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Research article

Minimal homothetical and translation lightlike graphs in $\mathbb{R}^{n+2}_q$

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Abstract: In this paper, homothetical and translation lightlike graphs, which are generalizations of homothetical and translation lightlike hypersurfaces are investigated in the semi-Euclidean space $\mathbb{R}^{n+2}_q$, respectively. We prove that all homothetical and all translation lightlike graphs are locally the hyperplanes. According to this fact, both of these graphs are minimal.

Keywords: lightlike hypersurfaces; homothetical graphs; translation graphs; semi-Euclidean space; minimality
Mathematics Subject Classification: 53B25, 53B30

1. Introduction and preliminaries

Lightlike (degenerate, null) submanifolds of a semi-Riemannian manifold have a very important place in physics. Especially, with its use in general relativity, it has attracted the attention of many scientists and there are many studies on this subject. Lightlike submanifolds were extensively studied by Duggal and Bejancu [1]. Semi-Riemanian submanifolds whose induced metric degenerates are called lightlike submanifolds. Lightlike hypersurfaces are lightlike submanifolds in semi-Euclidean space and have been studied by many mathematicians [1–10].

Let M be a hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $\overline{M}$ of index $q \in \{1, \ldots, n+1\}$, $n > 0$. Let $\overline{g}$ be the semi-Riemannian metric on $\overline{M}$. $\overline{g}$ induces on M a symmetric tensor field g of type (0,2). The radical (null) space of T_uM is

$$Rad\, T_uM = \{\xi_u \in T_uM : \; g_u(\xi_u, X_u) = 0, \; \forall X_u \in T_uM\},$$

where T_uM is the tangent space to M at $u \in M$. Since

$$T_uM^{\perp} = \{ V_u \in T_u\overline{M} : \; \overline{g}_u(V_u, W_u) = 0, \; \forall W_u \in T_uM\},$$

we have

$$Rad\, T_uM = T_uM \, \cap \, T_uM^{\perp} \; [1].$$

Definition 1.1. *Let M be a hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $\overline{M}$, $n>0$. If $Rad\,T_uM \neq \{0\}$ for any $u \in M$, M is called a lightlike (degenerate) hypersurface of $\overline{M}$ [1].*

If M is a lightlike hypersurface of $\overline{M}$, $T_uM^{\perp}$ is an one-dimensional vector subspace of the tangent space T_uM. Each m-dimensional subspace in T_uM that does not contain the subspace $T_uM^{\perp}$ is orthogonal to $T_uM^{\perp}$ and called a screen space at point u. The vector bundle that is constituted by choosing a screen space each point of M is said to be a screen distribution on M, denoted by $S(TM)$. Thus we have

$$TM = S(TM) \perp TM^{\perp}.$$

$T\overline{M}_{/M}$ is a vector bundle that has M as base space and assigns $T_u\overline{M}$ to each point u of M. g_u is non-degenerate on $S(T_uM)$. If a subspace is non-degenerate, its complementary orthogonal subspace is also non-degenerate and is uniquely determined. Thus, the vector bundle that is determined by the complementary orthogonal subspace is called the orthogonal complementary vector bundle to $S(TM)$ in $T\overline{M}_{/M}$, denoted by $S(TM)^{\perp}$. Also we have

$$T\overline{M}_{/M} = S(TM) \perp S(TM)^{\perp} \text{ [1].}$$

Theorem 1.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero $\xi \in \Gamma(TM^{\perp})$ on a coordinate neighbourhood $\mathcal{U} \in M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ with the following properties:*

$$\overline{g}(N,\xi) = 1,$$

and

$$\overline{g}(N,N) = \overline{g}(N,W) = 0, \quad \forall W \in \Gamma(S(TM)_{/\mathcal{U}}).$$

The space that is the union of subspaces spanned by the vector N_u at each point $u \in M$ is a lightlike vector bundle and is called the lightlike transversal vector bundle of M with respect to $S(TM)$. It is denoted by $tr(TM)$. $tr(TM)_u$ is the subspace spanned by the vector N_u. Hence we have

$$T\overline{M}_{/M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM) \text{ [1].}$$

