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Research article

Evaluation of fractional-order equal width equations with the exponential-decay kernel

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Abstract: In this article we consider the homotopy perturbation transform method to investigate the fractional-order equal-width equations. The homotopy perturbation transform method is a mixture of the homotopy perturbation method and the Yang transform. The fractional-order derivative are defined in the sense of Caputo-Fabrizio operator. Several fractions of solutions are calculated which define some valuable evolution of the given problems. The homotopy perturbation transform method results are compared with actual results and good agreement is found. The suggested method can be used to investigate the fractional perspective analysis of problems in a variety of applied sciences.

Keywords: homotopy perturbation method; Yang transform; equal width equations; Caputo-Fabrizio operator
Mathematics Subject Classification: 34A34, 35A20, 35A22, 44A10, 33B15

1. Introduction

Many academics have explored fractional evaluation equations over the last century due to their vast relevance in various fields of science and technology. It has been observed that fractional order equations can be used to describe numerous physical systems and to address a variety of problems. Consequently, achieving more productive results in fractional calculus [1–6] is essential to achieving the whole objective. Caputo deemed the fractional Caputo derivative [7] to be the many suitable technique for discovering fractional systems because it accurately incorporates beginning conditions that are absent from numerous specific designs [8]. Oldham et al. conclude that fractional-order derivative and integral can be utilized to demonstrate significantly more useful synthetic issues than current approaches [9]. In addition, the literature contains further agreements on fractional theories and applications, such as fractal mathematics [10–13].

Numerous scholars have worked on partial evaluation equations in recent years due to its extensive applications in numerous scientific and technological fields. These fractional equations are suitable for describing a variety of remarkable phenomena in classical dynamic, acoustics, electrodynamics, material science, plasma physics, electrostatics, viscoelasticity, optoelectronic frameworks, and so on [14, 15]. The fractional non-linear equal width (EW) equations are extremely important partial differential equations that show various complicated non-linear phenomena in the area of science, solid state physics, particularly plasma waves, astrophysics, materials science, chemical physics, etc. The EW equations characterized the behaviour of nonlinear waves in a variety of nonlinear schemes, including magnetohydrodynamic waves in nanoparticle waves in plasma, surface waves in fluid flow, cold plasma, shallow water waves, acoustic waves in enharmonic crystal, etc [16–19].

He formulated the homotopy perturbation method (HPM) [20] in 1999, which is a mixture of the homotopy technique and the classical perturbation method and has been broadly utilized both on linear and non-linear equations [21, 22]. The significance of the homotopy perturbation method lies in the fact that it does not require a small parameter in the equation, thereby mitigating the disadvantages of conventional perturbation methods. The main aim of this paper is to implement integral transform named "Yang transformation" discovered by Yang [23] with homotopy perturbation method to solve nonlinear fractional order partial differential equations. We solve nonlinear EW equations through the homotopy perturbation transform method (HPTM). We obtain a power series solution within the context of a rapidly convergent series, and only a few iterations are required to obtain extremely efficient results. There is no need for discretization or linearization of the nonlinear problem, and only a few iterations are required to arrive at a solution that can be quickly estimated using these methods.

Due to the aforementioned tendency, the fractional-order nonlinear equal width equations are solved utilizing the HPTM. For renewability analytic technique, the Yang transform integrates the HPTM in an efficient manner. Several transforms are combined to form the Yang transform. Both of these technique produce interpretive results in the form of a convergent series. The fractional derivative operator of Caputo-Fabrizio is used to explain quantitative categorizations of nonlinear equal width equations. In modeling and enumeration investigations, the method provided have been proven effective. The exact-analytical findings are a valuable method for analyzing the dynamics of computationally challenging systems, particularly fractional PDEs. Using this approximate expression, financial and monetary phenomena can be examined.

2. Preliminaries concepts

In this part, we address several key ideas, conceptions, and terminologies related to fractional derivative operators involving index and exponential decay as a kernel, as well as the Yang transform specific repercussions.

Definition 2.1. If $\mathbb{P}(\eta) \in \mathbf{H}^1[0,T], T > 0$, then the Caputo-Fabrizio (CF) derivative is defined as follows [24]:

$$^{CF}D_\eta^\alpha[\mathbb{P}(\eta)] = \frac{N(\alpha)}{1-\alpha}\int_0^\eta \mathbb{P}^{'}(\varrho)K(\eta,\varrho)d\varrho, \quad 0 < \alpha \leq 1. \tag{2.1}$$

$N(\alpha)$ is the normalization term with $N(0) = N(1) = 1$. However, if $\mathbb{P}(\eta) \notin \mathbf{H}^1[0,T]$, then the above derivative is defined as follows:

$$^{CF}D_\eta^\alpha[\mathbb{P}(\eta)] = \frac{N(\alpha)}{1-\alpha}\int_0^\eta [\mathbb{P}(\eta) - \mathbb{P}(\varrho)]K(\eta,\varrho)d\varrho. \tag{2.2}$$

Definition 2.2. The fractional CF integral is given as [24]

$$^{CF}I_\eta^\alpha[\mathbb{P}(\eta)] = \frac{1-\alpha}{N(\alpha)}\mathbb{P}(\eta) + \frac{\alpha}{N(\alpha)}\int_0^\eta \mathbb{P}(\varrho)d\varrho, \quad \eta \geq 0, \alpha \in (0,1]. \tag{2.3}$$

Definition 2.3. For $N(\alpha) = 1$, shows the Laplace transform of CF derivative is defined as [24]:

$$L\left[^{CF}D_\eta^\alpha[\mathbb{P}(\eta)]\right] = \frac{vL[\mathbb{P}(\eta) - \mathbb{P}(0)]}{v + \alpha(1-v)}. \tag{2.4}$$

Definition 2.4. The Yang transform of $\mathbb{P}(\eta)$ is given as [26]

$$\mathbb{Y}[\mathbb{P}(\eta)] = \chi(v) = \int_0^\infty \mathbb{P}(\eta)e^{-\frac{\eta}{v}}d\eta. \ \eta > 0, \tag{2.5}$$

Remark. Yang transform of some useful function is define as below.

$$\begin{aligned} \mathbb{Y}[1] =& v, \\ \mathbb{Y}[\eta] =& v^2, \\ \mathbb{Y}[\eta^i] =& \Gamma(i+1)v^{i+1}. \end{aligned} \tag{2.6}$$

