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Research article

Statistical convergence of new type difference sequences with Caputo fractional derivative

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Abstract: In this study, we discuss the idea of difference operators $\Delta_p^{\alpha}, \Delta_p^{(\alpha)}$ $(\alpha \in \mathbb{R})$ and examine some properties of these operators. We also describe the concepts of ordered statistical convergence and lacunary statistical by using the Δ_p^{α}-difference operator. We examine some features of these sequence spaces and present some inclusion theorems. We obtain the Caputo fractional derivative in this work.

Keywords: difference operators; statistical convergence; sequence spaces; fractional difference operators
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1. Introduction

In this section, we will give the basic definitions and concepts that we need for other sections.

Assume that w is the set of all complex sequences $v = (v_k)_{k=1}^{\infty}$ and ℓ_∞, c and c_0 describes the Banach spaces of sequences and normed by $\|v\|_\infty = \sup_k |v_k|$.

In 1981, the difference sequence spaces $E(\Delta)$ were proposed in [1]. These are Banach spaces with norm:$\|v\|_\Delta = |v_1| + \|\Delta v\|_\infty$. He showed that $E \subseteq E(\Delta)$, since there exists a sequence $v_k = (k)$ $(k = 1, 2, 3, ...)$ for which $\Delta v_k = 1$, so that although v is not convergent but, it is Δ-convergent. Later, Et and Çolak [2] defined generalized difference sequence spaces. Recently, Ioan [3] introduced Δ_p^m and discussed the concept of p-convexity of this difference sequence.

Later on, Karakaş et al. [4] defined and discussed some basic topological and algebraic properties of the sequence spaces $E\left(\Delta_p^m\right)$ for $E = \ell_\infty, c$ and c_0, where $p, m \in \mathbb{N}$, $\Delta_p v = (pv_k - v_{k+1})$ and $\Delta_p^m v = \left(\Delta_p^m v_k\right) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} p^{m-i} v_{k+i}$. In the case $v \in E\left(\Delta_p^m\right)$ (for $E = \ell_\infty, c$ and c_0), we call Δ_p^m-bounded, Δ_p^m-convergent and Δ_p^m-zero, respectively. The sequence spaces $\ell_\infty\left(\Delta_p^m\right), c\left(\Delta_p^m\right)$ and $c_0\left(\Delta_p^m\right)$ are Banach spaces with norm

$$\|v\|_{\Delta_p^m} = \sum_{i=1}^{m} \left|v_i\right| + \left\|\Delta_p^m v\right\|_\infty .$$

The statistical convergence was discussed in [5] firstly. Later, this idea was given by Steinhaus [6] and Fast [7] for complex number sequences. Recently, many mathematicians have studied the topological properties of this type of convergence and its relation to summability (see [8–10]).

We describe the natural density of a subset C of $\mathbb{N}$ as:

$$\delta(C) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in C\}| ,$$

if the limit exists, where $|.|$ is cardinality of set C.

Let $v = (v_k)_{k=1}^{\infty}$ be a sequence of complex numbers.The sequence v is statistically convergent to complex number ℓ if, for every positive number ε, $\delta(\{k \in \mathbb{N}: |v_k - \ell| \geq \varepsilon\})$ has natural density zero. We define ℓ as the statistical limit of (v_k). Then, we have $S_t - \lim v_k = \ell$. We describe the spaces of all statistically convergent by S_t. It is easily seen that the statistical limit is necessarily unique.

The ordered statistical convergence was presented by Gadjiev and Orhan [11]. After that Çolak [12] defined and studied the concepts of β-density and statistical convergence of order β. Also, this has been studied by many mathematicians in recent years (see [4, 8, 13]).

Lacunary sequence was described by Freedman et al. [14] as follows:

By Lacunary sequence $\Phi = (u_r)$; $r = 0, 1, 2, 3, ...$, where $u_0 = 0$,and $h_r = u_r - u_{r-1} \to \infty$ as $r \to \infty$. We denote by $I_r = (u_{r-1}, u_r]$ the intervals determined by Φ and $q_r = \frac{u_r}{u_{r-1}}$ for $r = 0, 1, 2, ...$

Fridy and Orhan [15] have defined novel type of statistical convergence. In addition, the relationship of this concept with summability was given by Fridy and Orhan [16]. Later, Lacunary statistical convergence of order β was defined by Sengül and Et [17] as follows:

Let $\Phi = (u_r)$ be a lacunary sequence, $v = (v_k) \in w$ and $0 < \beta \leq 1$. Let there exsit ℓ such that for $\varepsilon > 0$. Then, we can say that the sequence $v = (v_k)$ is S_Φ^β-lacunary statistically convergent of order β.

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} |\{k \in I_r : |v_k - \ell| \geq \varepsilon\}| = 0,$$

where $I_r = (u_{r-1}, u_r]$. In the case, we write $S_\Phi^\beta - \lim v_k = \ell$.

The definition of a modulus function is given by Nakano [18] as follows:

We assume that ϕ fulfils the following conditions

i) $\phi(v) = 0$ if and only if $v = 0$,

ii) $\phi(v + v^*) \leq \phi(v) + \phi(v^*)$, for all $v, v^* \geq 0$,

iii) ϕ is increasing,

iv) $\lim_{z\to 0^+} \phi(v) = 0$.

A modulus function can be bounded and unbounded.

We will give information about fractional difference sequences.

Recently, the topological properties of fractional difference sequences were first studied by Baliarsingh [19]. Later, different properties of fractional difference sequences were examined by the same author and his colleagues (see for details [20, 21]).

Baliarsingh [19], Baliarsingh and Dutta [21] defined the generalized fractional difference operator $\Delta^\alpha : w \to w$ as follows:

$$(\Delta^{\alpha} v_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}, \tag{1.1}$$

where $\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1} dt$.

