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Research article

On moment convergence for some order statistics

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Abstract: By exploring the uniform integrability of a sequence of some order statistics (OSs), we obtain the moment convergence conclusion of the sequence under some weak conditions even when the corresponding population of interest has no moment of any positive order. As an application, we embody the range of applications of a theorem presented in a reference dealing with the approximation of the difference between the moment of a sequence of normalized OSs and the corresponding moment of a standard normal distribution. By the aid of the embodied theorem, we explore the infinitesimal type of the moments of errors when we estimate some population quantiles by relative OSs. Finally, by the obtained conclusion, we can easily get a combination formula which seems hard to be proved in other methods.

Keywords: convergence in distribution; convergence in probability; moment convergence; order statistic; uniform integrability
Mathematics Subject Classification: 62F10, 62G30

1. Introduction

Order statistic (OS) plays an important role in nonparametric statistics. Under the assumption of large sample size, relative investigations are mainly focused on asymptotic distributions of some functions of these OSs. Among these studies, the elegant one provided by Bahadur in 1966 (see [1]) is the central limit theorem on OSs. As was revealed there, under the situation of an absolute continuous population, the sequence of some normalized OSs usually has an asymptotic standard normal distribution. That is useful in the construction of a confidence interval for estimating some certain quantile of the population. Comparatively, study on some moment convergence of the mentioned sequence is also significant, for instance, if we utilize a sample quantile as an asymptotic unbiased estimator for the corresponding quantile of the population, then the analysis of the second moment convergence of the sequence is significant if we want to make an approximation of the mean square error of the estimate.

However, the analysis of moment convergence of OSs is usually very difficult, the reason, as was interpreted by Thomas and Sreekumar in [2], may lie in the fact that the moment of OS is usually very difficult to obtain.

For a random sequence, although it is well-known that the convergence in distribution does not necessarily guarantee the corresponding moment convergence, usually, that obstacle can be sufficiently overcome by the additional requirement of the uniform integrability of the sequence. For instance , we can see [3] as a reference dealing with some extreme OSs under some populations. In that article Wang et al. discussed uniform integrability of the sequence of some normalized extreme OSs and derived equivalent moment expressions there.

Here in the following theorem we discuss the moment convergence for some common OSs rather than extreme ones.

Theorem 1. *For a population X distributed according to a continuous probability density function (pdf) $f(x)$, let $p \in (0,1)$ and x_p be the $p-$quantile of X satisfying $f(x_p) > 0$. Let $(X_1, ..., X_n)$ be a random sample arising from X and $X_{i:n}$ be the $i-$th OS. If the cumulative distribution function (cdf) $F(x)$ of X has an inverse function $G(x)$ satisfying*

$$|G(x)| \leq B \cdot x^{-q}(1-x)^{-q} \tag{1.1}$$

for some constants $B > 0, q \geq 0$ and all $x \in (0,1)$, then for arbitrary $\delta > 0$, we have

$$\lim_{n\to\infty} EX_{i:n}^{\delta} = x_p^{\delta},$$

provided $\lim_{n\to+\infty} i/n = p$ or equivalently rewritten as $i/n = p + o(1)$.

Remark 1. *Now we use the symbol $\lceil z \rceil$ for the integer part of a positive number z and $m_{n,p}$ for the p-quantile of a random sample $(X_1, \cdots, X_n)$, namely, $m_{n,p} = (X_{pn:n} + X_{pn+1:n})/2$ if pn is an integer and $m_{n,p} = X_{\lceil pn+1 \rceil:n}$ otherwise. As both limiting conclusions $\lim_{n\to\infty} EX^{\delta}_{\lceil pn \rceil:n} = x_p^{\delta}$ and $\lim_{n\to\infty} EX^{\delta}_{\lceil pn+1 \rceil:n} = x_p^{\delta}$ hold under the conditions of Theorem 1 and $m^{\delta}_{n,p}$ is always squeezed by $X^{\delta}_{\lceil pn \rceil:n}$ and $X^{\delta}_{\lceil pn+1 \rceil:n}$, according to the Sandwich Theorem, we have $\lim_{n\to\infty} Em^{\delta}_{n,p} = x_p^{\delta}$.*

Remark 2. *For a continuous function $H(x)$ where $x \in (0,1)$, if*

$$\lim_{x\to 0+} H(x) = \lim_{x\to 1-} H(x) = 0,$$

then there is a constant $C > 0$ such that the inequality $|H(x)| \leq C$ holds for all $x \in (0,1)$. By that reason, the condition (1.1) can be replaced by the statement that there exists some constant $V \geq 0$ such that

$$\lim_{x\to 0+} G(x)x^V(1-x)^V = \lim_{x\to 1-} G(x)x^V(1-x)^V = 0.$$

Remark 3. *As the conclusion is on moment convergence of OSs, one may think that the moment of the population X in Theorem 1 should exist. That is a misunderstanding because the existence of the moment of the population is actually unnecessary. We can verify that by a population according to the well-known Cauchy distribution $X \sim f(x) = \frac{1}{\pi(1+x^2)}$ where $x \in (-\infty, +\infty)$, in this case, the moment EX of the population does not exist whereas the required conditions in Theorem 1 are satisfied. Even* *for some population without any moment of positive order, the conclusion of Theorem 1 still holds, for instance, if* $f(x) = \frac{1}{x(\ln(x))^2} I_{[e,\infty)}(x)$ *(where the symbol* $I_A(x)$ *or* I_A *stands for the indicator function of a set A), then we have the conclusion*

$$G(x) = e^{\frac{1}{x-1}} I_{(0,1)}(x),$$

which leads to

$$\lim_{x\to 0+} G(x)x(1-x) = \lim_{x\to 1-} G(x)x(1-x) = 0,$$

and therefore the condition (1.1) holds, thus we can see that Theorem 1 is workable. That denies the statement in the final part of paper [4] exclaiming that under the situation $X \sim f(x) = \frac{1}{x(\ln(x))^2} I_{[e,\infty)}(x)$ *any OS does not have any moment of positive order.*

According to Theorem 1, we known that the OS $X_{i:n}$ of interest is an asymptotic unbiased estimator of the corresponding population quantile x_p. Now we explore the infinitesimal type of the mean error of the estimate and derive

Theorem 2. *Let* $(X_1, ..., X_n)$ *be a random sample from X who possesses a continuous pdf* $f(x)$*. Let* $p \in (0, 1)$ *and* x_p *be the p−quantile of X satisfying* $f(x_p) > 0$ *and* $X_{i:n}$ *be the i−th OS. If the cdf* $F(x)$ *of X has an inverse function* $G(x)$ *with a continuous derivative function* $G'''(x)$ *in* $(0, 1)$ *and there is a constant* $U \geq 0$ *such that*

$$\lim_{x\to 0+} \left(G'''(x) \cdot x^U (1-x)^U\right) = \lim_{x\to 1-} \left(G'''(x) \cdot x^U (1-x)^U\right) = 0, \tag{1.2}$$

then under the assumption $i/n = p + O(n^{-1})$ *which indicates the existence of the limit* $\lim_{x\to 0+} \frac{i/n-p}{1/n}$*, the following proposition stands*

$$|E(X_{i:n} - x_p)| = O(1/n). \tag{1.3}$$

Remark 4. *Obviously we can see that* $|E(m_{n,p} - x_p)| = O(1/n)$ *under the conditions of Theorem 2.*