Lemma 2.5. (Laplace-Yang duality).
Let the Laplace transformation of $\mathbb{P}(\eta)$ is $F(v)$, then $\chi(v) = F(1/v)$.
Proof. For proof, see reference [25].
Lemma 2.6. Let $\mathbb{P}(\eta)$ is function of continuous; then, the CF derivative of Yang transform of $\mathbb{P}(\eta)$ is defined by [25]

$$\mathbb{Y}[\mathbb{P}(\eta)] = \frac{\mathbb{Y}[\mathbb{P}(\eta) - v\mathbb{P}(0)]}{1 + \alpha(v-1)}. \tag{2.7}$$

Proof. The fractional CF Laplace transform is expressed as

$$L[\mathbb{P}(\eta)] = \frac{L[v\mathbb{P}(\eta) - \mathbb{P}(0)]}{v + \alpha(1-v)}. \tag{2.8}$$

Also, we have that the connection among Laplace and Yang property, i.e., $\chi(v) = F(1/v)$. To obtain the desired conclusion, we substitute v with $1/v$ in Eq (2.8), obtaining

$$\begin{aligned} \mathbb{Y}[\mathbb{P}(\eta)] =& \frac{\frac{1}{v}\mathbb{Y}[\mathbb{P}(\eta) - \mathbb{P}(0)]}{\frac{1}{v} + \alpha(1 - \frac{1}{v})}, \\ \mathbb{Y}[\mathbb{P}(\eta)] =& \frac{\mathbb{Y}[\mathbb{P}(\eta) - v\mathbb{P}(0)]}{1 + \alpha(v-1)}. \end{aligned} \tag{2.9}$$

The proof is completed.

3. Methodology of the HPTM

The HPTM method for solving generic nonlinear fractional CF partial differential equations. Take the generic non-linear CF partial differential equations with nonlinear function as an example. $N(\mathbb{V}(\psi,\eta))$ with linear fractional notation $L(\mathbb{V}(\psi,\eta))$ is the same as [25]

$$\begin{cases} {}^{CF}D_{\eta}^{\alpha}\mathbb{V}(\varsigma,\eta)+L(\mathbb{V}(\varsigma,\eta))+N(\mathbb{V}(\varsigma,\eta))=g(\varsigma,\eta), \\ \mathbb{V}(\varsigma,0)=h(\varsigma), \end{cases} \tag{3.1}$$

the source term is $g(\varsigma,\eta)$. Applying Yang transform to Eq (3.1), we get

$$\frac{\mathbb{Y}[\mathbb{V}(\varsigma,\eta)-v\mathbb{V}(\varsigma,0)]}{1+\alpha(v-1)}=-\mathbb{Y}[L(\mathbb{V}(\varsigma,\eta))+N(\mathbb{V}(\varsigma,\eta))]+\mathbb{Y}[g(\varsigma,\eta)],$$

$$\mathbb{Y}[\mathbb{V}(\varsigma,\eta)]=vh(\varsigma)-(1+\alpha(v-1))[\mathbb{Y}[L(\mathbb{V}(\varsigma,\eta))+N(\mathbb{V}(\varsigma,\eta))]+\mathbb{Y}[g(\varsigma,\eta)]. \tag{3.2}$$

Implement inverse Yang transformation, we obtain

$$\mathbb{V}(\varsigma,\eta)=\mathbb{V}(\varsigma,0)-\mathbb{Y}^{-1}[(1+\alpha(v-1))[\mathbb{Y}[L(\mathbb{V}(\varsigma,\eta))+N(\mathbb{V}(\varsigma,\eta))]+\mathbb{Y}[g(\varsigma,\eta)]], \tag{3.3}$$

where the source term is $\mathbb{V}(\varsigma,\eta)$. Now, we use HPTM

$$\mathbb{V}(\varsigma,\eta)=\sum_{i=0}^{\infty}\rho^{i}\mathbb{V}_{i}(\varsigma,\eta). \tag{3.4}$$

We decompose the nonlinear term $N(\mathbb{V}(\varsigma,\eta))$ as

$$N(\mathbb{V}(\varsigma,\eta))=\sum_{i=0}^{\infty}\rho^{i}H_{i}(\mathbb{V}), \tag{3.5}$$

where He's polynomial is $H_i(\mathbb{V})$:

$$H_i(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\cdots,\mathbb{V}_i)=\frac{1}{\Gamma(i+1)}\frac{\partial^i}{\partial\rho^i}\left[N\left(\sum_{i=0}^{\infty}\rho^i\mathbb{V}_i\right)\right]_{\rho=0},\quad i=1,2,3,\cdots. \tag{3.6}$$

Substituting Eqs (3.4) and (3.5) in Eq (3.3), we obtain

$$\sum_{i=0}^{\infty}\rho^{i}\mathbb{V}_{i}(\varsigma,\eta)=\mathbb{V}(\varsigma,\eta)-\rho\left(\mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left[L\sum_{i=0}^{\infty}\rho^{i}\mathbb{V}_{i}(\varsigma,\eta)+N\sum_{i=0}^{\infty}\rho^{i}H_{i}(\mathbb{V})\right]\right]\right). \tag{3.7}$$

We acquire the following terms by comparing coefficients: of ρ in (3.7):

$$\begin{aligned} \rho^0 : \mathbb{V}_0(\varsigma,\eta) =& \mathbb{V}(\varsigma,\eta), \\ \rho^1 : \mathbb{V}_1(\varsigma,\eta) =& \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left[L(\mathbb{V}_0(\varsigma,\eta))+H_0(\mathbb{V})\right]\right], \\ \rho^2 : \mathbb{V}_2(\varsigma,\eta) =& \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left[L(\mathbb{V}_1(\varsigma,\eta))+H_1(\mathbb{V})\right]\right], \\ \rho^3 : \mathbb{V}_3(\varsigma,\eta) =& \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left[L(\mathbb{V}_2(\varsigma,\eta))+H_2(\mathbb{V})\right]\right], \\ &\vdots \\ \rho^i : \mathbb{V}_i(\varsigma,\eta) =& \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left[L(\mathbb{V}_i(\varsigma,\eta))+H_i(\mathbb{V})\right]\right]. \end{aligned} \tag{3.8}$$

As a solution, the achieved result of Eq (3.1) is given as:

$$\mathbb{V}(\varsigma,\eta) = \mathbb{V}_0(\varsigma,\eta) + \mathbb{V}_1(\varsigma,\eta) + \cdots . \tag{3.9}$$

Convergence and error analysis

The following theorems are crucial to convergence and error analysis method that handle the fundamental frameworks (3.1).