In this study we acquired the following results by applying the Δ_p^{α} difference operator on the Eq (1.1) and the results of Baliarsingh and Dutta [21] studies:

$$(\Delta_p^{\alpha} v_k) = \sum_{i=0}^{\infty} (-1)^i p^{(\alpha-i)} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i},$$

$$(\Delta_p^{(\alpha)} v_k) = \sum_{i=0}^{\infty} (-1)^i p^{(\alpha-i)} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k-i},$$

$$(\Delta_p^{-\alpha} v_k) = \sum_{i=0}^{\infty} (-1)^i p^{(-\alpha-i)} \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} v_{k+i},$$

$$(\Delta_p^{(-\alpha)} v_k) = \sum_{i=0}^{\infty} (-1)^i p^{(-\alpha-i)} \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} v_{k-i}.$$

Especially, for $\alpha = \frac{1}{2}$, it procure that

- $\Delta_p^{\frac{1}{2}} v_k = p^{1/2} v_k - \frac{p^{-1/2}}{2} v_{k+1} - \frac{p^{-3/2}}{8} v_{k+2} - \frac{p^{-5/2}}{16} v_{k+3} - \frac{5p^{-7/2}}{128} v_{k+4} - \frac{7p^{-9/2}}{256} v_{k+5} - \frac{21p^{-11/2}}{1024} v_{k+6} + \ldots.$
- $\Delta_p^{(\frac{1}{2})} v_k = p^{1/2} v_k - \frac{p^{-1/2}}{2} v_{k-1} - \frac{p^{-3/2}}{8} v_{k-2} - \frac{p^{-5/2}}{16} v_{k-3} - \frac{5p^{-7/2}}{128} v_{k-4} - \frac{7p^{-9/2}}{256} v_{k-5} - \frac{21p^{-11/2}}{1024} v_{k-6} + \ldots.$
- $\Delta_p^{-\frac{1}{2}} v_k = p^{-1/2} v_k + \frac{p^{-3/2}}{2} v_{k+1} + \frac{3p^{-5/2}}{8} v_{k+2} + \frac{5p^{-7/2}}{16} v_{k+3} + \frac{35p^{-9/2}}{128} v_{k+4} + \frac{63p^{-11/2}}{256} v_{k+5} + \frac{231p^{-13/2}}{1024} v_{k+6} + \ldots.$
- $\Delta_p^{(-\frac{1}{2})} v_k = p^{-1/2} v_k + \frac{p^{-3/2}}{2} v_{k-1} + \frac{3p^{-5/2}}{8} v_{k-2} + \frac{5p^{-7/2}}{16} v_{k-3} + \frac{35p^{-9/2}}{128} v_{k-4} + \frac{63p^{-11/2}}{256} v_{k-5} + \frac{231p^{-13/2}}{1024} v_{k-6} + \ldots.$

We define the operators, $\Delta_p^{\alpha}, \Delta_p^{(\alpha)}, \Delta_p^{-\alpha}$ and $\Delta_p^{(-\alpha)}$ can be explicit as triangles as follows:

$$\Delta_p^{\alpha} = \begin{pmatrix} p^{\alpha} & -p^{(\alpha-1)}\alpha & \frac{p^{(\alpha-2)}\alpha(\alpha-1)}{2!} & -\frac{p^{(\alpha-3)}\alpha(\alpha-1)(\alpha-2)}{3!} & \ldots \\ 0 & p^{\alpha} & -p^{(\alpha-1)}\alpha & \frac{p^{(\alpha-2)}\alpha(\alpha-1)}{2!} & \ldots \\ 0 & 0 & p^{\alpha} & -p^{(\alpha-1)}\alpha & \ldots \\ 0 & 0 & 0 & p^{\alpha} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta_p^{(\alpha)} = \begin{pmatrix} p^{\alpha} & 0 & 0 & 0 & \ldots \\ -p^{(\alpha-1)}\alpha & p^{\alpha} & 0 & 0 & \ldots \\ \frac{p^{(\alpha-2)}\alpha(\alpha-1)}{2!} & -p^{(\alpha-1)}\alpha & p^{\alpha} & 0 & \ldots \\ -\frac{p^{(\alpha-3)}\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{p^{(\alpha-2)}\alpha(\alpha-1)}{2!} & -p^{(\alpha-1)}\alpha & p^{\alpha} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta_p^{-\alpha} = \begin{pmatrix} p^{-\alpha} & p^{(-\alpha-1)}\alpha & \frac{p^{(-\alpha-2)}\alpha(\alpha+1)}{2!} & \frac{p^{(-\alpha-3)}\alpha(\alpha+1)(\alpha+2)}{3!} & ... \\ 0 & p^{-\alpha} & p^{(-\alpha-1)}\alpha & \frac{p^{(-\alpha-2)}\alpha(\alpha+1)}{2!} & ... \\ 0 & 0 & p^{-\alpha} & p^{(-\alpha-1)}\alpha & ... \\ 0 & 0 & 0 & p^{-\alpha} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta_p^{(-\alpha)} = \begin{pmatrix} p^{-\alpha} & 0 & 0 & 0 & ... \\ p^{(-\alpha-1)}\alpha & p^{-\alpha} & 0 & 0 & ... \\ \frac{p^{(-\alpha-2)}\alpha(\alpha+1)}{2!} & p^{(-\alpha-1)}\alpha & p^{-\alpha} & 0 & ... \\ \frac{p^{(-\alpha-3)}\alpha(\alpha+1)(\alpha+2)}{3!} & \frac{p^{(-\alpha-2)}\alpha(\alpha+1)}{2!} & p^{(-\alpha-1)}\alpha & p^{-\alpha} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note. Without loss of generality, we assume throughout that the series defined in (1.1) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (1.1) reduces to a unite sum i.e.,

$$(\Delta^\alpha v_k) = \sum_{i=0}^{\alpha}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}.$$

At the same time, when we take some notations of Δ_p^α and $\Delta_p^{(\alpha)}$ privately, we see that we will obtain generalized private operators as follows:

i) If $\alpha = 1, p = 1$ then the operator Δ_p^α turns to Δ and $(\Delta v_k) = v_k - v_{k+1}$, described by Kızmaz [1].

ii) If $\alpha = m \in \mathbb{N}, p = 1$ then the operator Δ_p^α reduces to Δ^m and $\Delta^m v = (\Delta^m v_k) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} v_{k+j}$, described by Et and Çolak [2].

iii) If $\alpha = m \in \mathbb{N}, p \in \mathbb{N}$ then the operator Δ_p^β reduces to Δ_p^m and $\Delta_p^m v = (\Delta_p^m v_k) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} p^{m-j} v_{k+j}$, described by Karakaş et al. [4].

iv) If $\alpha = 1, p = 1$ then the operator $\Delta_p^{(\alpha)}$ reduces to $\Delta^{(1)}$ and $(\Delta v_k) = v_k - v_{k-1}$, described by Malkowsky and Parashar [22].

v) If $\alpha = m \in \mathbb{N}, p = 1$ then the operator $\Delta_p^{(\alpha)}$ reduces to $\Delta^{(m)}$ and $\Delta^{(m)} v = (\Delta^{(m)} v_k) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} v_{k-j}$, described by Et [23].

