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Research article

Hilbert series of mixed braid monoid $MB_{2,2}$

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Abstract: Hilbert series is a simplest way to calculate the dimension and the degree of an algebraic variety by an explicit polynomial equation. The mixed braid group $B_{m,n}$ is a subgroup of the Artin braid group B_{m+n}. In this paper we find the ambiguity-free presentation and the Hilbert series of canonical words of mixed braid monoid $MB_{2,2}$.

Keywords: braid group; mixed braid monoid; canonical words; complete presentation; Hilbert series
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1. Introduction

Hilbert series helps us in expressing the growth of dimension of homogenous components of a graded algebra. Presently, these ideas are extended to graded algebras, filtered algebras, and graded or filtered modules over these algebras, monoid in commutative algebra and coherent sheaves over projective schemes in algebraic geometry. The Hilbert series is a particular case of the Hilbert-Poincaré series of a graded vector space. It is also of a great significance in computational algebraic geometry due to the easiest way for computing the dimension and the degree of an algebraic variety defined by explicit polynomial equations. The Hilbert series is also helpful in providing us the important invariants of the algebraic varieties. This particular article focuses on the presentation and Hilbert series of the mixed braid monoid $MB_{2,2}$.

The classical braid group B_{m+1}, given by Artin [3], possesses the following presentation:

$$B_{m+1} = \left\langle a_1, \ldots, a_m \;\middle|\; \begin{array}{l} a_i a_j = a_j a_i \text{ if } |i-j| \geq 2 \\ a_{i+1}\, a_i\, a_{i+1} = a_i\, a_{i+1}\, a_i \text{ if } 1 \leq i \leq m-1 \end{array} \right\rangle.$$

The braid monoid MB_{m+1} has the same presentation as the braid group B_{m+1}. The braid group B_{m+1} admits another presentation known as the band presentation given by Birman et al. in [5].

Definition 1. *[7] Let U be a set. A square matrix $G = (g_{uv})_{u,v \in U}$ is called Coxeter matrix over U such that $m_{uu} = 1$ and $m_{uv} = m_{vu} \in \{2, 3, ..., \infty\}$ for all $u, v \in U, u \neq v$.*

Definition 2. *[7] A Coxeter graph Γ is a labeled graph. It can be defined as*

i. A set of vertices of Γ is denoted by U.

ii. Two vertices $u, v \in U, u \neq v$ are connected by an edge if $m_{uv} \geq 3$. The labeling of edge is m_{uv} if $m_{uv} \geq 4$.

Definition 3. *[7] Let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of the Coxeter graph Γ. Then the group defined by*

$$W = \langle s \in S \mid s^2 = 1\,, (st)^{m_{st}} = 1 \ \text{for all} \ s, t \in S,\ s \neq t,\ m_{st} \neq \infty \rangle$$

is called Coxeter group (of type Γ).

Definition 4. *[12] In mixed braid group $B_{m,n}$ the first index m denotes trivial strings in the braid group and the next n strings shows the braiding by itself and with the m strings. The mixed braid group $B_{m,n}$ generated by $m + n$ strands is defined by*

$$B_{m,n} = \left\langle \begin{array}{c} \alpha_1, \ldots, \alpha_m, \\ \sigma_1, \ldots, \sigma_{n-1} \end{array} \left| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{if} \mid i - j \mid \geq 2 \\ \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i \ \text{if}\ 1 \leq i \leq n-1 \\ \alpha_i\sigma_j = \sigma_j\alpha_i \ \text{if}\ j \geq 2 \ \text{and}\ 1 \leq i \leq m \\ \alpha_i\sigma_1\alpha_i\sigma_1 = \sigma_1\alpha_i\sigma_1\alpha_i \ \text{if}\ 1 \leq i \leq m \\ \alpha_l\,(\sigma_1\,\alpha_m\,\sigma_1^{-1}) = (\sigma_1\,\alpha_m\,\sigma_1^{-1})\,\alpha_l \ \text{if}\ \ l < m \end{array} \right. \right\rangle.$$

If we remove the last n strands of Artin group B_{m+n} then we have only identity braid with m strands. The collection of all such elements of Artin group B_{m+n} will be denoted by $B_{m,n}$.