Definition 1.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ and $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$. If $X, Y \in \Gamma(TM)$, then $\overline{\nabla}_X Y \in \Gamma(T\overline{M})$. Using the decomposition $T\overline{M}_{/M} = TM \oplus tr(TM)$, we obtain the formulas*

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V,$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while $h(X,Y)$ and $\nabla^t_X V$ belong to $\Gamma(tr(TM))$. It is easy to check that ∇ is a torsion-free linear form on M , h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, which has range $\Gamma(tr(TM))$, A_V is a $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$. We call ∇ and ∇^t the induced connections on M and $tr(TM)$, respectively. Consistent with the classical theory of Riemannian hypersurfaces we call h and A_V the second fundamental form and the shape operator respectively, of the lightlike immersion of M in $\overline{M}$. Also, we name the above equations the Gauss and the Weingarten formulae, respectively [1].

Definition 1.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Next, if P denotes the projection morphism of TM on $S(TM)$ with respect to the decomposition $TM = S(TM) \perp TM^{\perp}$, we obtain*

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \ \forall X, Y \in \Gamma(TM).$$

and

$$\overline{\nabla}_X U = -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t_X U, \ U \in \Gamma(TM^{\perp}),$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$ while $\overset{*}{h}(X, PY)$ and $\overset{*}{\nabla}{}^t_X U$ belong to $\Gamma(TM^{\perp})$. It follows that $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on vector bundles $S(TM)$ and $TM^{\perp}$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and $\overset{*}{A}_U$ is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$. We call $\overset{*}{h}$ and $\overset{*}{A}_U$ the second fundamental form and the shape operator of the screen distribution $S(TM)$, respectively. Also, the above equations are the Gauss and the Weingarten equation for the screen distribution $S(TM)$ [1].

Proposition 1.1. *On any lightlike Monge hypersurfaces M of R^{n+2}_q, the shape operators $A_{\overline{N}}$ and $\overset{*}{A}_\xi$ of M and of the natural screen distribution are related by*

$$A_{\overline{N}} = \frac{1}{2}\overset{*}{A}_\xi \ [1].$$

Definition 1.4. *Let ξ be a normal null section. The trace of $-\overset{*}{A}_\xi$ is called the lightlike mean curvature H_ξ on M associated with ξ. Then*

$$H_\xi = trace(-\overset{*}{A}_\xi) = -trace(\overset{*}{A}_\xi).$$

One of the good properties of the lightlike mean curvature is that it does not depend on the screen distribution chosen, but only of the local normal null section ξ [2].

Let $y_0, y_1, \ldots, y_{n+1}$ be coordinate functions in $\mathbb{R}^{n+2}$ and $x_1, x_2, \ldots, x_{n+1}$ be coordinate functions in $\mathbb{R}^{n+1}$. Let M be a lightlike Monge hypersurface determined by the transformation

$$\psi(x_1, x_2, \ldots, x_{n+1}) = (F(x_1, x_2, \ldots, x_{n+1}), x_1, x_2, \ldots, x_{n+1}),$$

where $F : D \to R$ is a smooth function and D is an open subset of $\mathbb{R}^{n+1}$. We have

$$\partial_\alpha \circ \psi = F_{x_\alpha} \frac{\partial}{\partial y_0} \circ \psi + \frac{\partial}{\partial y_\alpha} \circ \psi, \qquad 1 \leq \alpha \leq n+1,$$

where $\partial_1, \partial_2, \ldots, \partial_{n+1}$ are coordinate frame fields on M. Since $g(\partial_\alpha \circ \psi, \xi) = 0$ for each α, we have

$$\xi \circ \psi = \frac{\partial}{\partial y_0} \circ \psi - \sum_{s=1}^{q-1} F_{x_s} \frac{\partial}{\partial y_s} \circ \psi + \sum_{\alpha=q}^{n+1} F_{x_\alpha} \frac{\partial}{\partial y_\alpha} \circ \psi,$$

where ξ is the normal vector field on M. The vector field $\overline{N}$ determined by the equation

$$\overline{N} = -\frac{\partial}{\partial y_0} + \frac{1}{2}\xi$$

satisfies the conditions of Theorem 1.1 and spans the vector bundle $tr(TM)$. $\overline{N}$ is called the natural lightlike transversal vector bundle of M [1].