We organized this study as follows:

In the second part, we will define $\Delta_p^\alpha, \Delta_p^{(\alpha)}$ difference operators and examine some properties of this operators.

In the third chapter, the concept of β. order statistical convergence for Δ_p^α difference sequences will be defined and some topological properties of this convergence will be examined. In addition, β. order lacunary statistical concepts will be defined and the relationship of this convergence with β. order statistical convergence will be examined.In the last section, we define the relationships between the Caputo derivative and the Riemann-Liouville derivatives using the difference operator Δ_p^α. In the discussion section, we emphasized the importance of the study.

2. Main results

In this section, we will define the fractional difference operators $\Delta_p^{\alpha}, \Delta_p^{(\alpha)}$, $(\alpha \in \mathbb{R})$ by making use of the fractional difference operator, and we will give some properties of this operator.

Theorem 2.1. *The operators $E : w \to w$ for $E \in \left\{\Delta_p^{\alpha}, \Delta_p^{(\alpha)}, \Delta_p^{-\alpha}, \Delta_p^{(-\alpha)}\right\}$ are linear over $\mathbb{C}$.*

Proof. The proof of the theorem is easily illustrated by the technique used by Baliarsingh and Dutta in [21]. Therefore, we have omitted. □

Theorem 2.2. *We have*
i) $\Delta_p^{\alpha} \circ \Delta_p^{\gamma} \equiv \Delta_p^{\gamma} \circ \Delta_p^{\alpha} \equiv \Delta_p^{\alpha+\gamma}$.
ii) $\Delta_p^{(\alpha)} \circ \Delta_p^{(\gamma)} \equiv \Delta_p^{(\gamma)} \circ \Delta_p^{(\alpha)} \equiv \Delta_p^{(\alpha+\gamma)}$.

Proof. The proof can be seen obviously from Theorem 2.1. Therefore, we have omitted it. □

Theorem 2.3. *If α be a proper fraction, then*
i) $\Delta_p^{\alpha} \circ \Delta_p^{-\alpha} \equiv \Delta_p^{-\alpha} \circ \Delta_p^{\alpha} \equiv Id$.
ii) $\Delta_p^{(\alpha)} \circ \Delta_p^{(-\alpha)} \equiv \Delta_p^{(-\alpha)} \circ \Delta_p^{(\alpha)} \equiv Id$,
where Id is the identity operator in w.

Proof. *i)* Suppose $v \in w$ and for $\alpha = 1$, we have

$$\begin{aligned} (\Delta_p^{-1} \circ \Delta_p)v_k &= \Delta_p^{-1}(\Delta_p v_k) \\ &= \Delta_p^{-1}(pv_k - v_{k+1}...) \\ &= v_k \equiv Id. \end{aligned}$$

ii) The proof is done as in (*i*). □

Theorem 2.4. *Let α be a natural number and $v \in w$, then,*
i) $(\Delta_p^{\alpha} v_k) = (-1)^{\alpha}(\Delta_p^{(\alpha)} v_{k+\alpha})$,
ii) $(\Delta_p^{(\alpha)} v_k) = (-1)^{\alpha}(\Delta_p^{\alpha} v_{k-\alpha})$.

Proof. *i)* The induction method is used to prove the theorem. We have $\Delta v_k = v_k - v_{k+1} = (-1)(v_{k+1} - v_k) = \Delta^{(1)} v_{k+1}$ for $\alpha = 1, p = 1$ and $v \in w$.

The Basis step is now complete. Let us assume that the theorem is true for a natural number s, i.e.,

$$(\Delta^s v_k) = (-1)^s(\Delta^{(s)} v_{k+s}).$$

Now, we take

$$\begin{aligned} \Delta^{s+1} v_k &= \Delta(\Delta^s v_k) \\ &= \Delta((-1)^s \Delta^{(s)} v_{k+s}), \text{(by the assumption).} \\ &= (-1)^s \Delta^{(s)} v_{k+s} - (-1)^s \Delta^{(s)} v_{k+s+1,} \\ &= (-1)^{s+1}\left[\Delta^{(s)} v_{k+s+1} - \Delta^{(s)} v_{k+s}\right] \\ &= (-1)^{s+1} \Delta^{(s)} v_{k+s+1,} \text{(by Theorem 2.2).} \end{aligned}$$

This complement the proof.
ii) The proof is similar to (*i*). □

Theorem 2.5. *For α be a proper fraction and $v \in w$, we get*

$$(\Delta_p^{\alpha} + \Delta_p^{-\alpha})v_k = 2v_k + \sum_{i=1}^{\infty} p^{(\alpha-i)} \frac{(\alpha^+)_{i-1} + (-1)^i(\alpha^-)_{i-1}}{i!} v_{k+i},$$

where

$$(\alpha^+)_{i-1} = \alpha(\alpha+1)(\alpha+2)...(\alpha+i-1)$$

and

$$(\alpha^-)_{i-1} = \alpha(\alpha-1)(\alpha-2)...(\alpha-i+1).$$

Proof. The proof is straightforward from the definition, so we omit it. □

Let $v = (v_k)$ and $z = (z_k)$ be two sequences in w. We define the product of v and z as $vz = (v_k z_k)$. Now, the first forward and backward differences of vz are given by $\Delta_p(vz) = (pv_k z_k - v_{k+1}z_{k+1})$ and $\Delta_p^{(1)}(vz) = (pv_k z_k - v_{k-1}z_{k-1})$, respectively. The basic objective of this part is to find the -th difference of product sequence vz where is a positive integer. So, we state the following theorems.

