $p = 3$, $n = 1$, $0 = s_0 < t_1 = \frac{1}{8} < s_1 = \frac{7}{8} < t_2 = 1$, $\mu_i = 4$ and $D_x F_i(t, u(t) - u(t_{i+1})) = t(u(t) - u(t_{i+1}))^5$. By simple calculations, we obtain $C \approx 1.3282$ and $A^* \approx 0.0511$. It is obvious that all conditions of Theorem 1.2 hold. So the problem (4.2) has at least two nontrivial weak solutions.

□


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## Conflict of interest

The author declares no conflict of interest.

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