Theorem 3.1. Let $\mathbb{V}(\varsigma,\eta)$ be the exact solution of (3.1) and let $\mathbb{V}_i(\varsigma,\eta) \in H$ and $\sigma \in (0,1)$, where the Hilbert space show by H. Then, the obtained solution $\sum_{i=0}^{\infty} \mathbb{V}_i(\varsigma,\eta)$ will convergence $\mathbb{V}(\varsigma,\eta)$ if $\mathbb{V}_i(\varsigma,\eta) \leq \mathbb{V}_{i-1}(\varsigma,\eta)\ \ \forall i > A$, i.e., for any $\omega > 0 \exists A > 0$, such that $\|\mathbb{V}_{i+n}(\varsigma,\eta)\| \leq \beta, \forall i,n \in N$ [25].

Proof. We make a sequence of $\sum_{i=0}^{\infty} \mathbb{V}_i(\varsigma,\eta)$.

$$\begin{aligned}
C_0(\varsigma,\eta) =& \mathbb{V}_0(\varsigma,\eta),\\
C_1(\varsigma,\eta) =& \mathbb{V}_0(\varsigma,\eta) + \mathbb{V}_1(\varsigma,\eta),\\
C_2(\varsigma,\eta) =& \mathbb{V}_0(\varsigma,\eta) + \mathbb{V}_1(\varsigma,\eta) + \mathbb{V}_2(\varsigma,\eta),\\
C_3(\varsigma,\eta) =& \mathbb{V}_0(\varsigma,\eta) + \mathbb{V}_1(\varsigma,\eta) + \mathbb{V}_2(\varsigma,\eta) + \mathbb{V}_3(\varsigma,\eta),\\
&\vdots\\
C_i(\varsigma,\eta) =& \mathbb{V}_0(\varsigma,\eta) + \mathbb{V}_1(\varsigma,\eta) + \mathbb{V}_2(\varsigma,\eta) + \cdots + \mathbb{V}_i(\varsigma,\eta),
\end{aligned} \tag{3.10}$$

To produce the proper result, we must establish that $C_i(\varsigma,\eta)$ constitutes a "Cauchy sequences". Consider, for instance,

$$\begin{aligned}
\|C_{i+1}(\varsigma,\eta) - C_i(\varsigma,\eta)\| = \|\mathbb{V}_{i+1}(\varsigma,\eta)\| \leq \sigma\|\mathbb{V}_i(\varsigma,\eta)\| \leq \sigma^2\|\mathbb{V}_{i-1}(\varsigma,\eta)\| \leq \sigma^3\|\mathbb{V}_{i-2}(\varsigma,\eta)\| \cdots\\
\leq \sigma_{i+1}\|\mathbb{V}_0(\varsigma,\eta)\|.
\end{aligned} \tag{3.11}$$

For $i,n \in N$, we acquire

$$\begin{aligned}
\|C_i(\varsigma,\eta) - C_n(\varsigma,\eta)\| =& \|\mathbb{V}_{i+n}(\varsigma,\eta)\| = \|C_i(\varsigma,\eta) - C_{i-1}(\varsigma,\eta) + (C_{i-1}(\varsigma,\eta) - C_{i-2}(\varsigma,\eta))\\
&+ (C_{i-2}(\varsigma,\eta) - C_{i-3}(\varsigma,\eta)) + \cdots + (C_{n+1}(\varsigma,\eta) - C_n(\varsigma,\eta))\|\\
\leq& \|C_i(\varsigma,\eta) - C_{i-1}(\varsigma,\eta)\| + \|(C_{i-1}(\varsigma,\eta) - C_{i-2}(\varsigma,\eta))\|\\
&+ \|(C_{i-2}(\varsigma,\eta) - C_{i-3}(\varsigma,\eta))\| + \cdots + \|(C_{n+1}(\varsigma,\eta) - C_n(\varsigma,\eta))\|\\
\leq& \sigma^i\|\mathbb{V}_0(\varsigma,\eta)\| + \sigma^{i-1}\|\mathbb{V}_0(\varsigma,\eta)\| + \cdots + \sigma^{i+1}\|\mathbb{V}_0(\varsigma,\eta)\|\\
=& \|\mathbb{V}_0(\varsigma,\eta)\|(\sigma^i + \sigma^{i-1} + \sigma^{i+1})\\
=& \|\mathbb{V}_0(\varsigma,\eta)\|\frac{1-\sigma^{i-n}}{1-\sigma^{i+1}}\sigma^{n+1}.
\end{aligned} \tag{3.12}$$

Since $0 < \sigma < 1$, and $\mathbb{V}_0(\varsigma,\eta)$ is bounded, let us take $\beta = 1 - \sigma/(1-\sigma_{i-n})\sigma^{n+1}\|\mathbb{V}_0(\varsigma,\eta)\|$, and we obtain Thus, $\{\mathbb{V}_i(\varsigma,\eta)\}_{i=0}^{\infty}$ forms a "Cauchy sequence" in H. It follows that the sequence $\{\mathbb{V}_i(\varsigma,\eta)\}_{i=0}^{\infty}$ is a convergent sequence with the limit $\lim_{i\to\infty} \mathbb{V}_i(\varsigma,\eta) = \mathbb{V}(\varsigma,\eta)$ for $\exists \mathbb{V}(\varsigma,\eta) \in \mathcal{H}$. Hence, this ends the proof.

Theorem 3.2. Let $\sum_{h=0}^{k} \mathbb{V}_h(\varsigma,\eta)$ is finite and $\mathbb{V}(\varsigma,\eta)$ represents the obtained series solution. Let $\sigma > 0$ such that $\|\mathbb{V}_{h+1}(\varsigma,\eta)\| \leq \|\mathbb{V}_h(\varsigma,\eta)\|$, then the following relation gives the maximum absolute error [25].

$$\|\mathbb{V}(\varsigma,\eta) - \sum_{h=0}^{k} \mathbb{V}_h(\varsigma,\eta)\| < \frac{\sigma^{k+1}}{1-\sigma}\|\mathbb{V}_0(\varsigma,\eta)\|. \tag{3.13}$$

Proof. Since $\sum_{h=0}^{k}\mathbb{V}_h(\varsigma,\eta)$ is finite, this implies that $\sum_{h=0}^{k}\mathbb{V}_h(\varsigma,\eta)<\infty$.
Consider

$$\begin{aligned}
\|\mathbb{V}(\varsigma,\eta)-\sum_{h=0}^{k}\mathbb{V}_h(\varsigma,\eta)\| =& \|\sum_{h=k+1}^{\infty}\mathbb{V}_h(\varsigma,\eta)\| \\
\leq& \sum_{h=k+1}^{\infty}\|\mathbb{V}_h(\varsigma,\eta)\| \\
\leq& \sum_{h=k+1}^{\infty}\sigma^h\|\mathbb{V}_0(\varsigma,\eta)\| \\
\leq& \sigma^{k+1}(1+\sigma+\sigma^2+\cdots)\|\mathbb{V}_0(\varsigma,\eta)\| \\
\leq& \frac{\sigma^{k+1}}{1-\sigma}\|\mathbb{V}_0(\varsigma,\eta)\|.
\end{aligned} \tag{3.14}$$

This ends the theorem's proof.