The proof of the following Theorem is straightforward , so we choose to state these results without proof.

Theorem 2.6. *(Leibnitz theorem) Let α be a positive integer and $v, z \in w$,*

$$\Delta_p^{\alpha}(v_k z_k) = \sum_{i=0}^{\infty} \binom{\alpha}{i} p^{\alpha-i} \Delta_p^i v_k \Delta_p^{\alpha-i} z_{k+i}$$

in particular, if α is an integer, then

$$\begin{aligned} \Delta_p^{\alpha}(v_k z_k) &= p^{\alpha} v_k \Delta_p^{\alpha} z_k + \alpha p^{\alpha-1} \Delta_p v_k \Delta_p^{\alpha-1} z_{k+1} + \frac{\alpha(\alpha-1)}{2!} p^{\alpha-2} \Delta_p^2 v_k \Delta_p^{\alpha-2} z_{k+2} + \\ &\quad ... + \Delta_p^{\alpha} v_k z_{k+\alpha}. \end{aligned}$$

Using the above theorems, we get the following results.

Corollary 2.1. *i) If $(v_k) = (1, 1, 1, ...)$, then $\Delta_p^{\alpha} v_k = \Delta_p^{(\alpha)} v_k = \alpha(p-1)^{\alpha}$.*
ii) If $(v_k) = (1, 0, 1, 0, ...)$, then $\Delta_p^{\alpha} v_k = \Delta_p^{(\alpha)} v_k = \frac{(p-1)^{\alpha} + (-1)^k (p+1)^{\alpha}}{2}$.
iii) If $(v_k) = \left(\frac{1}{2^k}\right)$ then $\Delta_p^{\alpha} v_k = \Delta_p^{(\alpha)} v_k = \frac{(2p-1)^{\alpha}}{2^{\alpha+k}}$. In particular, $\Delta_p^{-1} v_k = \frac{p-2}{2^{k-1}}$.
iv) If $(v_k) = \left(z^k\right)$ for $|z| < 1$, then $\Delta_p^{\alpha} v_k = \Delta_p^{(\alpha)} v_k = z^k (p-z)^{\alpha}$ and $\Delta_p^{-\alpha} v_k = \Delta_p^{(\alpha)} v_k = \frac{z^k p^{k-1}}{(p-z)^{\alpha}}$.

3. Δ_p^{α}-statistical convergence

In this section, we also describe the concepts of ordered statistical convergence and lacunary statistical by using difference operator Δ_p^{α}. We examined some properties of these sequence spaces and gave some inclusion theorems.

Now we will define the concepts of ordered statistical convergence and lacunary statistical convergence with the help of the difference operator Δ_p^{α}.

Definition 3.1. *Let $v = (v_k) \in w, 0 < \beta \leq 1$ and α be a proper fraction. The sequence $v = (v_k)$ is said to be Δ_p^α-statistically convergent of order β if there is a real number ℓ such that,*

$$\lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha v_k - \ell \right| \geq \varepsilon \right\} \right| = 0$$

for every $\varepsilon > 0$. In this case we write $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k \to \ell$. The set of Δ_p^α-statistically convergent sequences of order β will be denoted by $S_t^\beta \left(\Delta_p^\alpha\right)$. In the case $\ell = 0$, we shall write $S_0^\beta \left(\Delta_p^\alpha\right)$.

The Δ_p^α-statistical convergence of order β is well defined for $0 < \beta \leq 1$, but it is not well defined for $\beta > 1$. For this let $v = (v_k)$ be defined as

$$v_k = \begin{cases} 1, & k = 2n \ (n = 1, 2, 3...) \\ 0, & k \neq 2n \text{ otherwise} \end{cases}.$$

Then we have for $\alpha = 1$

$$\Delta_p v_k = \begin{cases} p, & k = 2n \ (n = 1, 2, 3...) \\ 0, & k \neq 2n \text{ otherwise} \end{cases}.$$

Then both

$$\lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha v_k - p \right| \geq \varepsilon \right\} \right| \leq \lim_n \frac{n}{2n^\beta} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha v_k - 0 \right| \geq \varepsilon \right\} \right| \leq \lim_n \frac{n}{2n^\beta} = 0,$$

for $\beta > 1$, so that $v = (v_k)$ is Δ_p^α-statistically convergent of order β both to p and 0. However, this is not possible.

Theorem 3.1. *Let $\beta \in (0, 1]$, α be a proper fraction and $v = (v_k)$, $z = (z_k)$ be sequences of real sequences. Then,*

i) If $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_1$ and $c \in \mathbb{R}$, then $S_t^\beta - \lim_{k\to\infty} c\Delta_p^\alpha v_k = c\ell_1$.

ii) If $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_1$ and $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha z_k = \ell_2$, then $S_t^\beta - \lim_{k\to\infty} \left(\Delta_p^\alpha v_k + \Delta_p^\alpha z_k\right) = \ell_1 + \ell_2$.

Proof. *i*) In case $c = 0$ part of proof is trivial. To show $c \neq 0$.If $c \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha c v_k - c\ell_1 \right| \geq \varepsilon \right\} \right| \leq \lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha c v_k - \ell_1 \right| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

ii) Using the linear property of Δ_p^α operator difference, we get the following inequality:

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha (v_k + z_k) - (\ell_1 + \ell_2) \right| \geq \varepsilon \right\} \right| \\ \leq \ & \lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha v_k - \ell_1 \right| \geq \frac{\varepsilon}{2} \right\} \right| + \lim_{n\to\infty} \frac{1}{n^\beta} \left| \left\{ k \leq n : \left| \Delta_p^\alpha z_k - \ell_2 \right| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

□

Theorem 3.2. *Let $0 < \beta \leq 1$ and α be a proper fraction. If a sequence $v = (v_k)$ is Δ_p^α-statistically convergent of order β, then $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k$ is unique.*