Definition 5. *[12] The mixed braid monoid $MB_{m,n}$ has the following presentation:*

$$MB_{m,n} = \left\langle \begin{array}{c} \alpha_1, \ldots, \alpha_m, \\ \sigma_1, \ldots, \sigma_{n-1} \end{array} \left| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i\,, \mid i - j \mid \geq 2 \\ \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i \ \text{if}\ 1 \leq i \leq n-1 \\ \alpha_i\sigma_j = \sigma_j\alpha_i\,,\ j \geq 2 \ \text{and}\ 1 \leq i \leq m \\ \alpha_i\sigma_1\alpha_i\sigma_1 = \sigma_1\alpha_i\sigma_1\alpha_i\,,\ 1 \leq i \leq m \end{array} \right. \right\rangle.$$

In the below diagram of mixed braid monoid $MB_{m,n}$, the single bonds shows relation of degree 3, the double bonds shows relation of degree 4 and two generators commute if they are not joined by a bond (see Figure 1).

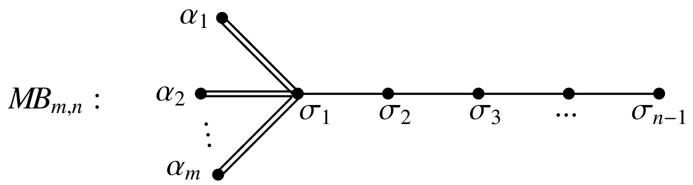

Figure 1. Coxeter graph of $MB_{m,m}$.

Conveniently α_i and σ_j are denoted by a single generator a_k, $1 \leq k \leq i+j$. Note that the mixed braid monoid $MB_{1,2}$, which is isomorphic to the Artin monoid of type B_2, is presented by

$$MB_{1,2} = \left\langle a_1, a_2 \,\middle|\, a_2\, a_1\, a_2\, a_1 = a_1\, a_2\, a_1\, a_2 \right\rangle.$$

The Coxeter diagram for $MB_{1,2}$ takes the form in Figure 2.

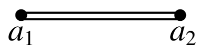

Figure 2. Coxeter graph of $MB_{1,2}$.

The complete presentation and Hilbert series for $MB_{1,2}$ are computed in [13]. This motivated us to compute the Hilbert series of $MB_{2,2}$.

Definition 6. *The classical presentation of $MB_{2,2}$ is given by*

$$MB_{2,2} = \left\langle a_1, a_2, a_3 \,\middle|\, a_2a_1a_2a_1 = a_1a_2a_1a_2,\ a_3a_2a_3a_2 = a_2a_3a_2a_3,\ a_3a_1 = a_1a_3 \right\rangle.$$

Let us denote the relations by

$$R_0 : a_3a_1 = a_1a_3, R_1 : a_2a_1a_2a_1 = a_1a_2a_1a_2$$

and

$$R_2 : a_3a_2a_3a_2 = a_2a_3a_2a_3.$$

The Coxeter diagram of $MB_{2,2}$ is given as Figure 3.

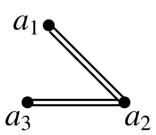

Figure 3. Coxeter graph of $MB_{2,2}$.

An example of a braid in $B_{2,2}$ (see Figure 4).

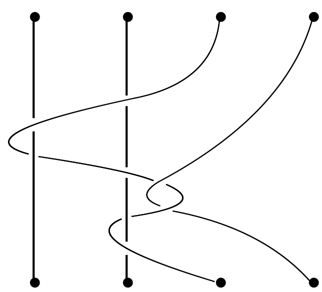

Figure 4. braid in $B_{2,2}$.

In 2003, Bokut [6] gave the non-commutative complete presentation of the braid monoid MB_{n+1} with the length-lexicographic order induced by $a_1 < \cdots < a_n$. In [10] we computed the Hilbert series of braid monoid MB_4 in band generators. In 1972, P. Deligne [8] proved that the Hilbert series of all the Artin monoids are rational functions. In [14] Saito computed the growth series of Artin monoids. In [13] we computed the Hilbert series of the Artin monoids $M(I_2(p))$, where $M(I_2(4))$ is isomorphic to $MB_{1,2}$. In [7] we showed that the growth rate of all the Artin monoids is less than 4. In [11] we proved that the growth rates are exponential and the growth rates for MB_3 and MB_4 are approximately 1.618 and 2.0868. In this paper we construct a linear system for canonical words and then we find Hilbert series and the growth rate of the mixed braid monoid $MB_{2,2}$. It is proved in [1] that the Hilbert series of the free G-graded F-algebra is a rational function. In [2] the authors proved that the Hilbert series of relatively free algebras is a rational function.