4. Implementation of the technique

4.1. Example

Consider the fractional nonlinear EW equation

$$D_\eta^\alpha\mathbb{V}+\mathbb{V}\mathbb{V}_\varsigma-\mathbb{V}_{\varsigma\varsigma\eta}=0,\quad \eta>0,\quad \varsigma\in R,\quad 0<\alpha\leq 1, \tag{4.1}$$

with the initial condition

$$\mathbb{V}(\varsigma,0)=3sech^2\left(\frac{\varsigma-15}{2}\right). \tag{4.2}$$

Employing the Yang transform on (4.1) with initial condition (4.2), we have

$$\begin{aligned}
\frac{1}{(1+\alpha(v-1))}\mathbb{Y}\Big(\mathbb{V}(\varsigma,\eta)\Big)-\frac{v}{(1+\alpha(v-1))}\mathbb{V}(\varsigma,0) &= \mathbb{Y}\Big[\mathbb{V}_{\varsigma\varsigma\eta}-\mathbb{V}\mathbb{V}_\varsigma\Big], \\
\mathbb{Y}\Big[\mathbb{V}(\varsigma,\eta)\Big] = v3sech^2\left(\frac{\varsigma-15}{2}\right)&+(1+\alpha(v-1))\mathbb{Y}\Big[\mathbb{V}_{\varsigma\varsigma\eta}-\mathbb{V}\mathbb{V}_\varsigma\Big].
\end{aligned} \tag{4.3}$$

Now using the inverse Yang transform we have

$$\mathbb{V}(\varsigma,\eta)=3sech^2\left(\frac{\varsigma-15}{2}\right)+\mathbb{Y}^{-1}\Big[(1+\alpha(v-1))\mathbb{Y}\Big\{\mathbb{V}_{\varsigma\varsigma\eta}-\mathbb{V}\mathbb{V}_\varsigma\Big\}\Big]. \tag{4.4}$$

Now we implemented HPM, we get

$$\sum_{\jmath=0}^{\infty}p^\jmath\mathbb{V}_\jmath(\varsigma,\eta)=3sech^2\left(\frac{\varsigma-15}{2}\right)+p\left[\mathbb{Y}^{-1}\left\{(1+\alpha(v-1))\mathbb{Y}\left(\left(\sum_{\jmath=0}^{\infty}p^\jmath\mathbb{V}_\jmath(\varsigma,\eta)_{\varsigma\varsigma\eta}\right)-\left(\sum_{\jmath=0}^{\infty}p^\jmath H_\jmath(\mathbb{V})\right)\right)\right\}\right]. \tag{4.5}$$

The nonlinear term can be find with the help of He's polynomials

$$\Sigma_{J=0}^{\infty} p^J H_J(\mathbb{V}) = \mathbb{V}\mathbb{V}_{\varsigma}. \tag{4.6}$$

The He's polynomials can be written as

$$\begin{aligned} H_0(\mathbb{V}) &= \mathbb{V}_0(\mathbb{V}_0)_{\varsigma}, \\ H_1(\mathbb{V}) &= \mathbb{V}_0(\mathbb{V}_1)_{\varsigma} + \mathbb{V}_1(\mathbb{V}_0)_{\varsigma}, \\ &\vdots \end{aligned}$$

Coefficients p comparing, we obtain as

$$\begin{aligned} p^0 &: \mathbb{V}_0(\varsigma,\eta) = 3sech^2\left(\frac{\varsigma - 15}{2}\right), \\ p^1 &: \mathbb{V}_1(\varsigma,\eta) = \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left\{(\mathbb{V}_0)_{\varsigma\varsigma\eta} - H_0\right\}\right], \\ p^1 &: \mathbb{V}_1(\varsigma,\eta) = 9sech^4\left(\frac{\varsigma - 15}{2}\right)\tanh\left(\frac{\varsigma - 15}{2}\right)\left(1+\alpha\eta-\alpha\right), \\ p^2 &: \mathbb{V}_2(\varsigma,\eta) = \mathbb{Y}^{-1}\left[(1+\alpha(v-1))\mathbb{Y}\left\{(\mathbb{V}_1)_{\varsigma\varsigma\eta} - H_1\right\}\right], \end{aligned}$$

$$p^2 : \mathbb{V}_2(\varsigma,\eta) = \frac{9}{4}\frac{1}{\cosh^1 2\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)}\Bigg[\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\Bigg\{-24\Big((1-\alpha)2\alpha\eta + (1-\alpha)^2 + \frac{\alpha^2\eta^2}{2}\Big)\cosh^3\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)$$
$$+30\Big((1-\alpha)2\alpha\eta + (1-\alpha)^2 + \frac{\alpha^2\eta^2}{2}\Big)\cosh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right) - 72\left(1+\alpha\eta-\alpha\right)\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\cosh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)$$
$$+135\left(1+\alpha\eta-\alpha\right)\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right) + 4\cosh^7\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\Bigg\}\left(1+\alpha\eta-\alpha\right)\Bigg],$$

$\vdots$

Provides the series form solution is $\mathbb{V}(\varsigma,\eta) = \Sigma_{k=0}^{\infty}\mathbb{V}_k(\varsigma,\eta)$

$$\mathbb{V}(\varsigma,\eta) = 3sech^2\left(\frac{\varsigma - 15}{2}\right) + 9sech^4\left(\frac{\varsigma - 15}{2}\right)\tanh\left(\frac{\varsigma - 15}{2}\right)\left(1+\alpha\eta-\alpha\right) +$$
$$\frac{9}{4}\frac{1}{\cosh^{12}\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)}\Bigg[\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\Bigg\{-24\Big((1-\alpha)2\alpha\eta + (1-\alpha)^2 + \frac{\alpha^2\eta^2}{2}\Big)\cosh^3\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)$$
$$+30\Big((1-\alpha)2\alpha\eta + (1-\alpha)^2 + \frac{\alpha^2\eta^2}{2}\Big)\cosh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right) - 72\left(1+\alpha\eta-\alpha\right)\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\cosh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)$$
$$+135\left(1+\alpha\eta-\alpha\right)\sinh\left(\frac{1}{2}\varsigma - \frac{15}{2}\right) + 4\cosh^7\left(\frac{1}{2}\varsigma - \frac{15}{2}\right)\Bigg\}\left(1+\alpha\eta-\alpha\right)\Bigg] + \cdots. \tag{4.7}$$