For i.i.d random variables(RVs) $X_1, ..., X_n$ with an identical expectation μ and a common finite standard deviation $\sigma > 0$, the famous Levy-Lindeberg central limit theorem reveals that the sequence of normalized sums

$$\left\{ \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}, n \geq 1 \right\}$$

converges in distribution to the standard normal distribution $N(0, 1^2)$ which we denote that as

$$\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{D} N(0, 1^2).$$

In 1964, Bengt presented his work [5] showing that if it is further assumed that $E|X_1|^k < +\infty$ for some specific positive k, then the m-th moment convergence conclusion

$$E\left(\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}\right)^m \to EZ^m, n \to +\infty, \tag{1.4}$$

holds for any positive m satisfying $m \leq k$. Here and throughout our paper, we denote Z a RV of standard normal distribution $N(0, 1^2)$.

Let $f(x)$ be a continuous pdf of a population X and x_r be the $r-$quantile of X satisfying $f(x_r) > 0$. Like the Levy-Lindeberg central limit theorem, Bahadur interpreted in [1] (1966) that for the OS $X_{i:n}$, following convergence conclusion holds

$$\frac{f(x_r)(X_{i:n} - x_r)}{\sqrt{r(1-r)/n}} \xrightarrow{D} N(0, 1^2),$$

provided $i/n \to r$ as $n \to \infty$.

Later in 1967, Peter studied moment convergence on similar topic. He obtained in [6] that for some $\varepsilon > 0$, $r \in (0, 1)$ and $p_n = i/n$, if the limit condition

$$\lim_{x\to\infty} x^{\varepsilon}[1 - F(x) + F(-x)] = 0$$

holds, then the conclusion

$$E\left(X_{i:n} - x_{\frac{i}{n+1}}\right)^k = \left[\frac{\sqrt{p_n(1-p_n)/n}}{f(x_{p_n})}\right]^k \int_{-\infty}^{\infty} \frac{x^k}{\sqrt{2\pi}} e^{-x^2/2} dx + o(n^{-k/2})$$

is workable for positive integer k and $rn \le i \le (1 - r)n$ as $n \to +\infty$.

In addition to the mentioned reference dealing with moment convergence on OSs, we find some more desirable conclusions on similar topic provided by Reiss in reference [7] in 1989, from which we excerpt the one of interest as what follows.

Theorem 3. *Respectively let $f(x)$ and $F(x)$ be the pdf and cdf of a population X. Let $p \in (0, 1)$ and x_p be the $p-$quantile of X satisfying $f(x_p) > 0$. Assume that on a neighborhood of x_p the cdf $F(x)$ has $m + 1$ bounded derivatives. If a positive integer i satisfies $i/n = p + O(n^{-1})$ and $E|X_{s:j}| < \infty$ holds for some positive integer j and $s \in \{1, ..., j\}$ and a measurable function $h(x)$ meets the requirement $|h(x)| \le |x|^k$ for some positive integer k, then*

$$Eh\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right) = \int_{-\infty}^{\infty} h(x) d\left(\Phi(x) + \varphi(x) \sum_{i=1}^{m-1} n^{-i/2} S_{i,n}(x)\right) + O(n^{-m/2}). \quad (1.5)$$

Here the function $\varphi(x)$ and $\Phi(x)$ are respectively the pdf and cdf of a standard normal distribution while $S_{i,n}(x)$, a polynomial of x with degree not more than $3i - 1$ and coefficients uniformly bounded over n, especially

$$S_{1,n}(x) = \left[\frac{2q-1}{3\sqrt{p(1-p)}} + \frac{\sqrt{p(1-p)} f'(x_p)}{2(f(x_p))^2}\right] x^2 + \frac{np - i + 1 - p}{\sqrt{p(1-p)}} + \frac{2(2p-1)}{3\sqrt{p(1-p)}}.$$

Remark 5. *By putting $h(x) = x^2$ and $m = 2$, we derive under the conditions of Theorem 3 that as $n \to +\infty$,*

$$E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^2 = \int_{-\infty}^{\infty} x^2 d\left(\Phi(x) + \varphi(x) n^{\frac{-1}{2}} S_{1,n}(x)\right) + O(n^{-1}) \to 1.$$

Therefore, we see that the sequence

$$\left\{E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^2, n \ge N_0\right\}$$

is uniformly bounded over $n \geq N_0$. Here N_0 is the positive integer number that the moment $EX_{i:n}^2$ exists when $n \geq N_0$. In accordance with the inequality $|E\xi| \leq \sqrt{E\xi^2}$ if only the moment $E\xi^2$ exists, the sequence

$$\left\{E\left(\frac{n^{1/2}f(x_p)(X_{i:n}-x_p)}{\sqrt{p(1-p)}}\right), n \geq N_0\right\}$$

is also uniformly bounded, say, by a number L over $n \in \{N_0, N_0+1, ...\}$. Now that

$$\left|E\frac{n^{1/2}f(x_p)(X_{i:n}-x_p)}{\sqrt{p(1-p)}}\right| \leq L, n \geq N_0,$$

we have

$$|E(X_{i:n}-x_p)| \leq L[\sqrt{p(1-p)}/f(x_p)]n^{-1/2}, n \geq N_0. \tag{1.6}$$

Under the conditions in Theorem 2, when we estimate a population quantile x_p by an OS $X_{i:n}$, usually the estimate is not likely unbiased, compared with the two conclusions (1.3) and (1.6), the result (1.3) in Theorem 2 is more accurate.

Remark 6. *For a random sample $(Y_1, Y_2, ..., Y_n)$ from a uniformly distributed population $Y \sim U[0,1]$, we write $Y_{i:n}$ the $i-$th OS. Obviously, conditions in Theorem 3 are fulfilled for any positive integer $m \geq 2$. That yields*

$$E\left(\frac{n^{1/2}(Y_{i:n}-p)}{\sqrt{p(1-p)}}\right)^2 = \int_{-\infty}^{\infty} x^2 d\left(\Phi(x)+\varphi(x)n^{-1/2}S_{1,n}(x)\right)+O(n^{-1}) = 1+O(n^{-1/2}),$$

and

$$\begin{aligned} E\left(\frac{n^{1/2}(Y_{i:n}-p)}{\sqrt{p(1-p)}}\right)^6 &= \int_{-\infty}^{\infty} x^6 d\left(\Phi(x)+\varphi(x)\sum_{i=1}^{5} n^{-i/2}S_{i,n}(x)\right)+O(n^{-3}) \\ &= \int_{-\infty}^{\infty} x^6\varphi(x)dx+\sum_{i=1}^{5}\alpha_i(n)n^{-i/2}+O(n^{-3}) \\ &= 15+\sum_{i=1}^{5}\alpha_i(n)n^{-i/2}+O(n^{-3}), \end{aligned}$$

where for each $i = 1, 2, ..., 5$, $\alpha_i(n)$ is uniformly bounded over n.