2. Complete presentation of $MB_{2,2}$

To get a canonical form of a word in an algebra the diamond lemma by G. Bergman [4] is very useful. To understand the concepts of ambiguities and canonical words, we start with his terminology.

Definition 7. *Let X be a nonempty set and X^* be the free monoid on X. Let w_1 and $w_2 \in X^*$, where $w_1 = x_1x_2\cdots x_r$, $w_2 = y_1y_2\cdots y_r$ with $x_i, y_i \in X$. Then $w_1 < w_2$ length-lexicographically if there is a $k \leq r$ such that $x_k < y_k$ and $x_i = y_i$ for all $i < k$.*

Definition 8. *Let $\mu = \nu$ be a relation in a monoid M and $\mu_1 = sw$ and $\mu_2 = wt$ be two words in M. Then the word of the form $\mu_1 \times_w \mu_2 = swt$ is said to be an ambiguity.*

If $\mu_1 t = s\mu_2$ (in the length-lexicographic order) then we say that the ambiguity swt is solvable. A presentation of M is said to be *complete* if and only if all the ambiguities are solvable. Corresponding to the relation $\mu = \nu$, the changes $\alpha\mu\beta \longrightarrow \alpha\nu\beta$ give a rewriting system. A complete presentation is equivalent to a confluent rewriting system. In a complete presentation of a monoid a word containing μ will be called *reducible word* and a word that does not contain μ will be called an *irreducible word* or *canonical word*. For example $a_3a_2a_3 = a_2a_3a_2$ is a basic relation in the braid monoid MB_4. A word $v = a_3^2a_2a_3$ contains $\alpha = a_3a_2a_3$. Hence v is a reducible word. Then $a_3^2a_2a_3 = a_2a_3a_2^2$ and $a_2a_3a_2^2$ is the canonical form of v. In a presentation of a monoid we fix a total order $a_1 < a_2 < \cdots < a_n$ on the generators.

Definition 9. *[9] Let G be a finitely generated group and A be a finite set of generators of G. The word length $l_A(g)$ of an element $g \in G$ is the smallest integer n for which there exist $a_1, \ldots, a_n \in A \cup A^{-1}$ such that $g = a_1 \cdots a_n$.*

Definition 10. *[9] The Hilbert function of a monoid M is given as $H(M, n) = a_n$, i.e., the number of elements of M of word length n. The Hilbert series of the monoid MB_n for arbitrary variable t is denoted by $H_M(t)$ and is defined by $H_M(t) = \sum_{n=0}^{\infty} a_n t^n$.*

Definition 11. *For a sequence of positive numbers $\{b_l\}_{l\geq 1}$ the rate of growth $r \in \mathbb{R}$ is given by*

$$\overline{\lim_{l}} \exp\Big(\frac{\log b_l}{l}\Big) = r.$$

Lemma 1. *[13] The following equations hold for the canonical words in $MI_2(4)$.*

(1) $P_1^{(4)} = \frac{t}{(1-t)} + \frac{t}{(1-t)}P_2^{(4)}$

(2) $P_2^{(4)} = t + tP_2^{(4)} + P_{21}^{(4)}$

(3) $P_{21}^{(4)} = tP_1^{(4)} - t^2P_{21}^{(4)} - \frac{t^3}{1-t}P_{212}^{(4)}$

(4) $P_{212}^{(4)} = t^2P_2^{(4)} - t^2P_{21}^{(4)}$.

The solution of the above equations is given in [13] which is as follows:

$$P_1^{(3)} = \frac{t}{(1-t)(1-t-t^2-t^3)},\ P_2^{(3)} = \frac{t(1+t+t^2)}{1-t-t^2-t^3},\ P_{21}^{(3)} = \frac{t^2(1+t)}{1-t-t^2-t^3},\ P_{212}^{(3)} = \frac{t^3}{1-t-t^2-t^3}.$$

Lemma 2. *[13] The Hilbert series of $MI_2(4)$ is given by*

$$P_M^{(3)}(t) \quad = \quad \frac{1}{(1-t)(1-2t)}.$$

While solving all the ambiguities we now give a complete presentation of $MB_{2,2}$.