Theorem 1.2. *Let M^{n+1} be a Monge hypersurface given by the transformation*

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1})$$

in the semi-Euclidean space $\mathbb{R}^{n+2}_q$. M^{n+1} is lightlike iff

$$\sum_{i=1}^{n+1} \varepsilon_i F_{x_i}^2 = 1, \tag{1.1}$$

where

$$\varepsilon_i = \begin{cases} -1, & 1 \le i \le q-1, \\ 1, & q \le i \le n+1, \end{cases}$$

and F_{x_i} is the partial derivative of the function F with respect to x_i for $i = 1, 2, \ldots, n+1$ [1].

Proposition 1.2. *Let M^{n+1} be a lightlike Monge hypersurface given by the transformation*

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1})$$

in the semi-Euclidean space $\mathbb{R}^{n+2}_q$. The lightlike mean curvature of M^{n+1} with respect to its normal section ξ is given by

$$H_\xi = \sum_{i=1}^{n+1} \varepsilon_i F_{x_i x_i},$$

where

$$\varepsilon_i = \begin{cases} -1, & 1 \le i \le q-1, \\ 1, & q \le i \le n+1, \end{cases}$$

and $F_{x_i x_i}$ is the second order partial derivative of the function F with respect to x_i (i=1, 2, . . ., n+1) [11].

Corollary 1.1. *Let M^{n+1} be a lightlike Monge hypersurface given by the transformation*

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1})$$

in the semi-Euclidean space $\mathbb{R}^{n+2}_q$. M^{n+1} is minimal iff

$$\sum_{i=1}^{n+1} \varepsilon_i F_{x_i x_i} = 0,$$

where

$$\varepsilon_i = \begin{cases} -1, & 1 \le i \le q-1, \\ 1, & q \le i \le n+1, \end{cases}$$

and $F_{x_i x_i}$ is the second order partial derivative of the function F with respect to x_i (i=1, 2, . . .,n+1) [11].

Homothetical and translation hypersurfaces are special Monge hypersurfaces. Many studies have been carried out with these hypersurfaces until today [11–16].

In semi-Euclidean space $\mathbb{R}^{n+2}_q$, homothetical and translation hypersurfaces are submanifolds with one codimension given by

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1}),\quad F(x_1,\ldots,x_{n+1}) = \prod_{i=1}^{n+1} f_i(x_i)$$

and

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1}),\ \ F(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n+1} f_i(x_i),$$

where $f_1, f_2, \ldots, f_{n+1}$ are smooth functions, respectively. Each function f_i depends on the one real variable x_i and is different from zero, for $1 \leq i \leq n+1$. Or else these hypersurfaces are locally the hyperplanes.

In the semi-Euclidean space $\mathbb{R}^{n+2}_q$, homothetical and translation graphs are $(n+1)$-dimensional Monge hypersurfaces given by

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1}),\ \ F(x_1,\ldots,x_{n+1}) = \prod_{i=1}^{n} f_i(x_i) f_{n+1}(u)$$

and

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}),\ x_1,\ldots,x_{n+1}),\ \ F(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n} f_i(x_i) + f_{n+1}(u),$$

where $u = \sum\limits_{i=1}^{n+1} c_i x_i$, c_i are constants, $c_{n+1} \neq 0$ and each f_i is a smooth function of one real variable for $i = 1, 2, \ldots, n+1$, respectively.

In [11], author proved that homothetical lightlike hypersurfaces are minimal. And than translation and homothetical lightlike hypersurfaces have been shown to be minimal by getting that they are locally the hyperplanes in the semi-Euclidean space [12].

In this paper we study homothetical and translation lightlike (degenerate) graphs which are generalizations of homothetical and translation lightlike (degenerate) hypersurfaces, respectively. We prove that all homothetical and all translation lightlike (degenerate) graphs are locally the hyperplanes. Also, both of these graphs are minimal.

2. Homothetical lightlike (degenerate) graphs of semi-Euclidean spaces

Previously one showed that every homothetical lightlike hypersurface is locally a hyperplane [12]. Now we will prove that a homothetical lightlike graph is locally a hyperplane in the following theorem.

