Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 17225–17235.
DOI: 10.3934/math.2022948
Received: 06 June 2022
Revised: 18 July 2022
Accepted: 19 July 2022
Published: 22 July 2022

Research article

New double-sum expansions for certain Mock theta functions

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Abstract: The study of expansions of certain mock theta functions in special functions theory has a long and quite significant history. Motivated by recent correlations between q-series and mock theta functions, we establish a new q-series transformation formula and derive the double-sum expansions for mock theta functions. As an application, we state new double-sum representations for certain mock theta functions.

Keywords: basic hypergeometric series; Mock theta functions; double sums
Mathematics Subject Classification: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction

Throughout this paper, we use the standard q-series notation in [5]:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), n = 1, 2, 3, \cdots .$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n,$$

where n is an integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} ;q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. $$

The theory of mock theta functions is a very important research area of the theory of the basic hypergeometric series. Mock theta function was first introduced by Ramanujan in his last letter to Hardy [24]. Ramanujan listed 17 mock theta functions and called them orders of 3, 5 and 7. But he did not say explicitly what he meant and did not also give exact definition of order. Until now, exact definition of order has still not been given. The study of mock theta functions has attracted many experts and scholars. Recently, Patkowski [21] gave some new expansions for Ramanujan's 10th-order, 7th-order, and 5th-order mock theta functions $\mathcal{F}_2(q^4)$, $\phi(q^4)$ and $\chi_1(q^4)$ by establishing some new Bailey pairs. In [19], Lovejoy and Osburn used Bailey pairs and Bailey transformation to obtain many mock theta functions in terms of q-hypergeometric double sums and gave connections to known single-sum mock theta functions. Then, Zhang and Li [25] derived some similar nice mock theta double sums by the same method on the previous basis. Patkowski [22] obtained double-sum expansions for mock theta functions of Andrew's third-order $\overline{\psi}_1(q)$ Ramanujan's 7th-order $\phi(q)$, and 10th-order $\mathcal{F}_2(q)$. Some more recent investigations on this subject can be found in [9, 10, 17, 18].

As an example, double-sum expansion for 10th-order mock theta function $\mathcal{F}_2(q)$ is restated as follows [22, Theorem 2.2, (2.13)]:

$$ \sum_{n\geq 0}\sum_{n\geq j\geq 0} \frac{(-1)^j q^{2n^2+2n+j^2+j}}{(-q;q)_{2n+1}(q^2;q^2)_{n-j}(q^2;q^2)_j(1-q^{2j+1})} = \mathcal{F}_2(q^2), $$

where $\mathcal{F}_2(q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}$.

In this paper, we make use of the following mock theta functions.

The second-order mock theta functions (see [20]):

$$ A(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, $$
$$ B(q) = \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}, $$
$$ \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}. $$

The third-order mock theta functions (see [7]):

$$ \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, $$
$$ \psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, $$

$$\nu(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

The sixth-order mock theta functions (see [3]):

$$\psi_6(q)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}(q;q^2)_{n-1}}{(-q;q)_{2n-1}},$$
$$\phi_6(q)=\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{n^2}(q;q^2)_{n}}{(-q;q)_{2n}},$$
$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$
$$\sigma(q)=\sum_{n=0}^{\infty}q^{(n+2)(n+1)/2}\frac{(-q;q)_n}{(q;q^2)_{n+1}}.$$

For more details about mock theta functions, the readers can refer to [1–4, 6, 7, 23, 24].

Based on the above research results, we continue to do some research on double-sum expansions for mock theta functions in this paper. The rest of this paper is arranged as follows. In Section 2, we first provide a new q-series transformation formula in terms of series rearrangement method. And then as applications, some new double sums for certain mock theta functions are given.

2. Main results

In this section, we give some double-sum representations for certain mock theta functions. First, in the following Proposition 1, we establish a new q-series transformation formula by means of q-series rearrangement.