As is above analyzed, we conclude that under the assumption $i/n = p + O(n^{-1})$,

$$E(Y_{i:n}-p)^2 \sim p(1-p)n^{-1} \text{ and } E(Y_{i:n}-p)^6 \sim 15p^3(1-p)^3n^{-3}. \tag{1.7}$$

Based on Theorems 1 and 3, here we give some alternative conditions to those in Theorem 3 to embody its range of applications including situations even when the population X in Theorem 3 has no definite moment of any positive order. We obtain:

Theorem 4. *Let $(X_1, ..., X_n)$ be a random sample derived from a population X who has a continuous pdf $f(x)$. Let $p \in (0,1)$ and x_p be the $p-$quantile of X satisfying $f(x_p) > 0$ on a neighborhood of x_p and the following three conditions hold,* *(i)* *The cdf* $F(x)$ *of* X *has an inverse function* $G(x)$ *satisfying*

$$|G(x)| \leq B \cdot x^{-Q}(1-x)^{-Q} \tag{1.8}$$

for some constants $B > 0, Q \geq 0$ *and all* $x \in (0,1)$.

(ii) $F(x)$ *has* $m+1$ *bounded derivatives where* m *is a positive integer.*

(iii) *Let* $i/n = p + O(n^{-1})$ *and* $a_{i:n} = x_p + O(n^{-1})$ *as* $n \to +\infty$.

Then the following limiting result holds as $n \to +\infty$

$$E\left(\frac{f(x_p)(X_{i:n} - a_{i:n})}{\sqrt{p(1-p)/n}}\right)^m = EZ^m + O(n^{-1/2}). \tag{1.9}$$

Remark 7. *For the mean* $\overline{X}_n$ *of the random sample* $(X_1, ..., X_n)$ *of a population* X *whose moment* EX^m *exists, according to conclusion (1.4), we see*

$$E(\overline{X}_n - \mu)^m = \left(\frac{\sigma}{\sqrt{n}}\right)^m EZ^m + o(n^{-m/2}),$$

which indicates that the $m-th$ *central moment of sample mean* $E(\overline{X}_n - \mu)^m$ *is usually of infinitesimal* $O(n^{-m/2})$.

Here under the conditions of Theorem 4, if $EX_{i,n} = x_p + O(n^{-1})$ *(we will verify in later section that for almost all continuous populations we may encounter, this assertion holds according to Theorem 2), then by Eq (1.9), we are sure that the central moment* $E(X_{i:n} - EX_{i:m})^m$ *is also of an infinitesimal* $O(n^{-m/2})$. *Moreover, by putting* $a_{i:n} = x_p$, *we derive under the assumptions of Theorem 4 that*

$$E\left(\frac{f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)/n}}\right)^m = EZ^m + O(n^{-1/2}).$$

Similar to Remark 1, we can also show by Sandwich Theorem that

$$E\left(\frac{f(x_p)(m_{n,p} - x_p)}{\sqrt{p(1-p)/n}}\right)^m = EZ^m + O(n^{-1/2}) \tag{1.10}$$

indicating that if we use the sample $p-$*quantile* $m_{n,p}$ *to estimate* x_p, *the corresponding population* $p-$*quantile, then* $E(m_{n,p} - x_p)^m = O(n^{-m/2})$.

For estimating a parameter of a population without an expectation, estimators based on functions of sample moments are always futile because of uncontrollable fluctuation. Alternatively, estimators obtained by some functions of OSs are usually workable. To find a desirable one of that kind, approximating some moment expressions of OSs is therefore significant. For instance, let a population X be distributed according to a pdf

$$f(x, \theta_1, \theta_2) = \frac{\theta_2}{\pi[\theta_2^2 + (x - \theta_1)^2]}, -\infty < x < +\infty, \tag{1.11}$$

where constants $\theta_2 > 0$ and θ_1 is unknown. Here $x_{0.56} = 0.19076\theta_2 + \theta_1$ and $x_{0.56} + x_{0.44} = 2x_{0.5} = 2\theta_1$. To estimate $x_{0.5} = \theta_1$, we now compare estimators $m_{n,0.5}$ and $(m_{n,0.56} + m_{n,0.44})/2$. Under large sample size, we deduce according to conclusion (1.10) that

$$\begin{aligned} E\left(\frac{m_{n,0.56}+m_{n,0.44}}{2}-\theta_1\right)^2 &= E\left(\frac{(m_{n,0.56}-x_{0.56})+(m_{n,0.44}-x_{0.44})}{2}\right)^2 \\ &\le \frac{E(m_{n,0.56}-x_{0.56})^2+E(m_{n,0.44}-x_{0.44})^2}{2} \\ &= \frac{0.44\times 0.56}{(f(x_{0.56}))^2}n^{-1}+O(n^{-3/2}) \\ &= 0.2554\pi\theta_2 n^{-1}+O(n^{-3/2}), \end{aligned}$$

whereas

$$E\left(m_{n,0.5}-\theta_1\right)^2 = 0.785\pi\theta_2 n^{-1} + O(n^{-3/2}).$$

Obviously, both estimators $m_{n,0.5}$ and $(m_{n,0.56} + m_{n,0.44})/2$ are unbiased for θ_1. For large n, the main part $0.2554\pi\theta_2 n^{-1}$ of the mean square error (MSE) $E[(m_{n,0.56} + m_{n,0.44})/2 - \theta_1]^2$ is even less than one-third of $0.785\pi\theta_2 n^{-1}$, the main part of the MSE $E\left(m_{n,0.5} - \theta_1\right)^2$. That is the fundamental reason why Sen obtained in [8] the conclusion that the named optimum mid-range $(m_{n,0.56} + m_{n,0.44})/2$ is more effective than the sample median $m_{n,0.5}$ in estimating θ_1.

By statistical comparison of the scores presented in following Table 1 standing for 30 returns of closing prices of German Stock Index(DAX), Mahdizadeh and Zamanzade reasonably applied the previously mentioned Cauchy distribution (1.11) as a stock market return distribution with θ_1 and θ_2 being respectively estimated as $\widehat{\theta_1} = 0.0009629174$ and $\widehat{\theta_2} = 0.003635871$ (see [9]).

Table 1. Scores for 30 returns of closing prices of DAX.