Theorem 2.1. *Let M^{n+1} be an $(n+1)$-dimensional homothetical graph of the semi-Euclidean space $\mathbb{R}^{n+2}_q$ determined by the following equations*

$$\psi(x_1,\ldots,x_{n+1}) = (F(x_1,\ldots,x_{n+1}), x_1,\ldots,x_{n+1}),\ \ F(x_1,\ldots,x_{n+1}) = \prod_{j=1}^{n} f_j(x_j) f_{n+1}(u), \tag{2.1}$$

where $u = \sum\limits_{i=1}^{n+1} c_i x_i$, c_i are constants for all $1 \leq i \leq n+1$, with $c_{n+1} \neq 0$ and $\sum\limits_{i=1}^{n+1} \varepsilon_i c_i^2 > 0$. M^{n+1} is lightlike, then it is locally a hyperplane.

Proof. Partial derivatives of the function F with respect to x_i are given by

$$F_i = f_i' \prod_{\substack{j=1\\ j\neq i}}^{n+1} f_j + c_i \prod_{j=1}^{n} f_j f_{n+1}', \quad F_{n+1} = c_{n+1} \prod_{j=1}^{n} f_j f_{n+1}' \tag{2.2}$$

for $1 \le i \le n$. Substitute this equations into (1.1), then we obtain the equation

$$\sum_{i=1}^{n} \varepsilon_i \frac{f_i'^2}{f_i^2} f_{n+1}^2 + \sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}'^2 + 2 \sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} f_{n+1} f_{n+1}' = \frac{1}{\prod_{j=1}^{n} f_i^2}, \tag{2.3}$$

where

$$\varepsilon_i = \begin{cases} -1, & 1 \le i \le q-1, \\ 1, & q \le i \le n+1, \end{cases}$$

and $F \neq 0$ in any point. Note that each function f_i is non-zero for $1 \le i \le n$. Otherwise $F = 0$. Derivative of the Eq (2.3) with respect to u, we get

$$\sum_{i=1}^{n} \varepsilon_i \frac{f_i'^2}{f_i^2} f_{n+1} f_{n+1}' + \sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}' f_{n+1}'' + \sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} (f_{n+1}'^2 + f_{n+1} f_{n+1}'') = 0. \tag{2.4}$$

Note that $f_{n+1} f_{n+1}' \neq 0$, otherwise M^{n+1} is a homothetical hypersurface. If we divide both sides of the Eq (2.4) by $f_{n+1} f_{n+1}' \neq 0$, then we get

$$\sum_{i=1}^{n} \varepsilon_i \frac{f_i'^2}{f_i^2} + \sum_{i=1}^{n+1} \varepsilon_i c_i^2 \frac{f_{n+1}''}{f_{n+1}} + \sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} \left(\frac{f_{n+1}'}{f_{n+1}} + \frac{f_{n+1}''}{f_{n+1}'} \right) = 0. \tag{2.5}$$

Derivative of the Eq (2.5) with respect to u, we find

$$\sum_{i=1}^{n+1} \varepsilon_i c_i^2 \left(\frac{f_{n+1}''}{f_{n+1}} \right)_u + \sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} \left(\frac{f_{n+1}'}{f_{n+1}} + \frac{f_{n+1}''}{f_{n+1}'} \right)_u = 0. \tag{2.6}$$

According to the Eq (2.6), we get following the cases to consider:

Case 1. $f_{n+1}'' = 0$.

Since $f_{n+1}'' = 0$, then

$$f_{n+1}' = a \quad \text{and} \quad f_{n+1} = au + b \tag{2.7}$$

with constants $a \neq 0, b$. If $a = 0$, then M^{n+1} is a homothetical hypersurface. Substitute this equations into (2.6), then we find

$$\sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} \frac{a^2}{(au+b)^2} = 0. \tag{2.8}$$