Proof. Assume that $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_1$ and $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_2$. Given $\varepsilon \geq 0$, consider the following sets:

$$C_1(\varepsilon) = \left\{k \in \mathbb{N} : \left|\Delta_p^\alpha v_k - \ell_1\right| \geq \frac{\varepsilon}{2}\right\}$$

and

$$C_2(\varepsilon) = \left\{k \in \mathbb{N} : \left|\Delta_p^\alpha v_k - \ell_2\right| \geq \frac{\varepsilon}{2}\right\},$$

hence we obtain $\delta^\beta(C_1(\varepsilon)) = 0$ since $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_1$ and $\delta^\beta(C_2(\varepsilon)) = 0$ since $S_t^\beta - \lim_{k\to\infty} \Delta_p^\alpha v_k = \ell_2$. Now, let $C(\varepsilon) = C_1(\varepsilon) \cup C_2(\varepsilon)$. Thus, we get $\delta^\beta(C(\varepsilon)) = 0$ which implies $\mathbb{N}/\delta^\beta(C(\varepsilon)) = 0$. Now, if $\mathbb{N}/C(\varepsilon)$, then we get

$$\begin{aligned} |\ell_1 - \ell_2| &\leq \left|\ell_1 - \Delta_p^\alpha v_k\right| + \left|\Delta_p^\alpha v_k - \ell_2\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So, we get $|\ell_1 - \ell_2| = 0$, i.e. $\ell_1 = \ell_2$. □

Theorem 3.3. *Let $0 < \beta \leq \gamma \leq 1$ and α be a proper fraction. Then $S_t^\beta\left(\Delta_p^\alpha\right) \subseteq S_t^\gamma\left(\Delta_p^\alpha\right)$ and the inclusion is strict for at least those β and γ for which there is a $k \in \mathbb{N}$ such that $\beta < \frac{1}{k} < \gamma$.*

Proof. The inclusion part of proof is trivial. To show the inclusion $S_t^\beta\left(\Delta_p^\alpha\right) \subseteq S_t^\gamma\left(\Delta_p^\alpha\right)$ is strict choose $\alpha = 1$ and defined a sequence $v = (v_k)$ by

$$v_k = \begin{cases} p, & k = n^3 \ (n = 1, 2, 3...), \\ 0, & k \neq n^3 \text{ otherwise.} \end{cases}$$

Then we have

$$\Delta_p v_k = \begin{cases} p^2, & k = n^3 \ (n = 1, 2, 3...), \\ -p, & k + 1 = n^3, \\ 0, & \text{otherwise,} \end{cases}$$

and so

$$\lim_{n\to\infty} \frac{1}{n^\gamma} \left|\left\{k \leq n : \left|\Delta_p^\alpha v_k - 0\right| \geq \varepsilon\right\}\right| \leq \lim_n \frac{2\sqrt[3]{n}}{n^\gamma} = 0,$$

hence $S_t^\gamma - \lim_{k\to\infty} \Delta_p^\alpha v_k = 0$, i.e $v \in S_t^\gamma\left(\Delta_p^\alpha\right)$ for $\frac{1}{3} < \gamma \leq 1$, but $v \notin S_t^\beta\left(\Delta_p^\alpha\right)$ for $0 < \beta \leq \frac{1}{3}$ so that the inclusion $S_t^\beta\left(\Delta_p^\alpha\right) \subset S_t^\gamma\left(\Delta_p^\alpha\right)$ is strict. This holds for $\frac{1}{3} = \beta < \gamma < \frac{1}{2}$ for example, but there is no a $k \in \mathbb{N}$ such that $\beta < \frac{1}{k} < \gamma$. Therefore, the condition $\beta < \frac{1}{k} < \gamma$ is sufficient but not necessary for strictness of inclusion $S_t^\beta\left(\Delta_p^\alpha\right) \subset S_t^\gamma\left(\Delta_p^\alpha\right)$. □

Corollary 3.1. *Let α be a proper fraction. If a sequence is Δ_p^α-statistically convergent of order β to ℓ, for some $0 < \beta \leq 1$, then it is Δ_p^α-statistically convergent to ℓ, that is $S_t^\beta\left(\Delta_p^\alpha\right) \subseteq S_t\left(\Delta_p^\alpha\right)$ and inclusion is strict at least for $0 < \beta < \frac{1}{2}$.*

Definition 3.2. *Let $\Phi = (u_r)$ be a lacunary sequence, $0 < \beta \leq 1$ and α be a proper fraction. The sequence $v = (v_k)$ is said to be Δ_p^α-lacunary statistically convergent of order β of fractional order α to the number ℓ, if there is a real number ℓ such that*

$$\lim_{n\to\infty}\frac{1}{h_r^{\beta}}\left|\left\{k\in I_r:\left|\Delta_p^{\alpha}v_k-\ell\right|\geq\varepsilon\right\}\right|=0,$$

for every $\varepsilon>0$. In this case we write $S_\Phi^\beta - \lim_{k\to\infty}\Delta_p^\alpha v_k \to \ell$. The set of Δ_p^α-lacunary statistically convergent sequences of order β will be denoted by $S_\Phi^\beta\left(\Delta_p^\alpha\right)$.

Theorem 3.4. *Let $\Phi=(u_r)$ be a lacunary sequence, $v=(v_k), z=(z_k)\in w, 0<\beta\leq 1$ and α be a proper fraction, then*

i) If $S_\Phi^\beta-\lim_{k\to\infty}\Delta_p^\alpha v_k=\ell_1$ and $c\in\mathbb{R}$, then $S_\Phi^\beta-\lim_{k\to\infty}c\Delta_p^\alpha v_k=c\ell_1$.

ii) If $S_\Phi^\beta-\lim_{k\to\infty}\Delta_p^\alpha v_k=\ell_1$ and $S_\Phi^\beta-\lim_{k\to\infty}\Delta_p^\alpha z_k=\ell_2$, then $S_\Phi^\beta-\lim_{k\to\infty}\left(\Delta_p^\alpha v_k+\Delta_p^\alpha z_k\right)=\ell_1+\ell_2$.