Theorem 1. *The braid monoid $MB_{2,2}$ has a complete presentation*

$$MB_{2,2} = \left\langle a_1, a_2, a_3 \;\middle|\; R_0, R_1, \ldots, R_6 \right\rangle,$$

where

$$\begin{aligned}
R_0^{(4)}&:\ a_3a_1 = a_1a_3,\\
R_1^{(4)}&:\ a_2a_1a_2a_1 = a_1a_2a_1a_2,\\
R_2^{(4)}&:\ a_2a_1^{n+1}a_2a_1a_2 = a_1a_2a_1a_2^2a_1^n,\\
R_3^{(4)}&:\ a_3a_2a_3a_2 = a_2a_3a_2a_3,\\
R_4^{(4)}&:\ a_3a_2^{n+1}a_3a_2a_3 = a_2a_3a_2a_3^2a_2^n,\\
R_5^{(4)}&:\ a_3a_2a_1^na_3a_2a_1a_2 = a_2a_3a_2a_1a_3a_2a_1^n,\\
R_6^{(4)}&:\ a_3a_2^{n+1}a_3a_2a_1^ma_3 = a_2a_3a_2a_3^2a_2^na_1^m,
\end{aligned}$$

where $n, m \geq 0$.

Proof. We denote the ambiguity formed by left sides of the relations R_i and R_j in $MB_{2,2}$ by $R_i - R_j = swt$ (say). If in the ambiguity $z = swt$, $L(z) = (sw)t$ and $R(z) = s(wt)$ are different lexicographically, then we get a new relation in $MB_{2,2}$ and if $L(z) = (sw)t$ and $R(z) = s(wt)$ are reduced to an identical word, then we say that ambiguity is solvable and no relation is formed. Here $L(z)$ denotes the canonical form of $(sw)t$ and $R(z)$ denotes the conical form of $s(wt)$. The above relations are obtained by solving the ambiguities involving the relations $R_0^{(4)}$, $R_1^{(4)}$ and $R_2^{(4)}$ and the new relations. In [13] we computed the relation (for $p = 4$) $R_3^{(4)}$, which is given by

$$R_3^{(4)}:\ a_2a_1^{n+1}a_2a_1a_2 = a_1a_2a_1a_2^2a_1^n.$$

For an ambiguity $R_2^{(4)} - R_2^{(4)} = a_3a_2a_3a_2a_3a_2 = w_1(say)$, we have

$$R(w_1) = a_3a_2\underline{a_3a_2a_3a_2} = a_3a_2^2a_3a_2a_3,\ L(w_1) = \underline{a_3a_2a_3a_2}a_3a_2 = a_2a_3a_2a_3^2a_2.$$

Hence we have a relation $R_{w_1}: a_3a_2^2a_3a_2a_3 = a_2a_3a_2a_3^2a_2$. Again by solving a new ambiguity $R_{w_1} - R_2^{(4)} = a_3a_2^2a_3a_2a_3a_2 = w_2$ we have

$$R(w_2) = a_3a_2^2\underline{a_3a_2a_3a_2} = a_3a_2^3a_3a_2a_3,\ L(w_2) = \underline{a_3a_2^2a_3a_2a_3}a_2 = a_2a_3a_2a_3^2a_2^2.$$

which gives another relation $R_{w_2}: a_3a_2^3a_3a_2a_3 = a_2a_3a_2a_3^2a_2^2$. By continuing the same process we have the general relation

$$R_4^{(4)}:\ a_3a_2^{n+1}a_3a_2a_3 = a_2a_3a_2a_3^2a_2^n,\quad n \geq 1.$$

In the ambiguity $R_2^{(4)} - R_1^{(4)} = a_3a_2a_3a_2a_1a_2a_1 = w_3$, we have

$$R(w_3) = a_3a_2a_3\underline{a_2a_1a_2a_1} = a_3a_2a_1a_3a_2a_1a_2,\ L(w_3) = \underline{a_3a_2a_3a_2}a_1a_2a_1 = a_2a_3a_2a_1a_3a_2a_1.$$