Since $a \neq 0$, then we have

$$\sum_{i=1}^{n} \varepsilon_i c_i \frac{f_i'}{f_i} = 0. \tag{2.9}$$

Since each f_i depends on a different variable, from the Eq (2.9), each function $\frac{f_i'}{f_i}$ is constant for $1 \le i \le n$. Thus, we can write

$$\frac{f_i'}{f_i} = a_i, \quad 1 \le i \le n, \tag{2.10}$$

with a constant a_i. From the Eqs (2.9) and (2.10), we have

$$\sum_{i=1}^{n} \varepsilon_i c_i a_i = 0. \tag{2.11}$$

Substitute the Eqs (2.7) and (2.10) into (2.4), then we obtain

$$\sum_{i=1}^{n} \varepsilon_i a_i^2 (au + b) + a \sum_{i=1}^{n} \varepsilon_i c_i a_i = 0. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\sum_{i=1}^{n} \varepsilon_i a_i^2 (au + b) = 0.$$

Since $au + b \neq 0$, we have

$$\sum_{i=1}^{n} \varepsilon_i a_i^2 = 0. \tag{2.13}$$

Substitute the Eqs (2.7) and (2.10) into (2.3), then we obtain

$$\sum_{i=1}^{n} \varepsilon_i a_i^2 (au + b)^2 + a^2 \sum_{i=1}^{n+1} \varepsilon_i c_i^2 + 2a \sum_{i=1}^{n} \varepsilon_i c_i a_i (au + b) = \frac{1}{\prod\limits_{i=1}^{n} f_i^2}. \tag{2.14}$$

From (2.11), (2.13) and (2.14), we get

$$a^2 \sum_{i=1}^{n+1} \varepsilon_i c_i^2 = \frac{1}{\prod\limits_{i=1}^{n} f_i^2}. \tag{2.15}$$

Since the left side of this equation is constant, the right side has to be constant. Also each function f_i is constant for $1 \leq i \leq n$. From (2.15), we can write

$$f_{n+1}(u) = au + b = \frac{1}{\left|\prod_{i=1}^{n} f_i\right| \sqrt{\sum_{i=1}^{n+1} \varepsilon_i c_i^2}} u + b \quad \text{with} \quad u = \sum_{i=1}^{n+1} c_i x_i \tag{2.16}$$

and

$$\psi(x_1, \ldots, x_{n+1}) = (\pm \frac{1}{\sqrt{\sum\limits_{i=1}^{n+1} \varepsilon_i c_i^2}} \sum_{i=1}^{n+1} c_i x_i + B, x_1, \ldots, x_{n+1}), \tag{2.17}$$

where $B = b \prod\limits_{i=1}^{n} f_i$, c_i are constants for $1 \leq i \leq n+1$ with $c_{n+1} \neq 0$ and $\sum\limits_{i=1}^{n+1} \varepsilon_i c_i^2 > 0$. Therefore, in this case the graph M^{n+1} is locally a hyperplane.

Case 2. $f''_{n+1} \neq 0$.

From $f''_{n+1} \neq 0$, according to the Eq (2.6), we have two possibilities.

Case 2a. Assume that $\left(\frac{f'_{n+1}}{f_{n+1}}+\frac{f''_{n+1}}{f'_{n+1}}\right)_u = 0$. Then there is a constant m, such that $\frac{f'_{n+1}}{f_{n+1}}+\frac{f''_{n+1}}{f'_{n+1}} = m$. Without loss of generality, suppose that $m=0$. Then we get the differential equation $\frac{f'_{n+1}}{f_{n+1}}+\frac{f''_{n+1}}{f'_{n+1}} = 0$. By solving the differential equation, we find

$$f_{n+1}(u) = d_1\sqrt{2u-d_2}, \tag{2.18}$$

with some constants $d_1 \neq 0$ and d_2. If $d_1 = 0$, then $f_{n+1}(u)$ would be vanish. From (2.18), we get

$$\left(\frac{f''_{n+1}}{f_{n+1}}\right)_u = \frac{4d_1^2}{(2u-d_2)^3} \neq 0. \tag{2.19}$$

From (2.6) and (2.19), we get $\sum\limits_{i=1}^{n+1}\varepsilon_i c_i^2 = 0$. Thus, we have a contradiction.