Theorem 3.5. *Let $\Phi=(u_r)$ be a lacunary sequence, $0<\beta\leq 1$ and α be a proper fraction. If $\lim_{r\to\infty}\inf q_r>1$, then $S_t^\beta\left(\Delta_p^\alpha\right)\subset S_\Phi^\beta\left(\Delta_p^\alpha\right)$.*

Proof. Suppose that $\lim_{r\to\infty}\inf q_r>1$; then there exists $\delta>0$ such that $q_r>1+\delta$ for sufficiently large r; which implies that

$$\frac{h_r}{u_r}\geq\frac{\delta}{\delta+1}\Rightarrow\left(\frac{h_r}{u_r}\right)^{\beta}\geq\left(\frac{\delta}{\delta+1}\right)^{\beta}\Rightarrow\frac{1}{u_r^{\beta}}\geq\frac{\delta^{\beta}}{(\delta+1)^{\beta}}\geq\frac{1}{h_r^{\beta}}.$$

If $v_k\to\ell\left[S^\beta\left(\Delta_p^\alpha\right)\right]$, then for every $\varepsilon>0$ and for sufficiently large r, we have

$$\begin{aligned}&\frac{1}{u_r^{\beta}}\left|\left\{k\leq u_r:\left|\Delta_p^{\alpha}v_k-\ell\right|\geq\varepsilon\right\}\right|\\ \geq\;&\frac{1}{k_r^{\beta}}\left|\left\{k\in I_r:\left|\Delta_p^{\alpha}v_k-\ell\right|\geq\varepsilon\right\}\right|\\ \geq\;&\frac{\delta^{\beta}}{(\delta+1)^{\beta}}\frac{1}{h_r^{\beta}}.\left|\left\{k\in I_r:\left|\Delta_p^{\alpha}v_k-\ell\right|\geq\varepsilon\right\}\right|,\end{aligned}$$

so $v\in S_\Phi^\beta\left(\Delta_p^m\right)$. □

Theorem 3.6. *Let $\Phi=(u_r)$ be a lacunary sequence, $0<\beta\leq 1$ and α be a proper fraction. If $\lim_{r\to\infty}\sup q_r<\infty$, then $S_\Phi^\beta\left(\Delta_p^\alpha\right)\subset S_t^\beta\left(\Delta_p^\alpha\right)$.*

Proof. The proof of this theorem can be easily done using the similar work of Fridy Orhan [15]. □

From Theorems 3.5 and 3.6 we get the following result.

Corollary 3.2. *Let $\Phi=(u_r)$ be a lacunary sequence, $0<\beta\leq 1$ and α be a proper fraction. Then $S_\Phi^\beta\left(\Delta_p^\alpha\right)=S_t^\beta\left(\Delta_p^\alpha\right)$ if $1<\lim_{r\to\infty}\inf q_r<\lim_{r\to\infty}\sup q_r<\infty$.*

Definition 3.3. *Let $\Phi=(u_r)$ be a lacunary sequence, $0<\beta\leq 1$, α be a proper fraction and $q\in\mathbb{R}^+$. A sequence (v_k) is said to be strongly $N_{\Phi,q}^\beta(\Delta_p^\alpha)$-summable (or strongly $N_{\Phi,q}(\Delta_p^\alpha)$-summable of order β) if there is a real number ℓ such that*

$$\lim_{k\to\infty}\frac{1}{h_r^{\beta}}\sum_{k\in I_r}\left|\Delta_p^{\alpha}v_k-\ell\right|^{q}=0,$$

where $I_r = (u_{r-1}, u_r]$.In this case, we write $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}) - \lim v_k = \ell$. The set of all strongly $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p})$-summable sequences will be denoted by

$$N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}) = \left\{ v = (v_k) : \lim_{k\to\infty} \frac{1}{h^{\beta}_r} \sum_{k\in I_r} \left|\Delta^{\alpha}_{p} v_k - \ell\right|^q = 0, \textit{ for some } \ell \right\}.$$

Theorem 3.7. *$N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p})$ is a Banach space normed by*

$$\|v_k\|_{\Delta^{\alpha}_{p},\Phi} = \sum_{i=1}^{\infty} \left|v_i\right| + \sup_r \left(\frac{1}{h^{\beta}_r} \sum_{k\in I_r} \left|\Delta^{\alpha}_{p} v_k\right|^q \right)^{\frac{1}{q}}, 1 \le q < \infty. \tag{3.1}$$

Proof. The proof of the theorem can be done similarly to Theorem 2.4 in the study of Sengül and Et [17]. □

Theorem 3.8. *$N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p})$ is a BK-space normed by (3.1).*

Now we will give the relationship between lacunary statistical convergence and lacunary summability defined with the help of the Δ^{α}_{p} operator with the following theorem.

Theorem 3.9. *Let $\Phi = (k_r)$ be a lacunary sequence, $v = (v_k), z = (z_k) \in w, 0 < \beta \le 1$, α be a proper fraction and $q \in \mathbb{R}^+$, then*
i) If $v_k \to \ell \left[N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p})\right]$, then $v_k \to \ell \left[S^{\beta}_{\Phi}\left(\Delta^{\alpha}_{p}\right)\right]$ and the inclusion is strict,
ii) If $v_k \to \ell \left[l_\infty\left(\Delta^{\alpha}_{p}\right)\right]$ and $z_k \to \ell \left[S_\Phi\left(\Delta^{\alpha}_{p}\right)\right]$, then $v_k \to \ell \left[N_{\Phi,q}(\Delta^{\alpha}_{p})\right]$.

Proof. The inclusion part of the proof is easy. In order to establish "the inclusion is strict", let Φ be given, choose $\alpha = m, \beta = 1; q = 1$ and define a sequence $v = (v_k)$ by Δ^m_p to be $1, 2, ..., [h_r]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\Delta^m_p = 0$ otherwise

It is clear that v is not Δ^m_p bounded. Since

$$\frac{1}{h_r}\left|\left\{k \in I_r : \left|\Delta^{\alpha}_{p} v_k - \ell\right| \ge \varepsilon\right\}\right| = \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0, \text{ as } r \to \infty$$

and

$$\frac{1}{h_r} \sum_{k\in I_r} \left|\Delta^{\alpha}_{p} v_k - 0\right| = \frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right] + 1\right)}{2h_r} \to \frac{1}{2}, \text{ as } r \to \infty.$$

From (1.1) we have $v \in S_\Phi\left(\Delta^m_p\right)$, $v \notin N_\Phi(\Delta^m_p)$. □

We will give the following relations between the lacunary summability concept defined with the help of the Δ^{α}_{p} operator according to the modulus function and the lacunary statistical convergence.