0.0011848	-0.0057591	-0.0051393	-0.0051781	0.0020043	0.0017787
0.0026787	-0.0066238	-0.0047866	-0.0052497	0.0004985	0.0068006
0.0016206	0.0007411	-0.0005060	0.0020992	-0.0056005	0.0110844
-0.0009192	0.0019014	-0.0042364	0.0146814	-0.0002242	0.0024545
-0.0003083	-0.0917876	0.0149552	0.0520705	0.0117482	0.0087458

Now we utilize $(m_{n,0.56} + m_{n,0.44})/2$ as a quick estimator of θ_1 and derive a value 0.00105955 which roughly closes to the estimate value 0.0009629174 in reference [9].

Even now there are many estimate problems (see [10] for a reference) dealing with situations when a population have no expectation, as above analysis, further study on moment convergence for some OSs may be promising.

2. Preparation of main proof

Lemma 1. *(see [11] and [12]) For a random sequence $\{\xi_1, \xi_2, \cdots\}$ converging in distribution to a RV ξ which we write as $\xi_n \xrightarrow{D} \xi$, if $d > 0$ is a constant and the following uniform integrability holds*

$$\lim_{s\to\infty}\sup_n E|\xi_n|^d I_{|\xi_n|^d \ge s} = 0,$$

then $\lim_{n\to\infty} E|\xi_n|^d = E|\xi|^d$ *and accordingly* $\lim_{n\to\infty} E{\xi_n}^d = E\xi^d$.

Remark 8. *As discarding some definite number of terms from* $\{\xi_1, \xi_2, \cdots\}$ *does not affect the conclusion* $\lim_{n\to\infty} E|\xi_n|^d = E|\xi|^d$, *the above condition* $\lim_{s\to+\infty} \sup_n E|\xi_n|^d I_{|\xi_n|^d \geq s} = 0$ *can be replaced by* $\lim_{s\to+\infty} \sup_{n\geq M} E|\xi_n^d| I_{|\xi_n^d|\geq s} = 0$ *for any positive constant* $M > 0$.

Lemma 2. *For* $p \in (0, 1)$ *and a random sample* $(\xi_1, \xi_2, \cdots, \xi_n)$ *from a population possessing a continuous pdf* $f(x)$, *if the p-quantile* x_p *of the population satisfies* $f(x_p) > 0$, *then for the i-th OS* $\xi_{i:n}$ *where* $i/n = p + o(1)$, *we have* $\xi_{i:n} \xrightarrow{D} x_p$.

Proof. Obviously, the sequence $\{\frac{f(x_p)(\xi_{i:n}-x_p)}{\sqrt{p(1-p)/n}}, n = 1, 2, ...\}$ has an asymptotic standard normal distribution $N(0, 1^2)$, thus we see that the statistic $\xi_{i:n}$ converges to x_p in probability. That leads to the conclusion $\xi_{i:n} \xrightarrow{D} x_p$ by the reason that, for a sequence of RVs, the convergence to a constant in probability is equivalent to the convergence in distribution.

3. Proof of theorems

Clarification before presenting the proof:

- Under the assumption $i/n = p + o(1)$ when $n \to \infty$, we would better think of i as a function of n and use the symbol a_n instead of i. Nevertheless, for simplicity concern, we prefer no adjustment.
- Throughout our paper, $C_1, C_2, \cdots$ are some suitable positive constants.

3.1. Proof of Theorem 1

As $\frac{i}{n} \to p \in (0, 1)$ when $n \to \infty$, we only need care large numbers n, i and $n - i$.

Let an integer $K > \delta q$ be given and $M > 0$ be such a number that if $n \geq M$, then all the following inequalities $i - 1 - \delta q > 0$, $n - i - \delta q > 0$, $n - i - K > 0$ and $\frac{i+K}{n} < v = \frac{1+p}{2}$ hold simultaneously. Here the existence of v in the last inequality is ensured by the fact $\frac{i+K}{n} \to p$ as $n \to \infty$.

According to Lemmas 1 and 2 as well as Remark 8, to prove Theorem 1 we only need to show that

$$\lim_{s^\delta\to+\infty} \sup_{n\geq M} E|X_{i:n}^\delta| I_{|X_{i:n}^\delta|\geq s^\delta} = 0. \tag{3.1}$$

That is

$$\lim_{s\to+\infty} \sup_{n\geq M} \int\limits_{|u|\geq s} |u|^\delta \frac{n!}{(i-1)!(n-i)!} F^{i-1}(u) f(u)[1 - F(u)]^{n-i} du = 0.$$

To show that equation, it suffices for us to prove respectively

$$\lim_{s\to+\infty} \sup_{n\geq M} \int_s^{+\infty} |u|^\delta \frac{n!}{(i-1)!(n-i)!} F^{i-1}(u) f(u)[1 - F(u)]^{n-i} du = 0$$

and

$$\lim_{s\to+\infty} \sup_{n\geq M} \int_{-\infty}^{-s} |u|^\delta \frac{n!}{(i-1)!(n-i)!} F^{i-1}(u) f(u)[1 - F(u)]^{n-i} du = 0.$$

Equivalently by putting $x = F(u)$, we need to prove respectively

$$\lim_{t\to 1-}\sup_{n\geq M}\int_t^1 |G^\delta(x)|\frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}dx = 0 \tag{3.2}$$

as well as

$$\lim_{t\to 0+}\sup_{n\geq M}\int_0^t |G^\delta(x)|\frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}dx = 0.$$

As both proofs are similar in fashion, we chose to prove the Eq (3.2) only. Actually, according to the given condition $|G(x)|\leq Bx^{-q}(1-x)^{-q}$, we see

$$\begin{aligned}
&\lim_{t\to 1-}\sup_{n\geq M}\int_t^1 |G^\delta(x)|\frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}dx\\
\leq\ & B^\delta\lim_{t\to 1-}\sup_{n\geq M}\int_t^1 \frac{n!}{(i-1)!(n-i)!}x^{i-1-\delta q}(1-x)^{n-i-\delta q}dx\\
\leq\ & B^\delta\lim_{t\to 1-}\sup_{n\geq M}\int_t^1 \frac{n!}{(i-1)!(n-i)!}(1-x)^{n-i-\delta q}dx\\
\leq\ & B^\delta\lim_{t\to 1-}\sup_{n\geq M}\frac{n!}{(i-1)!(n-i)!}(1-t)^{n-i-K}(1-t)^{K+1-\delta q}\\
\leq\ & B^\delta\lim_{t\to 1-}\sup_{n\geq M}\frac{n!(1-t)^{n-i-K}}{(i-1)!(n-i)!}\leq C_1\lim_{x\to 0+}\sup_{n\geq M}\frac{n!\times n}{i!(n-i)!}x^{n-i-K}.
\end{aligned} \tag{3.3}$$

Here the positive number $C_1 > 0$ exists because $n/i = 1/p + o(1)$ where $p \in (0, 1)$.