The Eq (4.7) put $\alpha = 1$, we obtain the solution of the suggested problem as:

$$\begin{aligned}\mathbb{V}(\varsigma,\eta) = {} & 3sech^2\left(\frac{\varsigma-15}{2}\right) + 9sech^4\left(\frac{\varsigma-15}{2}\right)\tanh\left(\frac{\varsigma-15}{2}\right)\eta + \\ & \frac{9}{4}\frac{1}{\cosh^{12}\left(\frac{1}{2}\varsigma-\frac{15}{2}\right)}\Bigg[\sinh\left(\frac{1}{2}\varsigma-\frac{15}{2}\right)\Bigg\{-24\eta^2\cosh^3\left(\frac{1}{2}\varsigma-\frac{15}{2}\right) \\ & + 30\eta^2\cosh\left(\frac{1}{2}\varsigma-\frac{15}{2}\right) - 72\eta\sinh\left(\frac{1}{2}\varsigma-\frac{15}{2}\right)\cosh\left(\frac{1}{2}\varsigma-\frac{15}{2}\right) \\ & + 135\eta\sinh\left(\frac{1}{2}\varsigma-\frac{15}{2}\right) + 4\cosh^7\left(\frac{1}{2}\varsigma-\frac{15}{2}\right)\Bigg\}\eta\Bigg] + \cdots .\end{aligned} \tag{4.8}$$

The exact result is:

$$\mathbb{V}(\varsigma,\eta) = 3sech^2\left(\frac{\varsigma-15-\eta}{2}\right). \tag{4.9}$$

Example 1, Figure 1 displays the evolution of the exact and HPTM solutions at $\alpha = 1$. Figure 2 shows that the different fractional order at $\alpha = 0.8$ and 0.6. In Figure 3, first graph the fractional order at $\alpha = 0.4$ and second graph show that the various fractional graph of Example 1. In Table 1, the illustrates a computational evaluation of the HPM [27] and the HPTM in accordance with absolute error, considering both fractional derivative operators into account.

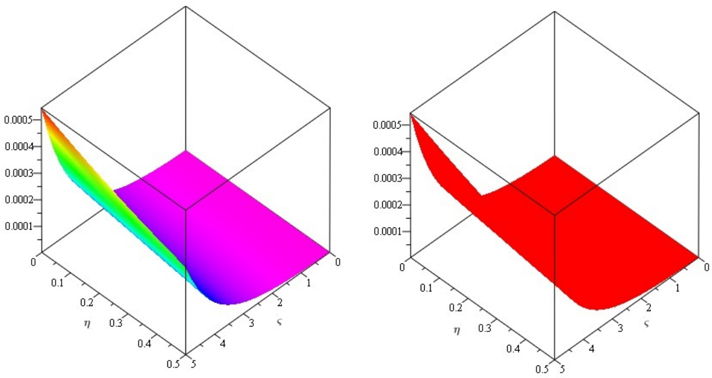

Figure 1. The exact and analytical solution graph of Example 1.

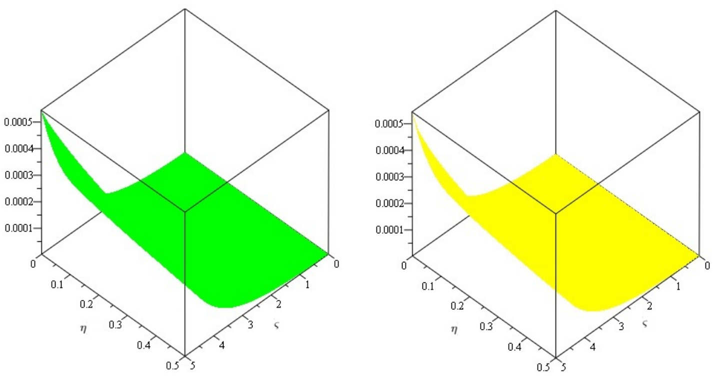

Figure 2. The fractional order graph of $\alpha = 0.8$ and 0.6 of Example 1.

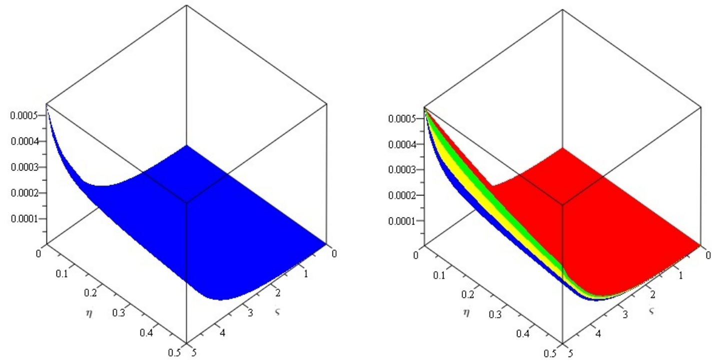

Figure 3. The first graph show that the fractional-order of $\alpha = 0.4$ and second graph of various fractional α of Example 1.

Table 1. Comparative analysis of HPM [27] and Current method (CM) solution of Example 1.