Definition 3.4. *Let $\Phi = (u_r)$ be a lacunary sequence, $0 < \beta \le 1$, α be a proper fraction and $q = (q_k)$ be a sequence of strictly positive real numbers. A sequence (v_k) is said to be strongly $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi)$-summable (or strongly $N_{\Phi,q}(\Delta^{\alpha}_{p}, \phi)$- summable of order β) if there is a real number ℓ such that*

$$\lim_{k\to\infty} \frac{1}{h^{\beta}_r} \sum_{k\in I_r} \left[\phi\left(\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right)\right]^{q_k} = 0,$$

where $I_r = (u_{r-1}, u_r]$ and ϕ is a modulus function. In this case, we write $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi) - \lim v_k = \ell$. The set of all strongly $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi)$-summable sequences will be described as:

$$N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi) = \left\{ v = (v_k) : \lim_{k\to\infty} \frac{1}{h^{\beta}_{r}} \sum_{k\in I_r} \left[\phi\left(\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right)\right]^{q_k} = 0, \textit{ for some } \ell \right\}.$$

Theorem 3.10. *Let $\beta, \eta \in (0,1]$ be reel number such that $\beta \leq \eta$, ϕ be a modulus function and $\Phi = (u_r)$ be a lacunary sequence, then $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi) \subset S^{\eta}_{\Phi}\left(\Delta^{\alpha}_{p}\right)$.*

Proof. Let $v \in N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi)$, $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_r$, $\left|\Delta^{\alpha}_{p} v_k - \ell\right| \geq \varepsilon$ and $\left|\Delta^{\alpha}_{p} v_k - \ell\right| < \varepsilon$ respectively. As $h^{\beta}_{r} \leq h^{\eta}_{r}$ for each r, Then we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{h^{\beta}_{r}} \sum_{k\in I_r} \left[\phi\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right]^{q_k} &\geq \frac{1}{h^{\eta}_{r}} \left[\sum_{1} \left[\phi\left(\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right)\right]^{q_k} + \sum_{2} \left[\phi\left(\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right)\right]^{q_k} \right] \\
&\geq \frac{1}{h^{\eta}_{r}} \sum_{1} \left[\phi\left(\left|\Delta^{\alpha}_{p} v_k - \ell\right|\right)\right]^{q_k} \\
&\geq \frac{1}{h^{\eta}_{r}} \sum_{1} [\phi(\varepsilon)]^{q_k} \\
&\geq \frac{1}{h^{\eta}_{r}} \sum_{1} \min\left([\phi(\varepsilon)]^{g}, [\phi(\varepsilon)]^{G}\right) \\
&= \frac{1}{h^{\eta}_{r}} \left|\left\{k \in I_r : \left|\Delta^{\alpha}_{p} v_k - \ell\right| \geq \varepsilon\right\}\right| \times \min\left([\phi(\varepsilon)]^{g}, [\phi(\varepsilon)]^{G}\right).
\end{aligned}$$

Hence $v \in S^{\eta}_{\Phi}\left(\Delta^{\alpha}_{p}\right)$. Where, $q = (q_k)$ is bounded and $0 < g = \inf_k q_k \leq q_k \leq \sup_k q_k = G < \infty$. □

Corollary 3.3. *Let $\beta \in (0,1]$ be reel number such that $\beta \leq \eta$, ϕ be a modulus function and $\Phi = (u_r)$ be a lacunary sequence, then $N^{\beta}_{\Phi,q}(\Delta^{\alpha}_{p}, \phi) \subset S^{\beta}_{\Phi}\left(\Delta^{\alpha}_{p}\right)$.*

4. Relation with Caputo and Riemann-Liouville derivative

Using $(\Delta^{\alpha}_{p} v_k) = \sum_{i=0}^{\infty} (-1)^i p^{(\alpha-i)} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}$ delta difference operator we defined in this section, we defined the relationships between Caputo derivative and Riemann-Liouville derivatives. We have the following definitons

i) Caputo Derivative: For $f(p) = p^{\alpha}$,

$$\begin{aligned}
{}^{c}_{0}D^{i}_{p} f(p) &= \int_{0}^{p} (p-\tau)^{-i} f'(\tau) d\tau \\
&= \frac{1}{\Gamma(1-i)} \int_{0}^{p} (p-\tau)^{-i} \alpha\tau^{\alpha-1} d\tau
\end{aligned}$$

$$= \frac{\alpha}{\Gamma(1-i)}\int_0^p (p-\tau)^{-i}\tau^{\alpha-1}d\tau$$

$$= \frac{\frac{\alpha}{\Gamma(1-i)}p^{\alpha-i}\Gamma(1-i)\Gamma(\alpha)}{\Gamma(\alpha-i+1)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}p^{\alpha-i},$$

$${}_0^cD_p^i f(p) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}p^{\alpha-i}.$$

ii) Riemann-Liouville Derivative: For $f(p) = p^\alpha$

$$\begin{aligned}
{}_0^{RL}D_p^i f(p) &= \frac{1}{\Gamma(1-i)}\frac{d}{dp}\int_0^p (p-\tau)^{-i} f(\tau)d\tau \\
&= \frac{1}{\Gamma(1-i)}\frac{d}{dp}\int_0^p (p-\tau)^{-i}\tau^{\alpha}d\tau \\
&= \frac{1}{\Gamma(1-i)}\frac{d}{dp}\left[\frac{p^{1-i+\alpha}\Gamma(1-i)\Gamma(1+\alpha)}{\Gamma(2-i+\alpha)}\right] \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(2-i+\alpha)}(1-i+\alpha)p^{\alpha-i} \\
&= \frac{\Gamma(1+\alpha)}{(1-i+\alpha)\Gamma(1-i+\alpha)}(1-i+\alpha)p^{\alpha-i} \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1-i+\alpha)}p^{\alpha-i},
\end{aligned}$$

$${}_0^{RL}D_p^i f(p) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}p^{\alpha-i},$$

(see [24]).