Now applying the Stirling's formula $n! = \sqrt{2\pi n}(n/e)^n e^{\frac{\theta}{12n}}$ where $\theta \in (0, 1)$ (see [13]), we have

$$\begin{aligned}
&\lim_{x\to 0+}\sup_{n\geq M}\frac{n!\times n}{i!(n-i)!}x^{n-i-K}\\
&\leq C_2\lim_{x\to 0+}\sup_{n\geq M}\frac{\sqrt{2\pi n}(n/e)^n\times n}{\sqrt{2\pi i}(i/e)^i\sqrt{2\pi(n-i)}((n-i)/e)^{n-i}}x^{n-i-K}\\
&\leq C_3\lim_{x\to 0+}\sup_{n\geq M}\frac{n^n\times n}{i^i\sqrt{n-i}(n-i)^{n-i}}x^{n-i-K}\\
&= C_3\lim_{x\to 0+}\sup_{n\geq M}\frac{n^n}{i^i(n-i)^{n-i}}\frac{n}{\sqrt{n-i}}x^{n-i-K}\\
&= C_3\lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left(\frac{i}{n}\right)^i\left(1-\frac{i}{n}\right)^{n-i}}\frac{n}{\sqrt{n-i}}x^{n-i-K}\\
&= C_3\lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[\left(\frac{i}{n}\right)^{\frac{i}{n}}\left(1-\frac{i}{n}\right)^{1-\frac{i}{n}}\right]^n}\frac{n}{\sqrt{n-i}}x^{n-i-K}.
\end{aligned} \tag{3.4}$$

Noting that

$$(\frac{i}{n})^{\frac{i}{n}}(1-\frac{i}{n})^{1-\frac{i}{n}} \to p^p(1-p)^{1-p},$$

as $n \to \infty$, we see that there exists a positive constant, say $Q > 0$ such that

$$(\frac{i}{n})^{\frac{i}{n}}(1-\frac{i}{n})^{1-\frac{i}{n}} \geq Q \cdot p^p(1-p)^{1-p}$$

for all n. Consequently,

$$\begin{aligned}
&\lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[\left(\frac{i}{n}\right)^{\frac{i}{n}}\left(1-\frac{i}{n}\right)^{1-\frac{i}{n}}\right]^n}\frac{n}{\sqrt{n-i}}x^{n-i-K}\\
&\leq \lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[Qp^p(1-p)^{1-p}\right]^n}\frac{n}{\sqrt{n-i}}x^{n-i-K}\\
&\leq C_4\lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[Qp^p(1-p)^{1-p}\right]^n}\sqrt{n}x^{n-i-K}. \qquad (3.5)
\end{aligned}$$

Due to the assumptions $\frac{i+K}{n} < v = \frac{1+p}{2} < 1$ as $n \geq M$, we derive

$$\begin{aligned}
&\lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[Qp^p(1-p)^{1-p}\right]^n}\sqrt{n}x^{n-i-K}\\
&\leq \lim_{x\to 0+}\sup_{n\geq M}\frac{1}{\left[Qp^p(1-p)^{1-p}\right]^n}\sqrt{n}x^{n-vn}\\
&= \lim_{x\to 0+}\sup_{n\geq M}\left[\frac{x^{1-v}}{Qp^p(1-p)^{1-p}}\right]^n\sqrt{n}\\
&\leq \lim_{u\to 0+}\sup_{n\geq 1}u^n\sqrt{n}. \qquad (3.6)
\end{aligned}$$

Finally, by the fact that if $u > 0$ is given sufficiently small, then the first term of the sequence $\left\{u^n\sqrt{n}, n \geq 1\right\}$ is the maximum, thus we can confirm

$$\lim_{u\to 0+}\sup_{n\geq 1}u^n\sqrt{n} = \lim_{u\to 0+}u = 0. \qquad (3.7)$$

Combining the five conclusions numbered from (3.3) to (3.7), we obtain Eq (3.2).

3.2. Proof of Theorem 2

Here we would like to assume $U > 1$ (or we may use $U + 2$ instead of U).

By the reason interpreted in Remark 2 and according to condition (1.2), we see that there is a constant $A > 0$ satisfying

$$|G'''(x)\cdot x^U(1-x)^U| \leq A. \qquad (3.8)$$

Now we define $Y = F(X)$ and $Y_{i:n} = F(X_{i:n})$ or equivalently $X = G(Y)$ and $X_{i:n} = G(Y_{i,n})$, we have $G(p) = x_p$. Obviously, the conclusions in Remark 6 are workable here.

By the Taylor expansion formula we have

$$G(Y_{i:n}) = G(p) + G'(p)(Y_{i:n}-p) + \frac{G''(p)}{2!}(Y_{i:n}-p)^2 + \frac{1}{3!}G'''(\xi)(Y_{i:n}-p)^3,$$

where

$$\xi \in (min(Y_{i:n}, p), max(Y_{i:n}, p)).$$

Noting that almost surely $0 < min(Y_{i:n}, p) < \xi < max(Y_{i:n}, p) < 1$, we obtain

$$\begin{aligned}
&|EG(Y_{i:n}) - G(p) - G'(p)E(Y_{i:n} - p) - \frac{G''(p)}{2}E(Y_{i:n} - p)^2| \\
&= |E[\frac{G'''(\xi)}{3!}(Y_{i:n} - p)^3]| \leq \frac{1}{6}|E[A\xi^{-U}(1-\xi)^{-U}(Y_{i:n} - p)^3]| \\
&\leq \frac{1}{6}|E\{A[p(1-p)]^{-U}Y_{i:n}^{-U}(1 - Y_{i:n})^{-U}(Y_{i:n} - p)^3\}| \\
&\leq \frac{1}{6}|E\{A[p(1-p)]^{-U}(Y_{i:n} - p)^3\}| \\
&\leq \frac{1}{6}A[p(1-p)]^{-U}\sqrt{E(Y_{i:n} - p)^6} = O(n^{-3/2})
\end{aligned} \tag{3.9}$$

by Eq (3.8). Here the last step is in accordance to (1.7).

Now we can draw the conclusion that

$$EG(Y_{i:n}) - G(p) - G'(p)E(Y_{i:n} - p) - \frac{1}{2}G''(p)E(Y_{i:n} - p)^2 = o(n^{-1}). \tag{3.10}$$

That is

$$EX_{i:n} - x_p - G'(p)(\frac{i}{n+1} - p) - \frac{1}{2}G''(p)E(Y_{i:n} - p)^2 = o(n^{-1}), \tag{3.11}$$

provided $i/n = p + O(n^{-1})$.

Still according to conclusion (1.7), we have

$$E(Y_{i:n} - p)^2 = O(n^{-1}).$$

Finally, as $i/n = p + O(n^{-1})$ also guarantees $i/(n+1) - p = O(n^{-1})$, we can complete the proof of $E(X_{i:n} - x_p) = O(n^{-1})$ or equivalently

$$|E(X_{i:n} - x_p)| = O(n^{-1})$$

by the assertion of (3.11).