Proposition 1. *For $|q|<1$ and $|\alpha ab|<1$, we have that*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q^2/a,-\lambda/q;q^2)_n(\alpha ab)^n}{(\alpha q^2,\lambda;q^2)_n}\\ &=\frac{(\alpha a,\alpha b;q^2)_\infty}{(\alpha q^2,\alpha ab/q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-\alpha q^{4n})(\alpha,q^2/a,q^2/b;q^2)_n(-\alpha ab/q^2)^nq^{n^2-n}}{(1-\alpha)(q^2,\alpha a,\alpha b;q^2)_n}\\ &\times\sum_{k=0}^{n}\frac{(q^{-2n},\alpha q^{2n},-\lambda/q;q^2)_kq^{4k}}{(q^2/b,\alpha q^2,\lambda;q^2)_k}.\end{aligned}\tag{2.1}$$

Proof. In terms of series rearrangement, the right-hand side of (2.1) equals that

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(1-\alpha q^{4n})(\alpha,q^2/a,q^2/b;q^2)_n(-\alpha ab/q^2)^nq^{n^2-n}}{(1-\alpha)(q^2,\alpha a,\alpha b;q^2)_n}\frac{(q^{-2n},\alpha q^{2n},-\lambda/q;q^2)_kq^{4k}}{(q^2/b,\alpha q^2,\lambda;q^2)_k}\\ &=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-\alpha q^{4n+4k})(\alpha,q^2/a,q^2/b;q^2)_{n+k}(-\alpha ab/q^2)^{n+k}q^{(n+k)^2-(n+k)}}{(1-\alpha)(q^2,\alpha a,\alpha b;q^2)_{n+k}}\end{aligned}$$

$$
\begin{aligned}
&\times \frac{(q^{-2(n+k)}, \alpha q^{2(n+k)}, -\lambda/q; q^2)_k q^{4k}}{(q^2/b, \alpha q^2, \lambda; q^2)_k} \\
&= \sum_{k=0}^{\infty} \frac{(\alpha, q^2/a, -\lambda/q; q^2)_k}{(q^2, \alpha a, \alpha b, \alpha q^2, \lambda; q^2)_k} (-\alpha ab)^k q^{k^2+k} \sum_{n=0}^{\infty} \frac{(1-\alpha q^{4n+4k})(\alpha q^{2k}, q^{2k+2}/a, q^{2k+2}/b; q^2)_n}{(1-\alpha)(q^{2k+2}, \alpha a q^{2k}, \alpha b q^{2k}; q^2)_n} \\
&\times (q^{-2(n+k)}, \alpha q^{2(n+k)}; q^2)_k (-\alpha ab)^n q^{n^2+2nk-3n} \\
&= \sum_{k=0}^{\infty} \frac{(1-\alpha q^{4k})(q^2/a, -\lambda/q; q^2)_k (\alpha; q^2)_{2k}}{(1-\alpha)(\alpha a, \alpha b, \alpha q^2, \lambda; q^2)_k} (\alpha ab)^k \sum_{n=0}^{\infty} \frac{(\alpha q^{4k}; q^2)_n (1-\alpha q^{4n+4k})}{(q^2; q^2)_n (1-\alpha q^{4k})} \\
&\times \frac{(q^{2k+2}/a, q^{2k+2}/b; q^2)_n}{(\alpha a q^{2k}, \alpha b q^{2k}; q^2)_n} \left(-\frac{\alpha ab}{q^2}\right)^n q^{n^2-n}.
\end{aligned} \tag{2.2}
$$

Let $q \to q^2, a \to \alpha q^{4k}, b \to q^{2k+2}/a, c \to q^{2k+2}/b, d \to \infty$ in the following sum of a very-well-poised ${}_6\phi_5$ series [5, II.20]

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d \end{matrix}; q, aq/bcd\right] \\
&= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.
\end{aligned} \tag{2.3}
$$

Then the second term of the right-hand side of (2.2) gives

$$
\frac{(\alpha q^{4k+2}, \alpha ab/q^2; q^2)_\infty}{(\alpha a q^{2k}, \alpha b q^{2k}; q^2)_\infty} = \frac{(\alpha q^2, \alpha ab/q^2; q^2)_\infty}{(\alpha a, \alpha b; q^2)_\infty} \frac{(\alpha a, \alpha b; q^2)_k}{(\alpha q^2; q^2)_{2k}}.
$$