Then,

$$\begin{aligned}
(\Delta_p^\alpha v_k) &= \sum_{i=0}^{\infty}(-1)^i p^{\alpha-i}\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}v_{k+i} \\
&= \sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(i+1)}\left({}_0^cD_p^i f(p)\right)v_{k+i} \\
&= \sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(i+1)}\left({}_0^{RL}D_p^i f(p)\right)v_{k+i},
\end{aligned}$$

we obtained the Caputo derivative in this work. This is new in the literature. We will try to get other fractional derivatives in the future works.

5. Conclusions

Recently, sequence spaces have been applied to different disciplines. For example, sequence spaces have been adapted to circuit and system analysis by Leake [25]. The difference operator was used by Kawamura et al. [26] in earthquake prediction. Convergence plays an important role in convex programming, mathematical modeling, and numerical analysis problems (see for details [27]). In addition, the concept of statistical convergence is one of the most studied subjects in recent years. Statistical convergence is related to probability theory in statistics, and this relationship has been demonstrated by many mathematicians. This convergence was used, especially in approximation theory (see [28]). In this study, we examined some inclusion theorems by defining a new difference operator and obtaining new statistical convergent and lacunary statistical convergent sequence spaces. The obtained results are important for the summability theory in classical analysis. Researchers working in this field can create new studies by taking advantage of this study.

Conflict of interest

The author declares that they have no competing interests.

References

1. H. Kızmaz, On certain sequence spaces, *Can. Math. Bull.*, **24** (1981), 169–176. https://doi.org/10.4153/CMB-1981-027-5
2. M. Et, R. Çolak, On some generalized difference sequence spaces, *Soochow Journal of Mathematics*, **21** (1995), 377–386.
3. I. Tincu, On some p-convex sequences, *Acta Universitatis Apulensis*, **11** (2006), 249–257.
4. A. Karakaş, Y. Altin, M. Et, Δ_p^m-statistical convergence of orderα, *Filomat*, **32** (2018), 5565–5572. https://doi.org/10.2298/FIL1816565K
5. A. Zygmund, *Trigonometric series*, Cambridge: Cambridge University Press, 1979. https://doi.org/10.1017/CBO9781316036587
6. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, **2** (1951), 73–74.
7. H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.
8. N. Aral, M. Et, Generalized difference sequence spaces of fractional order defined by Orlicz functions, *Commun. Fac. Sci. Univ.*, **69** (2020), 941–951. https://doi.org/10.31801/cfsuasmas.628863
9. M. Et, F. Nuray, Δ^m-statistical convergence, *Indian J. Pure Appl. Math.*, **32** (2001), 961–969.
10. J. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301–314. https://doi.org/10.1524/anly.1985.5.4.301
11. A. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.*, **32** (2002), 129–138. https://doi.org/10.1216/rmjm/1030539612
12. R. Çolak, Statistical convergence of order α, In: *Modern methods in analysis and its applications*, New Delhi: Anamaya Pub., 2010, 129.

13. A. Karakaş, Δ_p^m-lacunary statistical convergence of order β, *Proceedings of 8th International Conference on Recent Advances in Pure and Applied Mathematics*, 2021, 96.
14. A. Freedman, J. Sember, M. Raphael, Some Cesàro-type summability spaces, *P. Lond. Math. Soc.*, **37** (1978), 508–520. https://doi.org/10.1112/plms/s3-37.3.508
15. J. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific J. Math.*, **160** (1993), 43–51.
16. J. Fridy, C. Orhan, Lacunary statistical summability, *J. Math. Anal. Appl.*, **173** (1993), 497–504. https://doi.org/10.1006/jmaa.1993.1082
17. H. Sengül, M. Et, On lacunary statistical convergence of order α, *Acta Math. Sci.*, **34** (2014), 473–482. https://doi.org/10.1016/S0252-9602(14)60021-7
18. H. Nakano, Concave modulars, *J. Math. Soc. Japan*, **5** (1953), 29–49. https://doi.org/10.2969/jmsj/00510029
19. P. Baliarsingh, Some new difference sequence spaces of fractional order and their dual spaces, *Appl. Math. Comput.*, **219** (2013), 9737–9742. https://doi.org/10.1016/j.amc.2013.03.073
20. P. Baliarsingh, On a fractional difference operator, *Alex. Eng. J.*, **55** (2016), 1811–1816. https://doi.org/10.1016/j.aej.2016.03.037
21. P. Baliarsingh, S. Dutta, On the classes of fractional order difference sequence spaces and their matrix transformations, *Appl. Math. Comput.*, **250** (2015), 665–674. https://doi.org/10.1016/j.amc.2014.10.121
22. E. Malkowsky, S. Parashar, Matrix transformations in spaces of bounded and convergent difference sequences of order m, *Analysis*, **17** (1997), 87–98. https://doi.org/10.1524/anly.1997.17.1.87
23. M. Et, On some topological properties of generalized difference sequence spaces, *International Journal of Mathematics and Mathematical Sciences*, **24** (2000), 716581. https://doi.org/10.1155/S0161171200002325
24. D. Baleanu, A. Fernandez, A. Akgül, On a fractional operator combining proportional and classical differintegrals, *Mathematics*, **8** (2020), 360. https://doi.org/10.3390/math8030360
25. R. Leake, Monotone resolution sequence spaces and mappings, *IEEE T. Circuits*, **27** (1980), 800–804. https://doi.org/10.1109/TCS.1980.1084892
26. H. Kawamura, A. Tani, M. Yamada, K. Tsunoda, Real time prediction of earthquake ground motions and structural responses by statistic and fuzzy logic, *Proceedings of First International Symposium on Uncertainty Modeling and Analysis*, 1990, 534–538. https://doi.org/10.1109/ISUMA.1990.151311
27. S. Regmi, *Optimized iterative methods with applications in diverse disciplines*, New York: Nova Science Publisher, 2021.
28. G. Anastassiou, O. Duman, *Towards intelligent modeling: statistical approximation theory*, Berlin: Springer, 2011. http://dx.doi.org/10.1007/978-3-642-19826-7