Hence we have a relation $R_{w_3}: a_3a_2a_1a_3a_2a_1a_2 = a_2a_3a_2a_1a_3a_2a_1$. Again by solving a new ambiguity $R_{w_3} - R_1^{(4)} = a_3a_2a_1a_3a_2a_1a_2a_1 = w_4$ we have

$$R(w_4) = a_3a_2a_1a_3\underline{a_2a_1a_2a_1} = a_3a_2a_1^2a_3a_2a_1a_2,$$

$$L(w_4) = \underline{a_3a_2a_1a_3a_2a_1a_2}a_1 = a_2a_3a_2a_1a_3a_2a_1^2,$$

which gives another relation $R_{w_4}: a_3a_2a_1^2a_3a_2a_1a_2 = a_2a_3a_2a_1a_3a_2a_1^2$. By continuing the same process we have the general relation

$$R_5^{(4)}:\ a_3a_2a_1^na_3a_2a_1a_2 = a_2a_3a_2a_1a_3a_2a_1^n,\ \ n \geq 1.$$

In the ambiguity $R_4^{(4)} - R_0^{(4)} = a_3a_2^{n+1}a_3a_2a_3a_1 = w_5$, we have

$$R(w_5) = a_3a_2^{n+1}a_3a_2\underline{a_3a_1} = a_3a_2^{n+1}a_3a_2a_1a_3,\ L(w_5) = \underline{a_3a_2^{n+1}a_3a_2a_3}a_1 = a_2a_3a_2a_3^2a_2^na_1.$$

Hence we have a relation $R_{w_5}: a_3a_2^{n+1}a_3a_2a_1a_3 = a_2a_3a_2a_3^2a_2^na_1$. Again by solving a new ambiguity $R_{w_5} - R_0^{(4)} = a_3a_2^{n+1}a_3a_2a_3a_1a_3a_1 = w_6$ we have

$$R(w_6) = a_3a_2^{n+1}a_3a_2a_3a_1\underline{a_3a_1} = a_3a_2^{n+1}a_3a_2a_1^2a_3,$$

$$L(w_6) = \underline{a_3a_2^{n+1}a_3a_2a_3a_1a_3}a_1 = a_2a_3a_2a_3^2a_2^na_1^2,$$

which gives another relation $R_{w_6}: a_3a_2^{n+1}a_3a_2a_1^2a_3 = a_2a_3a_2a_3^2a_2^na_1^2$. By continuing the same process we have the general relation

$$R_6^{(4)}:\ a_3a_2^{n+1}a_3a_2a_1^ma_3 = a_2a_3a_2a_3^2a_2^na_1^m,\ \ n, m \geq 1.$$

All other ambiguities all solvable. Hence we have the complete set of relations.

3. Hilbert series of $MB_{2,2}$

For our convenience we use these notations for canonical and reducible words in $MB_{2,2}$. In general $B_*^{(4)}$ denotes the set of reducible words and $A_*^{(4)}$ denotes the set of canonical words in $MB_{2,2}$. Particularly, $B_i^{(4)}$ denotes the reducible word of $R_i^{(4)}$ in $MB_{2,2}$ and $A_{j(j-1)\dots k}^{(n+m)}$ denotes collection of all canonical words in $MB_{m,n}$ starting with $a_j a_{j-1}\dots a_k$. These sets are graded by length-lexicographic order, so we can compute the Hilbert series of these sets. The words $Xa_3 \times_3 a_3 Y$ and $Xa_3a_2 \times_{32} a_3a_2Y$ are equivalent to products Xa_3Y and Xa_3a_2Y, respectively. Let $Q_*^{(4)}$ denote the Hilbert series for $B_*^{(4)}$ and $P_*^{(4)}$ denote the Hilbert series for $A_*^{(4)}$. Let $H_M^{(m+n)}(t)$ denote the Hilbert series for the set

$$A^{(m+n)} = e \cup A_1^{(m+n)} \cup A_2^{(m+n)} \cup \cdots \cup A_k^{(m+n)}$$

(for some $k \in \mathbb{Z}$) for a monoid M. Then we have

$$H_M^{(m+n)}(t) = 1 + P_1^{(m+n)} + P_2^{(m+n)} + \cdots + P_k^{(m+n)}.$$

Theorem 2. *The following relations hold for the Hilbert series of reducible words in $MB_{2,2}$.*

$$a.Q_1^{(4)} = t^4,\ b.Q_2^{(4)} = \frac{t^6}{1-t},\ c.Q_3^{(4)} = t^4,\ d.Q_4^{(4)} = \frac{t^6}{1-t},\ e.Q_5^{(4)} = \frac{t^7}{1-t},\ f.Q_6^{(4)} = \frac{t^7}{(1-t)^2}.$$