The right-hand side of (2.2) yields

$$
\begin{aligned}
&\frac{(\alpha q^2, \alpha ab/q^2; q^2)_\infty}{(\alpha a, \alpha b; q^2)_\infty} \sum_{k=0}^{\infty} \frac{(1-\alpha q^{4k})(q^2/a, q^2/b, -\lambda/q; q^2)_k (\alpha; q^2)_{2k}}{(1-\alpha)(\alpha a, \alpha b, q^2/b, \alpha q^2, \lambda; q^2)_k} (\alpha ab)^k \frac{(\alpha a, \alpha b; q^2)_k}{(\alpha q^2; q^2)_{2k}} \\
&= \frac{(\alpha q^2, \alpha ab/q^2; q^2)_\infty}{(\alpha a, \alpha b; q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q^2/a, -\lambda/b; q^2)_k}{(\alpha q^2, \lambda; q^2)_k} (\alpha ab)^k.
\end{aligned}
$$

After some simplifications, we derive our desired result. This completes the proof. □

Next, as applications of the identity (2.1), in the following theorems we give some new double-sum representations for certain mock theta functions.

Theorem 1. *The following double-sum representation for the second-order mock theta function $A(q)$ is true:*

$$
A(q) = -q^2 \frac{(-q; q^2)_\infty^2}{(q; q^2)_\infty^2} \sum_{n \geq 0} \sum_{n \geq k \geq 0} \frac{(-1)^{n+k} q^{n^2+k^2-2n+3k-2nk} (1-q^{4n+1})(q; q^2)_{n+k} (-q^2; q^2)_n^2}{(1-q)(q^2; q^2)_{n-k} (-q; q^2)_n^2 (-q^2; q^2)_k (q; q^2)_{k+1}}. \tag{2.4}
$$

Proof. Let $a = b = -1, \alpha = q, \lambda = 0$ in (2.1). Then, we derive that

$$
\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^n}{(q^3; q^2)_n}
$$

$$
\begin{aligned}
=&\frac{(-q,-q;q^2)_\infty}{(q^3,q^{-1};q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{4n+1})(q,-q^2,-q^2;q^2)_n}{(1-q)(q^2,-q,-q;q^2)_n}(-1)^n q^{n^2-2n}\\
&\times\sum_{k=0}^{n}\frac{(q^{-2n},q^{2n+1};q^2)_k q^{4k}}{(-q^2,q^3;q^2)_k}.
\end{aligned}
$$

Multiplying both sides of the above equation by $\frac{q}{1-q}$ and after certain simplifications, we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n+1}}{(q;q^2)_{n+1}}\\
&=-\frac{q^2(-q;q^2)_\infty^2}{(1-q)(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{1-q^{4n+1}}{1-q}\frac{(q;q^2)_n(-q^2;q^2)_n^2}{(q^2;q^2)_n(-q;q^2)_n^2}(-1)^n q^{n^2-2n}\\
&\quad\times\sum_{k=0}^{n}\frac{(q^{-2n},q^{2n+1};q^2)_k q^{4k}}{(-q^2,q^3;q^2)_k}\\
&=-q^2\frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{n^2+k^2-2n+3k-2nk}(1-q^{4n+1})(q;q^2)_{n+k}(-q^2;q^2)_n^2}{(1-q)(q^2;q^2)_{n-k}(-q;q^2)_n^2(-q^2;q^2)_k(q;q^2)_{k+1}}.
\end{aligned}
$$

This completes the proof of the identity (2.4).

□

By taking $a=-q, b=-q^{-1}, \alpha=q, \lambda=0$ in (2.1), we deduce the following identity

Corollary 1. *The following double-sum representation for the second-order mock theta function $B(q)$ is true:*

$$
B(q)=-2q\frac{(-q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{n^2+k^2-2n+3k-2nk}(1-q^{4n+1})(q;q^2)_{n+k}(-q;q^2)_n(-q;q^2)_{n+1}}{(1-q)(q^2;q^2)_{n-k}(-q^2,-1;q^2)_n(q^2;q^4)_{k+1}}. \tag{2.5}
$$

Taking $\alpha=-1, \lambda=-q^2, b=-q^{-1}, a\to 0$ in (2.1), we can also attain the identity (2.6) for the second-order mock theta function $\mu(q)$

Corollary 2. *The following double-sum representation for the second-order mock theta function $\mu(q)$ is true:*

$$
\mu(q)=\frac{(q^{-1};q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}(1+q^{4n+1})\frac{(-1)^k q^{2n^2+k^2-3n+3k-2nk}(-1;q^2)_{n+k}(-q^3;q^2)_n(q^2;q^4)_k}{(q^2;q^2)_{n-k}(q^{-1};q^2)_n(-q^2;q)_{2k}(-q;q)_{2k}}. \tag{2.6}
$$

For certain third-order mock theta functions, we can also gain the similar conclusions as follows.

