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Research article

The fourth power mean value of one kind two-term exponential sums

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Abstract: In this paper, based on the analytic method and the properties of Gauss sums, we study the computational problems of the fourth power mean value of one kind two-term exponential sums through the classification and estimation of Dirichlet characters and give it a calculation formula or asymptotic formula in different conditions.

Keywords: the two-term exponential sums; the fourth power mean; Gauss sums
Mathematics Subject Classification: 11L03, 11L07

1. Introduction

The mean value calculation and upper bound estimation of exponential sums has always been a classical problem in analytic number theory. Exponential sum method plays an important role in many number theory problems. As a special kind of exponential sum, Gauss sum plays an important role not only in the study of analytic number theory, but also in cryptography. Any substantial progress in this field will play an important role in promoting the development of analytic number theory and cryptography. In this paper, we will estimate and calculate the fourth power mean value of one kind two-term exponential sums.

Let $q \geq 3$ be a positive integer. For any integers m and n, the two-term exponential sums $G(m, n, k, h; q)$ are defined as follows:

$$G(m, n, k, h; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),$$

where $e(y) = e^{2\pi iy}$ and $i^2 = -1$, k and h are both positive integers.

As a special form of exponential sums, we define the k-th Gauss sums $G(m, k; q)$ as follows:

$$G(m, k; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k}{q}\right).$$

In addition, another form of exponential sums

$$K(m,n;q)=\sum_{\substack{a=0\\(a,q)=1}}^{q-1} e\left(\frac{ma+n\overline{a}}{q}\right),$$

which are the famous Kloosterman sums, and $\overline{a}$ denotes the multiplicative inverse of a mod q, that means $a\overline{a}\equiv 1 \bmod q$.

The calculations of exponential sums are important in number theory, because they are related to many classical mathematical problems, such as Waring's problem. Many scholars have studied the properties of $G(m,n,k,h;q)$ in different forms, and obtained many interesting results, see [1–4]. For example, H. Zhang and W. P. Zhang [5] obtained an important result of $G(m,n,3,1;p)$ and proved an identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4=\begin{cases} 2p^3-p^2, & \text{if } 3\nmid p-1,\\ 2p^3-7p^2, & \text{if } 3\mid p-1,\end{cases}$$

where p is an odd prime and gcd $(n,\ p)=1$.

Z. Y. Chen and W. P. Zhang [6] studied the hybrid mean value of the 4-th Gauss sums and the Kloosterman sums, and obtain the following formula:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1} e\left(\frac{mc+\overline{c}}{p}\right)\right|^2=3p^3-3p^2+2p^{\frac{3}{2}}\alpha-3p,$$

where p is an odd prime with $p\equiv 5 \bmod 8$.

L. Chen and X. Wang [7] studied a kind of two-term exponential sums and obtained the following formula:

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4=\begin{cases} 2p^2(p-2), & \text{if } p=12k+7,\\ 2p^3, & \text{if } p=12k+11,\\ 2p(p^2-10p-2\alpha^2), & \text{if } p=24k+1,\\ 2p(p^2-4p-2\alpha^2), & \text{if } p=24k+5,\\ 2p(p^2-6p-2\alpha^2), & \text{if } p=24k+13,\\ 2p(p^2-8p-2\alpha^2), & \text{if } p=24k+17.\end{cases}$$

More relevant to it, W. P. Zhang and D. Han [8] studied a kind of two-term exponential sums and obtained an important result

$$\sum_{n=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{a^3+na}{p}\right)\right|^6=5p^4-8p^3-p^2.$$

The above formula gives the calculation formulae of the sixth mean value of exponential sums in this form when $k=3$ and $h=1$. Inspired by the conclusion, we intended to calculate higher-order forms, such as the case of $k=4$ and $h=1$ which are more meaningful. But this form of sixth mean value is too difficult to calculate, and some formulas have not been obtained yet. But yet, we can get a result for the fourth mean value in this condition:

$$\sum_{m=0}^{q-1} |C(1,m,4,1;q)|^4 = \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4+ma}{p}\right) \right|^4 .$$