Proof. Let $\alpha(i,\dots,j)$ be a word in $a_i, a_{i+1},\dots,a_j$ in $MB_{2,2}$, then $\Sigma\alpha(i,\dots,j)$ be a word in $a_{i+1}, a_{i+2},\dots,a_{j+1}$ in $MB_{2,2}$. $B_*^{(4)}$ denotes the reducible words in $MB_{2,2}$ corresponding to relation $R_*^{(4)}$. $Q_*^{(4)}$ denotes the Hilbert series of $B_*^{(4)}$. Hilbert series of $A_1^{(2)} = \{a_1, a_1^2,\dots\}$ is $P_1^{(2)} = t + t^2 + \cdots = \frac{t}{1-t}$. Therefore we have

a. $B_1^{(4)} = \{a_2a_1a_2a_1\}$. This implies $Q_1^{(4)} = t^4$.
b. Since $B_2^{(4)} = \{a_2a_1^{n+1}a_2a_1a_2\} = \{a_2a_1\} \times A_1^{(2)} \times \{a_2a_1a_2\}$. Hence $Q_2^{(4)} = \frac{t^6}{1-t}$.
c. Similarly $B_3^{(4)} = \{a_3a_2a_3a_2\}$ implies $Q_3^{(4)} = t^4$.
d. The decomposition $B_4^{(4)} = \{a_3a_2^{n+1}a_2a_2a_3\} = \{a_3a_2\} \times \Sigma A_1^{(2)} \times \{a_3a_2a_3\}$ gives $Q_4^{(4)} = \frac{t^6}{1-t}$.
e. $B_5^{(4)} = \{a_3a_2a_1^m a_3a_2a_1a_2\} = \{a_3a_2\} \times A_1^{(2)} \times \{a_3a_2a_1a_2\}$ gives $Q_5^{(4)} = \frac{t^7}{1-t}$.
f. $B_6^{(4)} = \{a_3a_2^{n+1}a_2a_2a_1^m a_3\} = \{a_3a_2\} \times \Sigma A_1^{(2)} \times \{a_3a_2\} \times A_1^{(2)} \times a_3$ gives $Q_6^{(4)} = \frac{t^7}{(1-t)^2}$.

Next we construct a linear system for canonical forms in $MB_{2,2}$.

Theorem 3. *The following relations hold for the Hilbert series of canonical words in $MB_{2,2}$.*

(1) $P_1^{(4)} = P_1^{(3)} + P_1^{(3)}P_3^{(4)}$.
(2) $P_2^{(4)} = P_2^{(3)} + P_2^{(3)}P_3^{(4)}$.
(3) $P_3^{(4)} = t + tP_3^{(4)} + P_{32}^{(4)}$.
(4) $P_{21}^{(4)} = P_{21}^{(3)} + P_{21}^{(3)}P_3^{(4)}$.
(5) $P_{32}^{(4)} = tP_2^{(4)} - t^2P_{32}^{(4)} - \frac{t^3}{1-t}P_{323}^{(4)} - \frac{t^3}{1-t}P_{3212}^{(4)} - \frac{t^6}{(1-t)^2}P_3^{(4)}$.
(6) $P_{212}^{(4)} = P_{212}^{(3)} + P_{212}^{(3)}P_3^{(4)}$
(7) $P_{323}^{(4)} = t^2P_3^{(4)} - t^2P_{32}^{(4)}$.
(8) $P_{3212}^{(4)} = tP_{212}^{(4)}$.

Proof. As we have already defined $A_*^{(4)}$ denotes the set of canonical words in $MB_{2,2}$ and $B_*^{(4)}$ denotes the set of reducible words in $MB_{2,2}$. Now we define the corresponding Hilbert series. Let $Q_*^{(4)}$ denotes the Hilbert series for $B_*^{(4)}$ and $P_*^{(4)}$ denotes the Hilbert series for $A_*^{(4)}$. Then using the set decomposition we have

(1) The decomposition of $A_1^{(4)}$ can be given as $A_1^{(4)} = A_1^{(3)} \cup (A_1^{(3)} \times A_3^{(4)})$. This implies Relation 1.

(2) It follows directly from $A_2^{(4)} = A_2^{(3)} \cup (A_2^{(3)} \times A_3^{(4)})$.

(3) The set $A_3^{(4)}$ can be written as $A_3^{(4)} = \{a_3\} \cup (a_3 \times A_3^{(4)}) \cup A_{32}^{(4)}$. This results the Relation 2.