Theorem 2. *The following double-sum representations for the third-order mock theta functions $\phi(q), \psi(q), \nu(q)$ hold true:*

$$
\phi(q)=-\frac{(1+q)(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2-3n+3k-2nk}(1-q^{4n-1})(q^{-1};q^2)_{n+k}(-q^2;q^2)_n}{(1-q)(q^2;q^2)_{n-k}(-q^{-1};q^2)_n(-q^2;q^2)_k^2} \tag{2.7}
$$

$$\psi(q) =\frac{q(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2-n+3k-2nk}(1-q^{4n+1})(q;q^2)_{n+k}(-q^2;q^2)_n}{(q^2;q^2)_{n-k}(-q;q^2)_n(-q^2;q^2)_k(q;q^2)_{k+1}} \tag{2.8}$$

$$\nu(q) =\frac{2(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2-2n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^3;q^2)_n}{(1+q)(q^2;q^2)_{n-k}(-1;q^2)_n(q^6;q^4)_k}. \tag{2.9}$$

Proof. For (2.7), taking $\alpha = q^{-1}, \lambda = -q^2, b = -1, a \to 0$ in (2.1), we attain that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}\\
&=\frac{(-q^{-1};q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{1-q^{4n-1}}{1-q^{-1}}\frac{(q^{-1},-q^2;q^2)_n}{(q^2,-q^{-1};q^2)_n}(-1)^n q^{2n^2-3n}\sum_{k=0}^{n}\frac{(q^{-2n},q^{2n-1};q^2)_k}{(-q^2,-q^2;q^2)_k}q^{4k}\\
&=-\frac{(1+q)(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{1-q^{4n-1}}{1-q}\frac{(-1)^{n+k}q^{2n^2+k^2-3n+3k-2nk}(q^{-1};q^2)_{n+k}(-q^2;q^2)_n}{(q^2;q^2)_{n-k}(-q^{-1};q^2)_n(-q^2;q^2)_k^2}.
\end{aligned}$$

This completes the proof of the identity (2.7).

For (2.8), set $\alpha = q, \lambda = 0, b = -1, a \to 0$ in (2.1). We deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^3;q^2)_n}\\
=&\frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{4n+1})(q,-q^2;q^2)_n}{(1-q)(q^2,-q;q^2)_n}(-1)^n q^{2n^2-n}\sum_{k=0}^{n}\frac{(q^{-2n},q^{2n+1};q^2)_k}{(-q^2,q^3;q^2)_k}q^{4k}\\
=&\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}(1-q^{4n+1})\frac{(q;q^2)_{n+k}(-q^2;q^2)_n}{(q^2;q^2)_{n-k}(-q;q^2)_n}\\
&\times\frac{(-1)^{n+k}q^{2n^2+k^2-n+3k-2nk}}{(-q^2,q^3;q^2)_k}.
\end{aligned}$$

Multiplying both sides of the above identity by $\frac{q}{1-q}$, we get the following double-sum representation for the third-order mock theta function $\psi(q)$:

$$\begin{aligned}
\psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+2n+1}}{(q;q^2)_{n+1}}=\frac{q(-q;q^2)_\infty}{(q;q^2)_\infty}&\sum_{n\geq 0}\sum_{n\geq k\geq 0}(1-q^{4n+1})\frac{(q;q^2)_{n+k}(-q^2;q^2)_n}{(q^2;q^2)_{n-k}(-q;q^2)_n}\\
&\times\frac{(-1)^{n+k}q^{2n^2+k^2-n+3k-2nk}}{(-q^2;q^2)_k(q;q^2)_{k+1}}.
\end{aligned}$$

This completes the proof of the identity (2.8).