3.3. Proof of Theorem 4

As $EZ = 0$, the proposition holds when $m = 1$, now we only consider the case of $m \geq 2$. By Theorem 1, we see $EX_{i:n}^2 \to x_p^2$, therefore $E|X_{s:j}|$ exists for some integer j and $s \in \{1, ..., j\}$ and Theorem 3 is workable here when we put $h(x) = x^m$. We derive

$$\begin{aligned}
&E\left(\frac{n^{1/2}f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^m = \int_{-\infty}^{\infty} x^m d\left(\Phi(x) + \varphi(x)\sum_{i=1}^{m-1} n^{-i/2}S_{i,n}(x)\right) + O(n^{-m/2}) \\
&= EZ^m + \sum_{i=1}^{m-1}\left(n^{-i/2}\int_{-\infty}^{\infty} x^m d\left(\varphi(x)S_{i,n}(x)\right)\right) + O(n^{-m/2}).
\end{aligned} \tag{3.12}$$

Moreover, for given positive integer $m \geq 2$, as the coefficients in polynomial $S_{i,n}(x)$ are uniformly bounded over n and $\varphi'(x) = -x\varphi(x)$, the sequence of the integrals

$$\left\{\int_{-\infty}^{\infty} x^m d\left(\varphi(x) S_{i,n}(x)\right), n = 1, 2, ...\right\}$$

is also uniformly bounded over n. That indicates that

$$E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^m = EZ^m + O(n^{-1/2}) \tag{3.13}$$

according to conclusion (3.12).

As a consequence, we can conclude that for explicitly given $m \geq 2$ the sequence

$$\left\{E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^m, n = 1, 2, ...\right\} \tag{3.14}$$

is uniformly bounded over n. Moreover,due to the inequality

$$\left|E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)\right| \leq \sqrt{E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^2},$$

we see that the sequence

$$\left\{E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right), n = 1, 2, ...\right\}$$

is also uniformly bounded over n.

Now that $a_{i:n} = x_p + O(n^{-1})$, we complete the proof by the following reasoning

$$\begin{aligned}
&E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - a_{i:n})}{\sqrt{p(1-p)}}\right)^m \\
&= E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}} + \frac{n^{1/2} f(x_p)(x_p - a_{i:n})}{\sqrt{p(1-p)}}\right)^m \\
&= \sum_{u=0}^{m}\left[\binom{m}{u}\left(\frac{n^{1/2} f(x_p)(x_p - a_{i:n})}{\sqrt{p(1-p)}}\right)^{m-u} E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^u\right] \\
&= \sum_{u=2}^{m}\left[\binom{m}{u}\left(\frac{n^{1/2} f(x_p)(x_p - a_{i:n})}{\sqrt{p(1-p)}}\right)^{m-u} E\left(\frac{n^{1/2} f(x_p)(X_{i:n} - x_p)}{\sqrt{p(1-p)}}\right)^u\right] + O(n^{-1/2}) \\
&= \sum_{u=2}^{m}\left[\binom{m}{u}\left(\frac{n^{1/2} f(x_p)(x_p - a_{i:n})}{\sqrt{p(1-p)}}\right)^{m-u} \left(EZ^u + O(n^{-1/2})\right)\right] + O(n^{-1/2}) \\
&= EZ^m + O(n^{-1/2}).
\end{aligned} \tag{3.15}$$

4. Some verifications and one application

4.1. Verification examples

Now we consider the applicability of our theorems obtained so far. As other conditions can be trivially or similarly verified, here we mainly focus on the verification of condition (1.2).

Example 1: Let the population X have a Cauchy distribution with a pdf $f(y) = \frac{1}{\pi(1+y^2)}, -\infty < y < +\infty$, correspondingly the inverse function of the cdf of X can be figured out to be

$$G(x) = -\frac{1}{tan(\pi x)}, 0 < x < 1,$$

satisfying

$$\lim_{x \to 0+} G'''(x)x^5(1-x)^5 = \lim_{x \to 1-} G'''(x)x^5(1-x)^5 = 0.$$

Example 2: For $X \sim f(x) = \frac{1}{x(\ln(x))^2} I_{[e,\infty)}(x)$, we have

$$G(x) = e^{\frac{1}{x-1}} I_{(0,1)}(x),$$

and

$$\lim_{x \to 0+} G'''(x)x(1-x) = \lim_{x \to 1-} G'''(x)x(1-x) = 0.$$

Example 3: For $X \sim N(0, 1^2)$, on that occasion, $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, $f'(y) = -yf(y)$ and $y = G(x) \Leftrightarrow x = F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, therefore, as $x \to 0+$, we have

$$\begin{aligned} \frac{(G(x))^2}{-\ln(x(1-x))} &\sim \frac{(G(x))^2}{-\ln x} = \frac{y^2}{-\ln(F(y))} \overset{y \to -\infty}{\sim} \frac{-2yF(y)}{f(y)} \\ &= -2\left[\frac{(yF(y))'}{(f(y))'}\right] = -2\left[\frac{F(y) + yf(y)}{-yf(y)}\right] = -2\left[\frac{F(y)}{-yf(y)} - 1\right]. \end{aligned}$$

Noting that as $x = F(y) \to 0+$ or equivalently $y \to -\infty$,

$$\frac{F(y)}{-yf(y)} \sim \frac{f(y)}{-f(y) - yf'(y)} = \frac{f(y)}{-f(y) + y^2 f(y)} = \frac{1}{-1 + y^2} \to 0, \tag{4.1}$$

we have as $x \to 0+$,

$$\frac{(G(x))^2}{-\ln(x(1-x))} \to 2.$$

By the same fashion, we can show as $x \to 1-$ that

$$\frac{(G(x))^2}{-\ln(x(1-x))} \to 2.$$

In conclusion, for $x \to 0+$ as well as for $x \to 1-$,

$$(G(x))^2 \sim -2\ln(x(1-x)). \tag{4.2}$$

Accordingly, there exists a positive $M > 0$ such that for all $x \in (0, 1)$,

$$(G(x))^2 \leq M|\ln(x(1-x))| = -M\ln(x(1-x)). \quad (4.3)$$

No matter if $x \to 0+$ or $x \to 1-$, we get

$$\begin{aligned} |G'''(x)| &= \left|\frac{-f''(y)f(y) + 3(f'(y))^2}{(f(y))^5}\right| = \left|\frac{-(y^2-1)(f(y))^2 + 3(-yf(y))^2}{(f(y))^5}\right| \\ &= \frac{2y^2+1}{(f(y))^3} \overset{|y|\to\infty}{\sim} \frac{2y^2}{(f(y))^3} = 2\frac{(G(x))^2}{(f(G(x)))^3} \sim \frac{-4\ln(x(1-x))}{(f(G(x)))^3}. \end{aligned} \quad (4.4)$$

Here the last step holds in accordance to Eq (4.2).