The problem in our paper is more complex than the results in [7] by L. Chen and X. Wang. First of all, these are two completely different sums, and in this case m is multiplied by a instead of a^4. Secondly, their summation over m can get rid of the fourth power part of the expansion. Finally, the summation over m in this paper can only deal with the first power part in the expansion process, which means that we have to deal with a higher order trigonometric sum and solve a simultaneous congruence equation. We not only give the essential relationship between the solution of the sum and the number of solutions of the system of congruence equations, but also give a concise and good-looking result in this high-dimensional case, this is also the difficulty of this paper.

Regarding this problem, we shall prove the following conclusions:

Theorem 1.1. *Let p be a prime with $p \equiv 1 \bmod 4$, then we have the asymptotic formula*

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4+ma}{p}\right) \right|^4 = 2p^3 + O(p^{\frac{5}{2}}).$$

Theorem 1.2. *Let $p > 3$ be a prime with $p \equiv 3 \bmod 4$, then we have the identity*

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4+ma}{p}\right) \right|^4 = \begin{cases} 2p^3 - 3p^2, & \text{if } p = 12k+7, \\ 2p^3 + p^2, & \text{if } p = 12k+11. \end{cases}$$

Corollary 1.1. *Let p be a prime, we have the asymptotic formula*

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4+ma}{p}\right) \right|^4 = 2p^3 + O(p^{\frac{5}{2}}).$$

Notes. If $p > 3$, the corollary is obvious according to the theorems, now we declaration the condition $p \leq 3$.

If $p = 2$, according to Euler's formula, we have

$$\begin{aligned} & \sum_{m=0}^{1} \left| \sum_{a=0}^{1} e\left(\frac{a^4+ma}{2}\right) \right|^4 \\ = & \sum_{m=0}^{1} \left| 1 + e\left(\frac{1+m}{2}\right) \right|^4 \\ = & \left| 1 + e\left(\frac{1}{2}\right) \right|^4 + 16 \\ = & |1 + cos\pi + isin\pi|^4 + 16 \\ = & 16 = 2p^3. \end{aligned}$$

Similarly, if $p = 3$ we can verify the corollary. So the corollary holds for all the primes.

2. Several lemmas

In this section, we shall give six lemmas that are necessary in the proof of our theorems. According to the elementary and analytical methods, we use the properties of Gauss sums and the character sums to obtain the lemmas as follows:

Lemma 2.1. *Let p be a prime with $p = 4k + 1$, λ be a fourth-order character mod p, then we have the estimation*

$$\tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(a^4+b^4-(a+b-1)^4-1) = O(p^{\frac{3}{2}}),$$

where $\tau(\lambda) = \sum_{a=0}^{p-1}\lambda(a)e\left(\frac{a}{p}\right)$ denotes the Gauss sums.

Proof. By using the properties of fourth-order character and complete residue systems, we know that when a and b are integers, and they pass through a complete residue systems, $a-1$, ab, ka, kb, $a+b$ and $2a+b$ also pass through a complete residue systems, then we have

$$\begin{aligned}
& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(a^4+b^4-(a+b-1)^4-1)\\
= \;& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(4(b-1)^3+6(b-1)^2+4(b-1)-4a(b-1)^3-6a^2(b-1)^2-4a^3(b-1))\\
= \;& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(4b^3+6b^2+4b-4ab^3-6a^2b^2-4a^3b)\\
= \;& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\lambda}(4b^3+6b^2+4b-4ab^4-6a^2b^4-4a^3b^4)\\
= \;& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\lambda}(b^4)\overline{\lambda}(4\overline{b}+6\overline{b}^2+4\overline{b}^3-4a-6a^2-4a^3).
\end{aligned}$$