For (2.9), taking $\alpha = -q, \lambda = 0, b = q^{-1}, a \to 0$ in (2.1), we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q^3;q^2)_n}=\frac{(-1;q^2)_\infty}{(-q^3;q^2)_\infty}&\sum_{n=0}^{\infty}\frac{(1+q^{4n+1})(-q,q^3;q^2)_n(-1)^n q^{2n^2-2n}}{(1+q)(q^2,-1;q^2)_n}\\
&\times\sum_{k=0}^{n}\frac{(q^{-2n},-q^{2n+1};q^2)_k q^{4k}}{(q^3,-q^3;q^2)_k}.
\end{aligned} \tag{2.10}$$

Multiplying both sides of the identity (2.10) by $\frac{1}{1+q}$ and after applications as seen above, we get our desired result (2.7). Thus we complete the proof of Theorem 2 by obtaining the Eq (2.9). □

Theorem 3. *The following double-sum representations for the third-order mock theta functions $\psi_6(q), \phi_6(q), \rho(q), \sigma(q)$ hold true:*

$$\psi_6(q) = \frac{q(q;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^k q^{2n^2+k^2-n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^4;q^4)_n(q^2;q^4)_k}{(q^2;q^2)_{n-k}(q;q)_{2n}(-q;q)_{2k}(-q;q)_{2k+1}} \tag{2.11}$$

$$\phi_6(q) = \frac{(q^{-1};q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^k q^{2n^2+k^2-3n+3k-2nk}(1+q^{4n-1})(-q^{-1};q^2)_{n+k}(-q^2;q^2)_n(q^2;q^4)_k}{(1+q^{-1})(q^2;q^2)_{n-k}(q^{-1};q^2)_n(-q;q)_{2k}^2} \tag{2.12}$$

$$\rho(q^2) = \frac{2(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k} q^{2n^2+k^2-2n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^3;q^2)_n(q^4;q^4)_k}{(q^2;q^2)_{n-k}(-1;q^2)_n(q^2;q)_{2k}(q^2;q^4)_{k+1}} \tag{2.13}$$

$$\sigma(q^2) = \frac{q^2(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k} q^{2n^2+k^2+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^2;q^4)_n(q^4;q^4)_k}{(q^2;q^2)_{n-k}(-q;q)_{2n}(q^2;q^4)_{k+1}(q;q)_{2k}}. \tag{2.14}$$

Proof. For (2.11), set $\alpha = -q, \lambda = -q^2, b = -1, a \to 0$ in (2.1). Then we obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2+2n}}{(-q^3,-q^2;q^2)_n} = \frac{(q;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1+q^{4n+1})(-q,-q^2;q^2)_n q^{2n^2-n}}{(1+q)(q^2,q;q^2)_n}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-2n},-q^{2n+1},q;q^2)_k q^{4k}}{(-q^2,-q^3,-q^2;q^2)_k}.$$

After suitable simplifications, we derive

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2+2n}}{(-q^2;q)_{2n}}$$
$$= \frac{(q;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^k q^{2n^2+k^2-n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^4;q^4)_n(q^2;q^4)_k}{(q^2;q^2)_{n-k}(q;q)_{2n}(-q;q)_{2k}(-q;q)_{2k+1}}.$$

Multiplying both sides of the above identity by $\frac{q}{1+q}$, we deduce that

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2+2n+1}}{(-q;q)_{2n+1}}$$
$$= \frac{q(q;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^k q^{2n^2+k^2-n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^4;q^4)_n(q^2;q^4)_k}{(q^2;q^2)_{n-k}(q;q)_{2n}(-q;q)_{2k}(-q;q)_{2k+1}}.$$

This completes the proof of (2.11).

For (2.12), set $\alpha = -q^{-1}, \lambda = -q^2, b = -1, a \to 0$ in (2.1). After some simplifications, we attain the identity (2.12). Therefore, we omit the proof.

For (2.13), taking $\alpha = -q, \lambda = q^3, b = q^{-1}, a \to 0$ in (2.1) and after some simplifications, we get that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q^6;q^4)_n}$$

$$= \frac{(-1;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1+q^{4n+1})(-q,q^3;q^2)_n q^{2n^2-2n}}{(1+q)(q^2,-1;q^2)_n}\sum_{k=0}^{n}\frac{(q^{-2n},-q^{2n+1},-q^2;q^2)_k q^{4k}}{(q^3,-q^3,q^3;q^2)_k}$$
$$= \frac{2(-q^2;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2-2n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^3;q^2)_n(-q^2;q^2)_k}{(1+q)(q^2;q^2)_{n-k}(-1;q^2)_n(q^3;q^2)_k(q^6;q^4)_k}.$$

Multiplying both sides of the above identity by $\frac{1}{1-q^2}$ and through a mass of complex computations, we find the following double-sum representation for the sixth-order mock theta function $\rho(q)$:

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q^2;q^4)_{n+1}}$$
$$= \frac{2(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2-2n+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^3;q^2)_n(q^4;q^4)_k}{(q^2;q^2)_{n-k}(-1;q^2)_n(q^2;q)_{2k}(q^2;q^4)_{k+1}}.$$

This completes the proof of the identity (2.13). For (2.14), take $\alpha=-q,\lambda=q^3,b=q,a\to 0$ in (2.1). Then we have that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+3n}(-q^2;q^2)_n}{(q^6;q^4)_n}$$
$$= \frac{(-q^2;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1+q^{4n+1})(-q,q;q^2)_n(-1)^n q^{2n^2}}{(1+q)(q^2,-q^2;q^2)_n}\sum_{k=0}^{n}\frac{(q^{-2n},-q^{2n+1},-q^2;q^2)_k}{(q,-q^3,q^3;q^2)_k}q^{4k}$$
$$= \frac{(-q^2;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(1+q^{4n+1})(-q;q^2)_{n+k}(q^2;q^4)_n(q^4;q^4)_k(-1)^{n+k}q^{2n^2+k^2+3k-2nk}}{(1+q)(q^2;q^2)_{n-k}(-q;q)_{2n}(q^6;q^4)_k(q;q)_{2k}}.$$

In order to get our desired result, we multiply both sides of the above identity by $\frac{q^2}{1-q^2}$. Thus, we have that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+3n+2}(-q^2;q^2)_n}{(q^2;q^4)_{n+1}}$$
$$= \frac{q^2(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq 0}\sum_{n\geq k\geq 0}\frac{(-1)^{n+k}q^{2n^2+k^2+3k-2nk}(1+q^{4n+1})(-q;q^2)_{n+k}(q^2;q^4)_n(q^4;q^4)_k}{(q^2;q^2)_{n-k}(-q;q)_{2n}(q^2;q^4)_{k+1}(q;q)_{2k}}.$$

□

3. Conclusions

Basic (or q-) polynomials and (or q-) hypergeometric functions are particularly applicable in many diverse areas of mathematics, physics and other sciences. Here in our present investigation, we have motivated by the work of Patkowski [22] and have studied the double-sum expansions for mock theta functions. Then in terms of series rearrangement method, we have established a new q-series transformation formula. As an applications, we have derived some new double-sum representations for certain mock theta functions.

Studies of the special functions and q-polynomials are widely using in different branches of mathematics. For example in [11], by make use of certain q-Chebyshev Polynomials, certain subclasses of analytic and bi-univalent functions have been defined systematically. Just like the class defined in [11], one can define a similar class by taking this newly established q-series transformation formula instead of q-Chebyshev Polynomials. These kind of investigations can also be found in [8, 12–16].

Conflict of interest

All authors declare no conflicts of interest in this paper.

Acknowledgments

The first-named author (Q. Hu) is supported by the Natural Science Foundation of Henan Province (No. 212300410211), the National Natural Science Foundation of China (Grant No. 12101287) and the National Project Cultivation Foundation of Luoyang Normal University (No. 2020-PYJJ-011).

References

1. G. E. Andrews, D. Hickerson, Ramanujan's "lost notebook": The sixth order mock theta functions, *Adv. Math.*, **89** (1991), 60–105. https://doi.org/10.1016/0001-8708(91)90083-J
2. G. E. Andrews, q-orthogonal polynomials, Rogers-Ramanujan identities, and mock theta functions, *Proc. Steklov Inst. Math.*, **276** (2012), 21–32. https://doi.org/10.1134/S0081543812010038
3. B. C. Berndt, S. H. Chan, Sixth order mock theta functions *Adv. Math.,* **216** (2007), 771–786. https://doi.org/10.1016/j.aim.2007.06.004
4. B. C. Berndt, R. A. Rankin, Ramanujan: Letters and Commentary, History of Mathematics, 9. American Mathematical Society, Providence, RI, London. Mathematical Society, London (1995), xiv+347pp.
5. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2Eds, Cambridge University Press, Cambridge, 2004. https://doi.org/10.1017/CBO9780511526251
6. B. Gordon, R. J. McIntosh, Some eighth order mock theta functions, *J. London Math. Soc.(2),* **62** (2000), 321–335. https://doi.org/10.1112/S0024610700008735
7. B. Gordon, R. J. McIntosh, A survey of classic mock theta functions, In Partitions, q-Series, and Modular Forms, *Developments in Math.,* **23** (2012), 95–144.
8. Q. X. Hu, H. M. Srivastava, B. Ahmad, N. Khan, M. G. Khan, W. K. Mashwani, et al., A subclass of multivalent Janowski type q-starlike functions and its consequences, *Symmetry*, **13** (2021), Article ID 1275, 1–14. https://doi.org/10.3390/sym13071275
9. Z. Jia, B. Khan, Q. Hu, D. Niu, Applications of generalized q-difference equations for general q-polynomials, *Symmetry*, **13** (2021), 1222. https://doi.org/10.3390/sym13071222