For $x \to 0+$ as well as for $x \to 1-$,

$$\begin{aligned} \frac{-4\ln(x(1-x))}{(f(G(x)))^3} &= 4\frac{-\ln(x(1-x))}{\left(\frac{1}{\sqrt{2\pi}}\right)^3 \exp(-\frac{3(G(x))^2}{2})} = \frac{4(\sqrt{2\pi})^3\,[-\ln(x(1-x))]}{\exp\left(-\frac{3(G(x))^2}{2}\right)} \\ &= 4(\sqrt{2\pi})^3\,[-\ln(x(1-x))]\left[\exp\left((G(x))^2\right)\right]^{\frac{3}{4}} \\ &\leq 4(\sqrt{2\pi})^3\,[-\ln(x(1-x))]\,[\exp(-M\ln(x(1-x)))]^{\frac{3}{4}} \\ &= 4(\sqrt{2\pi})^3\,[-\ln(x(1-x))]\,(x(1-x))^{\frac{-3M}{4}}. \end{aligned} \quad (4.5)$$

Thus we can see the achievement of condition (1.2) by

$$\lim_{x\to 0+}\left(G'''(x)\cdot x^M(1-x)^M\right) = \lim_{x\to 1-}\left(G'''(x)\cdot x^M(1-x)^M\right) = 0. \quad (4.6)$$

Remark 9. *For a RV X with a cdf $F(x)$ possessing an inverse function $G(x)$, we can prove that if $\sigma > 0$ and $\mu \in (-\infty, +\infty)$ are constants, then the cdf of the RV $\sigma X + \mu$ will have an inverse function $\sigma G(x) + \mu$. Thus for the general case $X \sim N(\mu, \sigma^2)$, we can still verify the condition (1.2).*

Example 4: For a population $X \sim U[a, b]$, $G(x) = (b-a)x + a$ is the inverse function of the cdf of X. As $G'''(x) = 0$, the assumption of condition (1.2) holds.

Generally, for any population distributed over an interval $[a, b]$ according to a continuous pdf $f(x)$, if $G'''(0+)$ and $G'''(1-)$ exist, then the condition (1.2) holds.

For length concern, here we only point out without detailed proof that for a population X according to a distribution such as Gamma distribution (including special cases such as the Exponential and the Chi-square distributions) and beta distribution and so on, the requirement of condition (1.2) can be satisfied.

4.2. An application in obtaining a combination formula

For a random sample $(X_1, ..., X_n)$ derived from a population X which is uniformly distributed over the interval $[0, 1]$, the moment of the i–th OS $EX_{i:n} = i/(n+1) \to p$ if $i/n \to p \in (0, 1)$ as $n \to \infty$. Let $a_{i:n} = i/n$. According to conclusion (1.9) where $f(x_p) = 1$ and $x_p = p \in (0, 1)$, we have for integer $m \geq 2$,

$$E(X_{i:n} - a_{i:n})^m = \int_0^1 \frac{(x-\frac{i}{n})^m n!}{(i-1)!(n-i)!} x^{i-1}(1-x)^{n-i}dx + o(n^{-m/2})$$

$$= \quad EZ^m\,(p(1-p))^{\frac{m}{2}}\,n^{-\frac{m}{2}} + o(n^{-m/2}). \tag{4.7}$$

That results in

$$\frac{n!\int_0^1 (nx-i)^m x^{i-1}(1-x)^{n-i}dx}{(i-1)!(n-i)!n^m} = EZ^m(p-p^2)^{\frac{m}{2}}n^{-\frac{m}{2}} + o(n^{-m/2}), \tag{4.8}$$

or equivalently

$$\frac{n!\sum_{j=0}^m [\binom{m}{j} n^j(-i)^{m-j}B(i+j, n+1-i)]}{(i-1)!(n-i)!n^m} = EZ^m(p-p^2)^{\frac{m}{2}}n^{-\frac{m}{2}} + o(n^{-m/2}).$$

Consequently we have the following equation

$$\frac{n!\sum_{j=0}^m [\binom{m}{j} n^j(-i)^{m-j}\frac{\Gamma(i+j)\Gamma(n+1-i)}{\Gamma(i+j+n+1-i)}]}{(i-1)!(n-i)!n^m} = EZ^m(p-p^2)^{\frac{m}{2}}n^{-\frac{m}{2}} + o(n^{-m/2}),$$

which yields

$$\frac{n!\sum_{j=0}^m [\binom{m}{j} n^j(-i)^{m-j}\frac{(i-1+j)!}{(n+j)!}]}{(i-1)!n^m} = EZ^m(p-p^2)^{\frac{m}{2}}n^{-\frac{m}{2}} + o(n^{-m/2}). \tag{4.9}$$

As $i/n \to p \in (0,1)$ when $n \to +\infty$, the above equation indicates that

$$\frac{\sum_{j=0}^m [\binom{m}{j} n^j(-i)^{m-j}\frac{(i-1+j)!(n+m)!}{(i-1)!(n+j)!}]}{n^{2m}} = EZ^m(p-p^2)^{\frac{m}{2}}n^{-\frac{m}{2}} + o(n^{-m/2}). \tag{4.10}$$

For convenience sake, now we denote $\sum_{k=u}^v = 0$ and $\prod_{k=u}^v = 1$ if $v < u$. Noting for given explicit integers $m \geq 2$ and $j \in \{0, 1, ..., m\}$ the expression

$$\binom{m}{j} n^j(-i)^{m-j}\frac{(i-1+j)!(n+m)!}{(i-1)!(n+j)!} = \binom{m}{j}(-1)^{m-j}\left(i^{m-j}\prod_{k=1}^{j}[(i-1)+k]\right)\left(n^j\prod_{k=j+1}^{m}(n+k)\right) \tag{4.11}$$

is a multinomial of i and n. We see that the nominator of the LHS of Eq (4.10) is also a multinomial which we now denote as

$$\sum_{j=0}^{m}\left[\binom{m}{j} n^j(-i)^{m-j}\frac{(i-1+j)!(n+m)!}{(i-1)!(n+j)!}\right] := \sum_{s=0}^{m}\sum_{t=0}^{m} a_{s,t}^{(m)} i^{m-s}n^{m-t}.$$

Equivalently, we derive

$$\sum_{j=0}^{m}\left\{\binom{m}{j}(-1)^{m-j}[i^{m-j}\prod_{k=1}^{j}(i-1+k)][n^j\prod_{k=j+1}^{m}(n+k)]\right\} = \sum_{k=0}^{2m}\sum_{s+t=k} a_{s,t}^{(m)} i^{m-s}n^{m-t}.$$

By Eq (4.10), we see for any given $p \in (0,1)$, if $i/n \to p \in (0,1)$ as $n \to +\infty$, then

$$\frac{\sum_{k=0}^{2m}\sum_{s+t=k} a_{s,t}^{(m)} i^{m-s}n^{m-t}}{n^{3m/2}} = EZ^m\,(p(1-p))^{\frac{m}{2}} + o(1). \tag{4.12}$$