Case 2b. Assume that $\left(\frac{f'_{n+1}}{f_{n+1}}+\frac{f''_{n+1}}{f'_{n+1}}\right)_u \neq 0$. Since $\sum\limits_{i=1}^{n+1}\varepsilon_i c_i^2 \neq 0$, if we rearrange the Eq (2.6), then there is a constant β such that

$$-\frac{\sum\limits_{i=1}^{n}\varepsilon_i c_i\frac{f'_i}{f_i}}{\sum\limits_{i=1}^{n+1}\varepsilon_i c_i^2} = \frac{\left(\frac{f''_{n+1}}{f_{n+1}}\right)_u}{\left(\frac{f'_{n+1}}{f_{n+1}}+\frac{f''_{n+1}}{f'_{n+1}}\right)_u} = \beta. \tag{2.20}$$

We assume that $\beta \neq 0$. From the Eq (2.20), we can write

$$\sum_{i=1}^{n}\varepsilon_i c_i\frac{f'_i}{f_i} = -\beta\sum_{i=1}^{n+1}\varepsilon_i c_i^2. \tag{2.21}$$

Since the right side of this equation is constant, the left side has to be constant. Also there are constants a_i such that the functions

$$\frac{f'_i}{f_i} = a_i, \tag{2.22}$$

for all $1 \leq i \leq n$. From (2.21) and (2.22), we obtain

$$\sum_{i=1}^{n}\varepsilon_i c_i a_i = -\beta\sum_{i=1}^{n+1}\varepsilon_i c_i^2. \tag{2.23}$$

If we subtract c_{n+1}^2 from this equation, then

$$c_{n+1}^2 = -\frac{1}{\varepsilon_{n+1}\beta}\sum_{i=1}^{n}\varepsilon_i(c_i a_i+\beta c_i^2). \tag{2.24}$$

Also, from this equation we can find $c_{n+1} = 0$ for suitable constants β, a_i, c_i, with $1 \leq i \leq n$. This is a contradiction with the definition of the homothetical graph.

Let β be zero in the Eq (2.20). From this equation, we get

$$\frac{f''_{n+1}}{f_{n+1}} = d \text{ and } \frac{f'_i}{f_i} = a_i \tag{2.25}$$

with $d \neq 0$, a_i constants for $1 \leq i \leq n$. If $d = 0$, then $f''_{n+1} = 0$. Because of the contradiction, d is different from zero. Substitute the equations in (2.25) into (2.5), then we obtain

$$\sum_{i=1}^{n} \varepsilon_i a_i^2 + d \sum_{i=1}^{n+1} \varepsilon_i c_i^2 = 0. \tag{2.26}$$

If we subtract c_{n+1}^2 from this equation, then

$$c_{n+1}^2 = -\frac{1}{\varepsilon_{n+1} d} \sum_{i=1}^{n} \varepsilon_i (a_i^2 + d c_i^2). \tag{2.27}$$

Also, from this equation we can find $c_{n+1} = 0$ for suitable constants d, a_i, c_i, with $1 \leq i \leq n$. This is a contradiction with the definition of the homothetical graph. Thus $f''_{n+1} \neq 0$ is not possible. Also the proof is completed. □

Corollary 2.1. *Homothetical lightlike (degenerate) graphs are minimal in the semi-Euclidean space* $\mathbb{R}_q^{n+2}$.

3. Translation lightlike (degenerate) graphs of semi-Euclidean spaces

In [12] one showed that every translation lightlike hypersurface is locally a hyperplane. Now we will prove that a translation lightlike graph is locally a hyperplane in the following theorem.

Theorem 3.1. *Let* M^{n+1} *be an* $(n+1)$*-dimensional translation graph of* $\mathbb{R}_q^{n+2}$ *determined by following equations*

$$\psi(x_1, \ldots, x_{n+1}) = (F(x_1, \ldots, x_{n+1}),\ x_1, \ldots, x_{n+1}),\ F(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} f_i(x_i) + f_{n+1}(u), \tag{3.1}$$

where $u = \sum_{i=1}^{n+1} c_i x_i$, c_i *are constants for all* $1 \leq i \leq n+1$, *with* $c_n \neq 0$ *and* $\sum_{i=1}^{n+1} \varepsilon_i c_i^2 \neq 0$. M^{n+1} *is lightlike, then it is locally a hyperplane.*