η	ς	$\lvert Exact - HPM \rvert$	$\lvert Exact - HPM \rvert$	$\lvert Exact - CM \rvert$	$\lvert Exact - CM \rvert$
		$\alpha = 0.6$	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 1$
	0.5	8.48379150×10^{-03}	1.5700000×10^{-08}	4.09610800×10^{-04}	1.5700000×10^{-08}
	1	1.69675830×10^{-02}	6.3000000×10^{-08}	1.63844300×10^{-03}	6.3000000×10^{-08}
	1.5	2.54513740×10^{-02}	1.4100000×10^{-07}	3.68649700×10^{-03}	1.4100000×10^{-07}
	2	3.39351660×10^{-02}	2.5200000×10^{-07}	6.55377200×10^{-03}	2.5200000×10^{-07}
	2.5	4.24189580×10^{-02}	3.9300000×10^{-07}	1.02402690×10^{-02}	3.9300000×10^{-07}
0.1	3	5.09027490×10^{-02}	5.6700000×10^{-07}	1.47459870×10^{-02}	5.6700000×10^{-07}
	3.5	5.93865400×10^{-02}	7.7000000×10^{-07}	2.00709300×10^{-02}	7.7000000×10^{-07}
	4	6.78703320×10^{-02}	1.0100000×10^{-06}	2.62150900×10^{-02}	1.0100000×10^{-06}
	4.5	7.63541240×10^{-02}	1.2700000×10^{-06}	3.31784700×10^{-02}	1.2700000×10^{-06}
	5	8.48379150×10^{-02}	1.5700000×10^{-06}	4.09610800×10^{-02}	1.5700000×10^{-06}
	0.5	1.27076980×10^{-02}	6.2500000×10^{-08}	6.81188800×10^{-04}	6.2500000×10^{-08}
	1	2.54153960×10^{-02}	2.5000000×10^{-07}	2.72475500×10^{-03}	2.5000000×10^{-07}
	1.5	3.81230940×10^{-02}	5.6300000×10^{-07}	6.13069900×10^{-03}	5.6300000×10^{-07}
	2	5.08307920×10^{-02}	1.0000000×10^{-06}	1.08990200×10^{-02}	1.0000000×10^{-06}
	2.5	6.35384900×10^{-02}	1.5630000×10^{-06}	1.70297190×10^{-02}	1.5630000×10^{-06}
0.2	3	7.62461880×10^{-02}	2.2500000×10^{-06}	2.45227950×10^{-02}	2.2500000×10^{-06}
	3.5	8.89538860×10^{-02}	3.0600000×10^{-06}	3.33782500×10^{-02}	3.0600000×10^{-06}
	4	1.01661584×10^{-01}	4.0000000×10^{-06}	4.35960800×10^{-02}	4.0000000×10^{-06}
	4.5	1.14369282×10^{-01}	5.0600000×10^{-06}	5.51762900×10^{-02}	5.0600000×10^{-06}
	5	1.2707698×10^{-01}	6.2500000×10^{-06}	6.81188800×10^{-02}	6.2500000×10^{-06}

4.2. Example

Consider the fractional nonlinear EW equation

$$D_\eta^\alpha \mathbb{V} + 3\mathbb{V}^2\mathbb{V}_\varsigma - \mathbb{V}_{\varsigma\varsigma\eta} = 0, \quad \eta > 0, \quad \varsigma \in R, \quad 0 < \alpha \leq 1, \tag{4.10}$$

with initial condition

$$\mathbb{V}(\varsigma, 0) = \frac{1}{4} sech(\varsigma - 30). \tag{4.11}$$

Incorporating Yang transform on (4.10), we get

$$\mathbb{Y}\Big[\mathbb{V}(\varsigma,\eta)\Big] = v\mathbb{V}(\varsigma,0) + (1+\alpha(v-1))\mathbb{Y}\Big[\mathbb{V}_{\varsigma\varsigma\eta} - 3\mathbb{V}^2\mathbb{V}_{\varsigma}\Big]. \tag{4.12}$$

Using the initial condition in Eq (4.12), we have

$$\mathbb{Y}\Big[\mathbb{V}(\varsigma,\eta)\Big] = v\frac{1}{4}sech(\varsigma-30) + (1+\alpha(v-1))\mathbb{Y}\Big[\mathbb{V}_{\varsigma\varsigma\eta} - 3\mathbb{V}^2\mathbb{V}_{\varsigma}\Big]. \tag{4.13}$$

By applying inverse Yang transform, we have

$$\mathbb{V}(\varsigma,\eta) = \frac{1}{4}sech(\varsigma-30) + \mathbb{Y}^{-1}\Big[(1+\alpha(v-1))\mathbb{Y}\Big\{\mathbb{V}_{\varsigma\varsigma\eta} - 3\mathbb{V}^2\mathbb{V}_{\varsigma}\Big\}\Big]. \tag{4.14}$$

Now we implemented HPM, we get

$$\Sigma_{J=0}^{\infty}p^J\mathbb{V}_J(\varsigma,\eta) = \frac{1}{4}sech(\varsigma-30) + p\Big[\mathbb{Y}^{-1}\Big\{(1+\alpha(v-1))\mathbb{Y}\Big(\Sigma_{J=0}^{\infty}p^J\mathbb{V}_J(\varsigma,\eta)_{\varsigma\varsigma\eta} - (\Sigma_{J=0}^{\infty}p^J H_J(\mathbb{V}))\Big)\Big\}\Big]. \tag{4.15}$$

The nonlinear term can be find with the help of He's polynomials

$$\Sigma_{J=0}^{\infty}p^J H_J(\mathbb{V}) = 3\mathbb{V}^2\mathbb{V}_{\varsigma}. \tag{4.16}$$

The He's polynomials can be written as

$$\begin{aligned}
H_0(\mathbb{V}) &= 3(\mathbb{V}_0)^2(\mathbb{V}_0)_{\varsigma},\\
H_1(\mathbb{V}) &= 3(\mathbb{V}_0)^2(\mathbb{V}_1)_{\varsigma} + 6\mathbb{V}_0\mathbb{V}_1(\mathbb{V}_0)_{\varsigma},\\
&\vdots
\end{aligned}$$

Coefficients p comparing, we obtain as

$$\begin{aligned}
p^0 &: \mathbb{V}_0(\varsigma,\eta) = \frac{1}{4}sech(\varsigma-30),\\
p^1 &: \mathbb{V}_1(\varsigma,\eta) = \mathbb{Y}^{-1}\Big[(1+\alpha(v-1))\mathbb{Y}\Big\{(\mathbb{V}_0)_{\varsigma\varsigma\eta} - H_0(\mathbb{V})\Big\}\Big],\\
p^1 &: \mathbb{V}_1(\varsigma,\eta) = \frac{3}{64}sech^3(\varsigma-30)\tanh(\varsigma-30)\Big(1+\alpha\eta-\alpha\Big),\\
&\vdots
\end{aligned}$$

provides the series form solution is

$$\begin{aligned}
\mathbb{V}(\varsigma,\eta) &= \Sigma_{m=0}^{\infty}\mathbb{V}_m(\varsigma,\eta),\\
\mathbb{V}(\varsigma,\eta) &= \frac{1}{4}sech(\varsigma-30) + \frac{3}{64}sech^3(\varsigma-30)\tanh(\varsigma-30)\Big(1+\alpha\eta-\alpha\Big) + \cdots.
\end{aligned} \tag{4.17}$$

The Eq (4.17) put $\alpha = 1$, we obtain the solution of the suggested problem as:

$$\mathbb{V}(\varsigma,\eta) = \frac{1}{4}sech(\varsigma-30) + \frac{3}{64}sech^3(\varsigma-30)\tanh(\varsigma-30) + \cdots. \tag{4.18}$$

The exact result is:

$$\mathbb{V}(\varsigma,\eta) = \frac{1}{4} sech\left(\varsigma - 30 - \frac{\eta}{4}\right). \tag{4.19}$$

Example 2, Figure 4 displays the evolution of the exact and HPTM solutions at $\alpha = 1$. Figure 5 shows that the different fractional order at $\alpha = 0.8$ and 0.6. In Figure 6, first graph the fractional order at $\alpha = 0.4$ and second graph show that the various fractional graph of Example 2.