(4) It follows from $A_{21}^{(4)} = A_{21}^{(3)} \cup (A_{21}^{(3)} \times A_3^{(4)})$.

(5) The set $A_{32}^{(4)}$ is decomposed as

$$A_{32}^{(4)} = \{a_3\} \times A_2^{(4)} \setminus (B_3^{(4)} \times_{32} A_{32}^{(4)}) \cup (B_{323}^{(4)} \times_{323} A_{323}^{(4)}) \cup (B_5^{(4)} \times_{3212} A_{3212}^{(4)}) \cup (B_6^{(4)} \times_3 A_3^{(4)})$$

which gives the required equation.

(6) The decomposition of $A_{212}^{(4)}$ can be written as $A_{212}^{(4)} = A_{212}^{(3)} \cup (A_{212}^{(3)} \times A_3^{(4)})$. This implies Relation 6.

(7) It follows from $A_{323}^{(4)} = \{a_3 a_2\} \times A_3^{(4)} \setminus (B_3^{(4)} \times_{32} A_{32}^{(4)})$.

(8) The decomposition of $A_{3212}^{(4)} = \{a_3\} \times A_{212}^{(4)}$ immediately gives us Relation 8.

Theorem 4. *Hilbert Series for the canonical words in $MB_{2,2}$ is given as*

$$H_{MB_{2,2}}(t) = \frac{1 - 2t + 2t^2 - 2t^3 + t^4 + t^6}{(1-t)(1 - 4t + 5t^2 - 5t^3 + 6t^4 - 3t^5 + 3t^6 - 4t^7 - t^8 - t^9 + t^{10})}.$$

Proof. As we have already computed the Hilbert series of canonical words and reducible words in Theorem 3 and in Theorem 2, respectively. Now we construct a linear system for canonical words in $MB_{2,2}$. For this we put the values of $P_1^{(3)}, P_2^{(3)}, P_{21}^{(3)}$ and $P_{212}^{(3)}$ form Lemma 1 in Theorem 3, we get following system of equations:

(1) $P_1^{(4)} = \frac{t}{(1-t)(1-t-t^2-t^3)} + \frac{t}{(1-t)(1-t-t^2-t^3)} P_3^{(4)}$

(2) $P_2^{(4)} = \frac{t(1+t+t^2)}{(1-t-t^2-t^3)} + \frac{t(1+t+t^2)}{(1-t-t^2-t^3)} P_3^{(4)}$

(3) $P_3^{(4)} = t + tP_3^{(4)} + P_{32}^{(4)}$

(4) $P_{21}^{(4)} = \frac{t^2(1+t)}{(1-t-t^2-t^3)} + \frac{t^2(1+t)}{(1-t-t^2-t^3)} P_3^{(4)}$

(5) $P_{32}^{(4)} = tP_2^{(4)} - t^2 P_{32}^{(4)} - \frac{t^3}{1-t} P_{323}^{(4)} - \frac{t^3}{1-t} P_{3212}^{(4)} - \frac{t^6}{(1-t)^2} P_3^{(4)}$

(6) $P_{212}^{(4)} = \frac{t^3}{1-t-t^2-t^3} + \frac{t^3}{1-t-t^2-t^3} P_3^{(4)}$

(7) $P_{323}^{(4)} = t^2 P_3^{(4)} - t^2 P_{32}^{(4)}$

(8) $P_{3212}^{(4)} = tP_{212}^{(4)}$.

Now we use Matrix Inversion Method to solve the system of these equations. Writing above equations in matrix form $AX = B$ we have

$$A = \begin{pmatrix} 1 & 0 & \frac{-t}{(1-t)(1-t-t^2-t^3)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-t(1+t+t^2)}{(1-t-t^2-t^3)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-t^2(1+t)}{(1-t-t^2-t^3)} & 1 & 0 & 0 & 0 & 0 \\ 0 & -t & \frac{t^6}{(1-t)^2} & 0 & 1+t^2 & 0 & \frac{t^3}{1-t} & \frac{t^3}{1-t} \\ 0 & 0 & \frac{-t^3}{1-t-t^2-t^3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t^2 & 0 & t^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & 0 & 1 \end{pmatrix}, X = \begin{pmatrix} P_1^{(4)} \\ P_2^{(4)} \\ P_3^{(4)} \\ P_{21}^{(4)} \\ P_{32}^{(4)} \\ P_{212}^{(4)} \\ P_{323}^{(4)} \\ P_{3212}^{(4)} \end{pmatrix}, B = \begin{pmatrix} \frac{t}{(1-t)(1-t-t^2-t^3)} \\ \frac{t(1+t+t^2)}{(1-t-t^2-t^3)} \\ t \\ \frac{t^2(1+t)}{(1-t-t^2-t^3)} \\ 0 \\ \frac{t^3}{(1-t-t^2-t^3)} \\ 0 \\ 0 \end{pmatrix}.$$