Proof. It is easy to check that

$$F_i = f'_i + c_i f'_n,\ \ F_{n+1} = c_{n+1} f'_{n+1} \tag{3.2}$$

for $i = 1, ..., n$. Substitute this equations into (1.1), then we obtain the equation

$$\sum_{i=1}^{n} \varepsilon_i f_i'^2 + \sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}'^2 + 2 \sum_{i=1}^{n} \varepsilon_i c_i f'_i f'_{n+1} = 1, \tag{3.3}$$

where

$$\varepsilon_i = \begin{cases} -1, & 1 \leq i \leq q-1 \\ 1, & q \leq i \leq n+1 \end{cases}$$

and $F \neq 0$ in any point. Derivative of the Eq (3.3) with respect to u, we find

$$\sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}' f_{n+1}'' + \sum_{i=1}^{n} \varepsilon_i c_i f_i' f_{n+1}'' = 0. \tag{3.4}$$

From this equation, we obtain

$$\left(\sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}' + \sum_{i=1}^{n} \varepsilon_i c_i f_i' \right) f_{n+1}'' = 0. \tag{3.5}$$

We assume that $f_{n+1}'' \neq 0$ and then from the Eq (3.5), we find

$$\sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}' + \sum_{i=1}^{n} \varepsilon_i c_i f_i' = 0. \tag{3.6}$$

Derivative of the Eq (3.6) with respect to u, we obtain

$$\sum_{i=1}^{n+1} \varepsilon_i c_i^2 f_{n+1}'' = 0. \tag{3.7}$$

Since $f_{n+1}'' \neq 0$, then

$$\sum_{i=1}^{n+1} \varepsilon_i c_i^2 = 0. \tag{3.8}$$

This is a contradiction with the assumption. From the Eq (3.7) it must be $f_{n+1}'' = 0$. Hence we find

$$f_{n+1}(u) = au + b \tag{3.9}$$

with constants $a \neq 0, b$. If $a = 0$, then M^{n+1} is a translation hypersurface. Substitute $f_{n+1}(u) = au + b$ into the Eq (3.3), we get

$$\sum_{i=1}^{n} \varepsilon_i f_i'^2 + a^2 \sum_{i=1}^{n+1} \varepsilon_i c_i^2 + 2a \sum_{i=1}^{n} \varepsilon_i c_i f_i' = 1. \tag{3.10}$$

Derivative of (3.10) with respect to x_i for all $i = 1, ..., n$, we obtain

$$(f_i' + ac_i) f_i'' = 0.$$

According to this equation, we get

$$f_i' + ac_i = 0 \text{ or } f_i'' = 0.$$

Also we obtain

$$f_i = -ac_i x_i + b_i, \tag{3.11}$$

where b_i are constants for all $i = 1, ..., n$. Substitute (3.9) and (3.11) into (3.3), then we obtain

$$a = \pm \frac{1}{c_{n+1}}. \tag{3.12}$$

Substitute (3.9), (3.11) and (3.12) into (3.1), then we have

$$F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n} f_i(x_i) + f_{n+1}(\sum_{i=1}^{n+1} c_i x_i) = \pm x_{n+1} + B$$

and

$$\psi(x_1, \dots, x_{n+1}) = (\pm x_{n+1} + B, x_1, x_2, \dots, x_{n+1}), \tag{3.13}$$

where $B = b + \sum_{i=1}^{n} b_i$, c_i are constants for all $1 \leq i \leq n+1$ with $c_{n+1} \neq 0$ and $\sum_{i=1}^{n+1} \varepsilon_i c_i^2 \neq 0$. Also, the translation lightlike graph M^{n+1} is locally a hyperplane in the semi-Euclidean space. □

Corollary 3.1. *Translation lightlike (degenerate) graphs are minimal in the semi-Euclidean space* $\mathbb{R}_q^{n+2}$.

4. Conclusions

In this paper, homothetical and translation lightlike (degenerate) graphs, which are generalizations of homothetical and translation lightlike (degenerate) hypersurfaces are investigated in the semi-Euclidean space $\mathbb{R}_q^{n+2}$, respectively. We prove that all homothetical and all translation lightlike (degenerate) graphs are locally the hyperplanes. As a result, both of these graphs are minimal.

Conflict of interest

The author declares no conflict of interest.

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