For λ be a fourth-order character mod p, then $\overline{\lambda}(b^4) = 1$, so we have

$$\begin{aligned}
& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\overline{\lambda}(4\overline{b}+6\overline{b}^2+4\overline{b}^3-4a-6a^2-4a^3)\\
= \;& \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(4b^3+6b^2+4b-4a^3-6a^2-4a)-\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}(-4a^3-6a^2-4a)\\
= \;& \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(8b^3+12b^2+8b-8a^3-12a^2-8a)+O(p)\\
= \;& \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}((2b)^3+3(2b)^2+4(2b)-(2a)^3-3(2a)^2-4(2a))+O(p).
\end{aligned}$$

According to the properties of the reduced residue system mod p, we are aware of that if a and b pass through a reduced residue system mod p respectively, obviously, $2a$ and $2b$ also pass through the same reduced residue system mod p. So, in the formula $((2b)^3 + 3(2b)^2 + 4(2b) - (2a)^3 - 3(2a)^2 - 4(2a))$ above, we change $2b$ into b, $2a$ into a, that equals $(b^3 + 3b^2 + 4b - a^3 - 3a^2 - 4a)$ mod p. Therefore, we can conclude that

$$\begin{aligned}
& \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}((2b)^3 + 3(2b)^2 + 4(2b) - (2a)^3 - 3(2a)^2 - 4(2a)) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3b^2 + 4b - a^3 - 3a^2 - 4a) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3b^2 + 3b + 1 + b + 1 - a^3 - 3a^2 - 3a - 1 - a - 1) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}((b+1)^3 + (b+1) - (a+1)^3 - (a+1)) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + b - a^3 - a) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3b^2a + 3ba^2 + a^3 + b + a - a^3 - a) + O(p) \\
= & \tau(\lambda)\lambda(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3b^2a + 3ba^2 + b) + O(p) \\
= & \tau(\lambda)\lambda(8)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3b(2a+b)^2 + 4b) + O(p) \\
= & \tau(\lambda)\overline{\lambda}(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3ba^2 + 4b) + O(p). \qquad (2.1)
\end{aligned}$$

Let $\chi_2 = (\frac{*}{p})$, where $(\frac{*}{p})$ be the Legendre symbol, directly by using of the conclusions in the reference [8, Lemma 1], we have

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3 + 3ba^2 + 4b) &= \frac{\tau(\lambda\chi_2)}{\tau(\lambda)}\sum_{b=0}^{p-1}C(1,0,2;p)\chi_2(3b^4 + 12b^2)\overline{\lambda}(b^3 + 4b) \\
&= \frac{C^2(1,0,2;p)}{p}\sum_{a=1}^{p-1}\overline{\lambda}(a)\chi_2(1-a)\sum_{b=0}^{p-1}\chi_2(3b^4 + 12b^2)\overline{\lambda}(b^3 + 4b) \\
&= \chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a)\chi_2(1-a)\sum_{b=0}^{p-1}\chi_2(b^2 + 4)\overline{\lambda}(b^3 + 4b) \\
&= \chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a)\chi_2(1-a)\sum_{b=0}^{p-1}\chi_2(b^2 + 4)\overline{\lambda}(b^2 + 4)\overline{\lambda}(b) \\
&= \chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a^3 - 2a^2 + a)\sum_{b=0}^{p-1}\lambda(b^3 + 4b), \qquad (2.2)
\end{aligned}$$

where

$$C(1,0,2;p)=\sum_{a=0}^{p-1}e\left(\frac{a^2}{p}\right)=\tau(\chi_2).$$

Therefore, we have

$$\begin{aligned}
&\tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(a^4+b^4-(a+b-1)^4-1)\\
=\ &\tau(\lambda)\overline{\lambda}(2)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(b^3+3ba^2+4b)\\
=\ &\tau(\lambda)\overline{\lambda}(2)\chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a)\chi_2(1-a)\sum_{b=0}^{p-1}\chi_2(b^2+4)\overline{\lambda}(b^3+4b)\\
=\ &\tau(\lambda)\overline{\lambda}(2)\chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a^3-2a^2+a)\sum_{b=0}^{p-1}\lambda(b^3+4b).
\end{aligned}$$