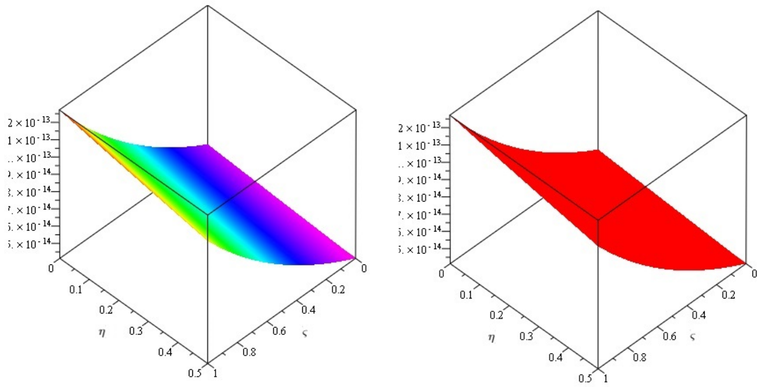

Figure 4. The exact and analytical solution graph of Example 2.

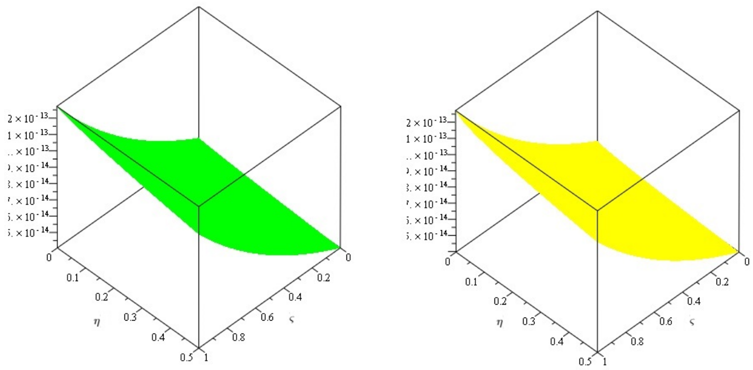

Figure 5. The fractional order graph of $\alpha = 0.8$ and 0.6 of Example 2.

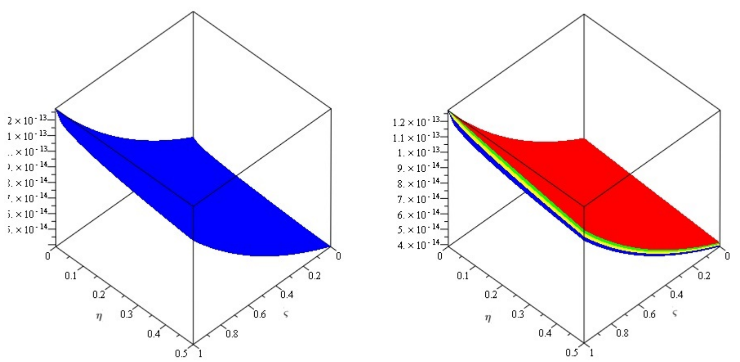

Figure 6. The first graph show that the fractional-order of $\alpha = 0.4$ and second graph of various fractional α of Example 2.

4.3. Example

Consider the fractional nonlinear EW equation

$$D_{\eta}^{\alpha}\mathbb{V} + \frac{12}{7}(\mathbb{V}^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}^6)_{\varsigma\varsigma\eta} = 0, \quad \eta > 0, \quad \varsigma \in R, \quad 0 < \alpha \leq 1, \tag{4.20}$$

with initial condition

$$\mathbb{V}(\varsigma, 0) = \cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right). \tag{4.21}$$

Using Yang transform on (4.20), we get

$$E\Big[\mathbb{V}(\varsigma, \eta)\Big] = v\mathbb{V}(\varsigma, 0) + (1 + \alpha(v-1))\mathbb{Y}\Big[\frac{12}{7}(\mathbb{V}^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}^6)_{\varsigma\varsigma\eta}\Big]. \tag{4.22}$$

Putting the initial condition in the Eq (4.22), we have

$$E\Big[\mathbb{V}(\varsigma, \eta)\Big] = v\cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right) + (1 + \alpha(v-1))\mathbb{Y}\Big[\frac{12}{7}(\mathbb{V}^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}^6)_{\varsigma\varsigma\eta}\Big]. \tag{4.23}$$

By applying inverse Yang transform, we have

$$\mathbb{V}(\varsigma, \eta) = \cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right) + \mathbb{Y}^{-1}\Big[(1 + \alpha(v-1))\mathbb{Y}\Big\{\frac{12}{7}(\mathbb{V}^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}^6)_{\varsigma\varsigma\eta}\Big\}\Big]. \tag{4.24}$$

Now we implemented HPM, we get

$$\Sigma_{J=0}^{\infty}p^J\mathbb{V}_J(\varsigma, \eta) = \cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right) + p\Big[\mathbb{Y}^{-1}\Big\{(1 + \alpha(v-1))\mathbb{Y}\Big(\Sigma_{J=0}^{\infty}p^J\mathbb{V}_J(\varsigma, \eta)_{\varsigma\varsigma\eta}\Big)\Big\}\Big]. \tag{4.25}$$

The nonlinear term can be find with the help of He's polynomials

$$\Sigma_{J=0}^{\infty}p^JH_J(\mathbb{V}) = \frac{12}{7}(\mathbb{V}^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}^6)_{\varsigma\varsigma\eta}. \tag{4.26}$$

The He's polynomials can be written as

$$H_0(\mathbb{V}) = \frac{12}{7}(\mathbb{V}_0^6)_{\varsigma} - \frac{3}{7}(\mathbb{V}_0^6)_{\varsigma\varsigma\eta}$$
$$\vdots$$