Noting that

$$\sum_{s+t=k} a_{s,t}^{(m)} i^{m-s} n^{m-t} = \left(\sum_{s+t=k} a_{s,t}^{(m)} p^{m-s} \right) n^{2m-k} + o(n^{2m-k}),$$

we see in accordance to (4.12) that

$$\frac{\sum_{k=0}^{2m} [(\sum_{s+t=k} a_{s,t}^{(m)} p^{m-s}) n^{2m-k} + o(n^{2m-k})]}{n^{3m/2}} = EZ^m (p - p^2)^{\frac{m}{2}} + o(1). \tag{4.13}$$

That indicates that if a non-negative integer k satisfies $2m - k > 3m/2$, or equivalently $0 \le k < m/2$, then the coefficient of n^{2m-k} in the nominator of LHS of Eq (4.13) must be zero for any given $p \in (0, 1)$, namely

$$\sum_{s+t=k} a_{s,t}^{(m)} p^{m-s} = 0, s + t = k < m/2$$

holds for any $p \in (0, 1)$. Thereby, for the case of non-negative integers s and t satisfying $s+t = k < m/2$, we see that the equation $a_{s,t}^{(m)} = 0$ surely holds.

It is funny to notice that for big m, we immediately have the following three corresponding equations

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} = 0,$$

$$\sum_{j=2}^{m} \binom{m}{j} (-1)^{m-j} \frac{j(j-1)}{2} = 0,$$

and

$$\sum_{j=2}^{m-1} \binom{m}{j} (-1)^{m-j} \frac{j(j-1)}{2} \frac{(m-j)(m+j+1)}{2} = 0,$$

according to the conclusions $a_{0,0}^{(m)} = 0$, $a_{1,0}^{(m)} = 0$ and $a_{1,1}^{(m)} = 0$.

As for the structure of $a_{s,t}^{(m)}$ when $s \ge 2$, $t \ge 1$ and $m > 2(s + t)$, obviously $s < m - t$ holds on this occasion and the term $a_{s,t}^{(m)} i^{m-s} n^{m-t}$ in the multinomial

$$\begin{aligned}
&\sum_{j=0}^{m} \left\{ \binom{m}{j} (-1)^{m-j} [i^{m-j} \prod_{k=1}^{j} (i - 1 + k)][n^j \prod_{k=j+1}^{m} (n + k)] \right\} \\
&= \sum_{j=0}^{m} \left\{ \binom{m}{j} (-1)^{m-j} [i^{m-j} \prod_{k=0}^{j-1} (i + k)][n^j \prod_{k=j+1}^{m} (n + k)] \right\} \\
&= \sum_{j=0}^{m} \left\{ \binom{m}{j} (-1)^{m-j} [i^{m-j+1} \prod_{k=1}^{j-1} (i + k)][n^j \prod_{k=j+1}^{m} (n + k)] \right\} \\
&= (\sum_{j=0}^{s} + \sum_{j=s+1}^{m-t} + \sum_{j=m-t+1}^{m}) \left\{ \binom{m}{j} (-1)^{m-j} [i^{m-j+1} \prod_{k=1}^{j-1} (i + k)][n^j \prod_{k=j+1}^{m} (n + k)] \right\}
\end{aligned}$$

is also the term $a_{s,t}^{(m)} i^{m-s} n^{m-t}$ in the multinomial

$$\sum_{j=s+1}^{m-t} \left\{ \binom{m}{j} (-1)^{m-j} [i^{m-j+1} \prod_{k=1}^{j-1} (i+k)][n^j \prod_{k=j+1}^{m} (n+k)] \right\}.$$

Noting for given $j \in \{s, ..., m-t\}$, the monomial

$$\left(\sum_{1 \le u_1 < u_2 < ... < u_s \le j-1} u_1 u_2 ... r_s \right) i^{m-s}$$

is the term with degree $m-s$ in the polynomial of i

$$[i^{m-j+1} \prod_{k=1}^{j-1} (i+k)],$$

while the monomial

$$\left(\sum_{j+1 \le v_1 < v_2 < ... < v_t \le m} v_1 v_2 ... v_t \right) n^{m-t}$$

is the term with degree $m-t$ in the polynomial of n

$$[n^j \prod_{k=j+1}^{m} (n+k)],$$

we see for $s+t < m/2$,

$$a_{s,t}^{(m)} = \sum_{j=s+1}^{m-t} \left(\binom{m}{j} (-1)^{m-j} \sum_{1 \le u_1 < ... < u_s \le j-1} u_1 ... u_s \sum_{j+1 \le v_1 < ... < v_t \le m} v_1 ... v_t \right).$$

Now that $a_{s,t}^{m} = 0$ holds provided $s+t = k < m/2$ according to Eq (4.13), we conclude the following Theorem.

Theorem 5. *If s,t and m are integers satisfying $s \ge 2$, $t \ge 1$ and $m > 2(s+t)$, then*

$$\sum_{j=s+1}^{m-t} \left(\binom{m}{j} (-1)^{m-j} \sum_{1 \le u_1 < u_2 < ... < u_s \le j-1} u_1 ... u_s \sum_{j+1 \le v_1 < v_2 < ... < v_t \le m} v_1 ... v_t \right) = 0.$$

Example 5: For big integer m, according to Theorem 5, we have $a_{2,1}^{(m)} = 0$ and $a_{2,2}^{(m)} = 0$. Correspondingly, we obtain equations

$$\sum_{j=3}^{m-1} \left(\binom{m}{j} (-1)^{m-j} \frac{(\sum_{i=1}^{j-1} i)^2 - (\sum_{i=1}^{j-1} i^2)}{2} \frac{(m+j+1)(m-j)}{2} \right) = 0,$$

and

$$\sum_{j=3}^{m-2} \left(\binom{m}{j} (-1)^{m-j} \frac{(\sum_{i=1}^{j-1} i)^2 - (\sum_{i=1}^{j-1} i^2)}{2} \frac{(\sum_{i=j+1}^{m} i)^2 - (\sum_{i=j+1}^{m} i^2)}{2} \right) = 0.$$

Both equations can be verified by the aid of Maple software.

5. Conclusions

Let real $\delta > 0$ and integer $m > 0$ be given. For a population satisfying condition (1.1), no matter if the population has an expectation or not, the moment of $X_{i:n}^{\delta}$ exists and the sequence $\{EX_{i:n}^{\delta}, n \geq 1\}$ converges for large i and n satisfying $i/n \to p \in (0, 1)$. Under some further trivial assumptions, for large integer n the $m-$th moment of the standardized sequence $\{X_{i:n}, n \geq 1\}$ can be approximated by the $m-$th moment of a standard normal distribution EZ^m.

Due to the fact that the existence requirement of some expectation $X_{s:j}$ in Theorem 3 has always been hard to be verified for a population without an expectation, for a long time, real-life world data corresponding to that population of interest has been unavailable in the vast majority of references. Now that the alternative condition (1.8) is presented, maybe things will improve in the future and we still have a long way to go.

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Conflict of interest

There exists no conflict of interest between authors.

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