By solving the above system of equations, we get

$$\begin{aligned}
P_1^{(4)} &= \frac{t-2t^2+2t^3-2t^4+t^5+t^7}{(1-t)(1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10})}, \\
P_2^{(4)} &= \frac{t-t^2+t^3-2t^4+t^5-t^6+2t^7+t^8+t^9}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_3^{(4)} &= \frac{t-2t^2+2t^3-3t^4+2t^5-t^6+2t^7-t^{10}}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_{21}^{(4)} &= \frac{t^2-t^3-t^6+t^7+t^8+t^9}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_{32}^{(4)} &= \frac{t^3-2t^4+2t^5-2t^6+t^7+t^9}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_{212}^{(4)} &= \frac{t^9+2t^8-t^5-t^3+t^2}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_{323}^{(4)} &= \frac{t^3-3t^4+3t^5-3t^6+3t^7-t^8+2t^9-2t^{10}-t^{11}-t^{12}}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}, \\
P_{3212}^{(4)} &= \frac{t^4-2t^5+2t^6-2t^7+t^8+t^{10}}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}}.
\end{aligned}$$

Hence the Hilbert series of $MB_{2,2}$ is computed as

$$\begin{aligned}
H_{MB_{2,2}}(t) &= 1+P_1^{(4)}+P_2^{(4)}+P_3^{(4)} \\
&= 1+\frac{t-2t^2+2t^3-2t^4+t^5+t^7}{(1-t)(1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10})} \\
&+ \frac{t-t^2+t^3-2t^4+t^5-t^6+2t^7+t^8+t^9}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}} \\
&+ \frac{t-2t^2+2t^3-3t^4+2t^5-t^6+2t^7-t^{10}}{1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10}} \\
&= \frac{1-2t+2t^2-2t^3+t^4+t^6}{(1-t)(1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10})} \\
&= 1+3t+8t^2+21t^3+53t^4+132t^5+327t^6+807t^7+1988t^8+\cdots.
\end{aligned}$$

4. Growth rate for $MB_{2,2}$

In the following remark we find the growth rate of $MB_{2,2}$. Using Maple we convert the Hilbert series into partial fraction and by expanding its each term.

Remark 1. *The partial fraction of above Hilbert series is calculated as*

$$\begin{aligned}
&\frac{1-2t+2t^2-2t^3+t^4+t^6}{(1-t)(1-4t+5t^2-5t^3+6t^4-3t^5+3t^6-4t^7-t^8-t^9+t^{10})}\\
=&\frac{-0.05443+0.01323t}{0.56047-1.44679t+t^2}+\frac{0.00582+0.00567t}{0.76790-0.22627t+t^2}\\
&+\frac{0.50}{-1.00+t}+\frac{0.07088+0.10684t}{1.26443+1.00955t+t^2}+\frac{0.03341+0.04181t}{2.14793+2.17401t+t^2}+\frac{0.06238}{2.10385-t}+\frac{0.60519}{0.40664-t}.
\end{aligned}$$

As the last term

$$\frac{0.60519}{0.40664-t}=1.48827(1+2.45917t+(2.45917)^2t^2+(2.45917)^3t^3+\cdots)$$

gives us the approximation of the series and the rest of the terms have negligible effect on it. Hence $a_l^{(4)}\approx 1.48827(2.45917)^l$*. Thus, in this case the growth function* $a_l^{(4)}$ *of* $MB_{2,2}$ *is also exponential and growth rate is approximately equal to* 2.45917.

5. Conclusions

The mixed braid group $B_{m,n}$ is the subgroup of the Artinian braid group B_{m+n}. In this article we compute the canonical forms(ambiguity free presentation) of the words in the braid monoid and the corresponding Hilbert series. This work can also be utilized to compute Hilbert series of braid monoid $MB_{m,n}$.

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Conflict of interest

Authors declare that there are no conflicts of intereset.

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