Coefficients p comparing, we obtain as

$$p^0 : \mathbb{V}_0(\varsigma, \eta) = \cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right),$$
$$p^1 : \mathbb{V}_1(\varsigma, \eta) = \mathbb{Y}^{-1}\Big[(1 + \alpha(v-1))\mathbb{Y}\Big\{H_0(\mathbb{V})\Big\}\Big],$$
$$p^1 : \mathbb{V}_1(\varsigma, \eta) = -\frac{24}{7}\cosh^{\frac{7}{5}}\left(\frac{5\varsigma}{6}\right)\sinh\left(\frac{5\varsigma}{6}\right)\Big(1 + \alpha\eta - \alpha\Big),$$
$$\vdots$$

The series form solution is

$$\mathbb{V}(\varsigma,\eta)=\sum_{m=0}^{\infty}\mathbb{V}_m(\varsigma,\eta)$$

$$\mathbb{V}(\varsigma,\eta)=\cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right)-\frac{24}{7}\cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right)\sinh\left(\frac{5\varsigma}{6}\right)\Big(1+\alpha\eta-\alpha\Big)+\cdots. \tag{4.27}$$

The Eq (4.27) put $\alpha=1$, we obtain the solution of the suggested problem as:

$$\mathbb{V}(\varsigma,\eta)=\cosh^{\frac{2}{5}}\left(\frac{5\varsigma}{6}\right)-\frac{24}{7}\cosh^{\frac{7}{5}}\left(\frac{5\varsigma}{6}\right)\sinh\left(\frac{5\varsigma}{6}\right)\eta+\cdots. \tag{4.28}$$

The exact result is:

$$\mathbb{V}(\varsigma,\eta)=\cosh^{\frac{2}{5}}\left\{\frac{5}{6}(\varsigma-\eta)\right\}. \tag{4.29}$$

Example 3, Figure 7 displays the evolution of the exact and HPTM solutions at $\alpha=1$. Figure 8 shows that the different fractional order at $\alpha=0.8$ and 0.6. In Figure 9, first graph the fractional order at $\alpha=0.4$ and second graph show that the various fractional graph of Example 3.

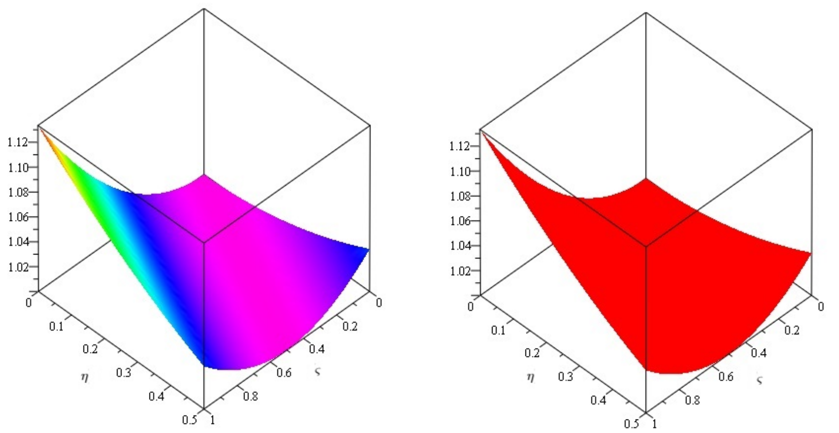

Figure 7. The exact and analytical solution graph of Example 3.

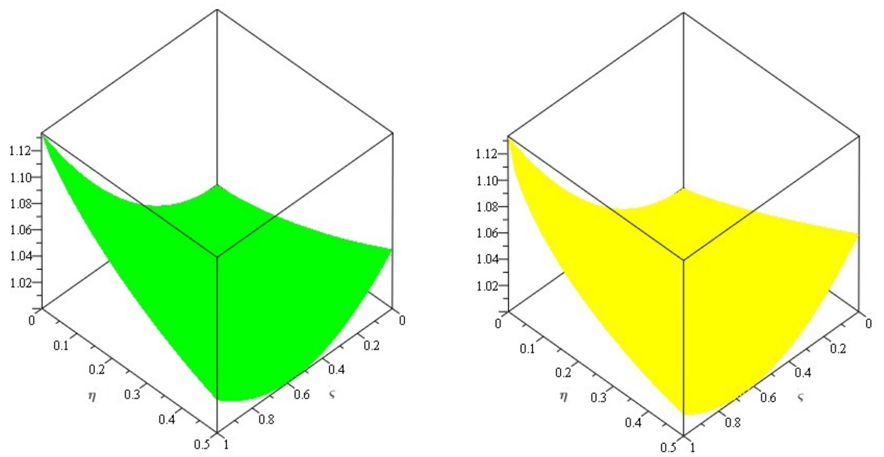

Figure 8. The fractional order graph of $\alpha=0.8$ and 0.6 of Example 3.

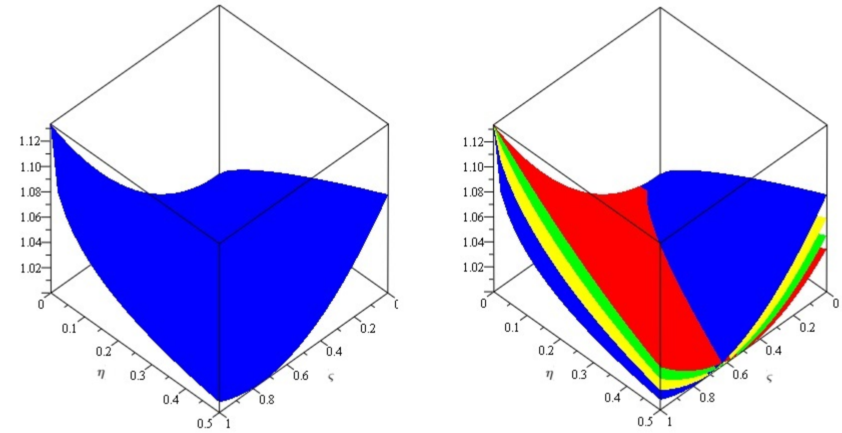

Figure 9. The first graph show that the fractional-order of $\alpha = 0.4$ and second graph of various fractional α of Example 3.

5. Conclusions

In this paper, we determined the fractional equal-width equations, applying an homotopy perturbation transform method. The solutions for some problems are investigated applied the given technique. The homotopy perturbation transform method solution is a good agreement with the exact solution of the suggested problems. The present technique are investigated the solutions of fractional-order examples. The figures analysis of the fractional-order results achieved has verify the convergence toward the results of the integer-order. The scheme effective and comprehensive execution is investigated and confirmed in an attempt to display that it may be applicable to other nonlinear evolutionary models that emerge in applied science.

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Conflict of interest

The authors declare that there is no conflict of interest.

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