10. Z. Jia, B. Khan, P. Agarwal, Q. Hu, X. Wang, Two new Bailey lattices and their applications, *Symmetry*, **13** (2021), 958. https://doi.org/10.3390/sym13060958

11. B. Khan, Z. G. Liu, T. G. Shaba, S. Araci, N. Khan, M. G. Khan, Applications of q-derivative operator to the subclass of Bi-Univalent functions involving q-Chebyshev Ppolynomials, *J. Math.*, **2022** (2022), 8162182. https://doi.org/10.1155/2022/8162182

12. B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, Applications of a certain integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Math.,* **6** (2021), 1024–1039. https://doi.org/10.3934/math.2021061

13. M. G. Khan, B. Ahmad, N. Khan, W. K. Mashwani, S. Arjika, B. Khan, et al., Applications of Mittag-Leffler type Poisson distribution to a subclass of analytic functions involving conic-type regions, *Dev. Geom. Funct. Theory,* **2021** (2021), Article ID 4343163, 9 pages. https://doi.org/10.1155/2021/4343163

14. W. K. Mashwani, B. Ahmad, N. Khan, M. G. Khan, S. Arjika, B. Khan, et al., Fourth Hankel determinant for a subclass of starlike functions based on modified sigmoid, *J. Funct. Space.*, **2021** (2021), Article ID 6116172. https://doi.org/10.1155/2021/6116172

15. M. S. U. Rehman, Q. Z. Ahmad, H. M. Srivastava, N. Khan, M. Darus, B. Khan, Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions, *AIMS Math.* **6** (2021), 1110–1125. https://doi.org/10.3934/math.2021067

16. L. Shi, B. Ahmad, N. Khan, M. G. Khan, S. Araci, W. K. Mashwani, et al., Coefficient estimates for a subclass of meromorphic multivalent q-close-to-convex functions, *Symmetry*, **2021**, *13*, 1840. https://doi.org/10.3390/sym13101840

17. H. M. Srivastava, J. Cao, S. Arjika, A note on generalized q-difference equations and their applications involving q-hypergeometric functions, *Symmetry*, **12** (2020), 1816. https://doi.org/10.3390/sym12111816

18. H. M. Srivastava, S. Arjika, A. S. Kelil, Some homogeneous q-difference operators and the associated generalized Hahn polynomials, *Appl. Set-Valued Anal. Optim.,* **1** (2019), 187–201. https://doi.org/10.48550/arXiv.1908.03207

19. J. Lovejoy, R. Osburn, Mock theta double sums, *Glasg. Math. J.*, **59** (2017), 323–348. https://doi.org/10.1017/S0017089516000197

20. R. J. McIntosh, Second order mock theta functions, *Can. Math. Bull.*, **50** (2007), 284–290. https://doi.org/10.4153/CMB-2007-028-9

21. A. E. Patkowski, On some new Bailey pairs and new expansions for some Mock theta functions, *Methods Appl. Anal.,* **23** (2016), 205–213. https://doi.org/10.7216/1300759920162310306

22. A. E. Patkowski, More on some Mock theta double sums, *Adv. in Appl. Math.,* **106** (2019), 20–27. https://doi.org/10.1016/j.aam.2019.01.006

23. S. Ramanujan, *Collected papers of Srinivasa Ramanujan*, Edited by G. H. Hardy, P. V. Seshu. Aiyar, B. M. Wilson, Third printing of the 1927 original. With a new preface and commentary by Bruce C. Berndt, AMS Chelsea Publishing; Providence, RI (2000), xxxviii+426pp.

24. S. Ramanujan *Collected Papers*, Cambridge University Press, 1972, reprinted Chelsea, New York, 1962.

25. Z. Zhang, X. Li Mock theta functions in terms of q-hypergeometric double sums, *Int. J. Number Theory*, **14** (2018), 1715–1728.

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