From Weil's classical work [9] we know that if χ is a q-th-order character to the prime modulo p, and if polynomial $f(x)$ is not a perfect q-th power modulo p, then we have the estimate

$$\sum_{x=N+1}^{N+H}\chi(f(x))=O(\sqrt{p}),$$

where N and H are any positive integers. So from the estimate we have

$$\begin{aligned}
&\tau(\lambda)\overline{\lambda}(2)\chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a^3-2a^2+a)\sum_{b=0}^{p-1}\lambda(b^3+4b)\\
=\ &O(p^{\frac{3}{2}}).
\end{aligned}\tag{2.3}$$

Similarly, we also have

$$\begin{aligned}
&\tau(\overline{\lambda})\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\lambda(a^4+b^4-(a+b-1)^4-1)\\
=\ &\tau(\overline{\lambda})\lambda(2)\chi_2(-3)\sum_{a=1}^{p-1}\lambda(a^3-2a^2+a)\sum_{b=0}^{p-1}\overline{\lambda}(b^3+4b)+O(p)\\
=\ &O(p^{\frac{3}{2}}).
\end{aligned}\tag{2.4}$$

This proves Lemma 2.1.

Lemma 2.2. *If $p>3$ is a prime, χ_2 denotes the Legendre symbol, then we have the following results:*

$$\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1)=\begin{cases}2\chi_2(6)p^{\frac{3}{2}}+O(p), & \text{if } p=4k+1,\\ \sqrt{p}\chi_2(4a^3+6a^2+4a), & \text{if } p=4k+3.\end{cases}$$

Proof. From the properties of reduced residue system mod p and Gauss sums, we have

$$\begin{aligned}
&\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1)\\
=\ &\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(4b^3+6b^2+4b-4ab^3-6a^2b^2-4a^3b)\\
=\ &\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\chi_2(4b^3+6b^2+4b-4ab^3-6a^2b^2-4a^3b)\\
=\ &\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(4b^3+6b^2+4b-4a^3-6a^2-4a)-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(8b^3+12b^2+8b-8a^3-12a^2-8a)-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2((2b)^3+3(2b)^2+4(2b)-(2a)^3-3(2a)^2-4(2a))\\
&-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2((b^3+3b^2+4b-a^3-3a^2-4a)-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2((b+1)^3+(b+1)-(a+1)^3-(a+1))-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(b^3+b-a^3-a)-\chi_2(-1)\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a). \qquad (2.5)
\end{aligned}$$

Let p be a prime with $p=4k+1$, note that $\chi_2(-1)=1$, then the formula (2.5) equals to

$$\begin{aligned}
&\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1)\\
=\ &\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=1}^{p-1}\chi_2(b^3+3ba^2+4b)-\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)p^{\frac{3}{2}}\cdot\sum_{\substack{b=1\\ b^3+4b\equiv 0 \bmod p}}^{p-1}\chi_2(3b)-\sum_{b=1}^{p-1}\chi_2(3b)-\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)\\
=\ &\chi_2(2)p^{\frac{3}{2}}\cdot\sum_{\substack{b=1\\ b^2+1\equiv 0 \bmod p}}^{p-1}\chi_2(6b)-\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a)
\end{aligned}$$

$$
\begin{aligned}
&= 2\chi_2(6)p^{\frac{3}{2}} - \sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a) \\
&= 2\chi_2(6)p^{\frac{3}{2}} + O(p). \qquad (2.6)
\end{aligned}
$$

On the other hand, let $p > 3$ be an odd prime with $p = 4k+3, \chi_2(-1) = -1$, then we have the identity

$$
\begin{aligned}
&\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(b^3+b-a^3-a) \\
&= \chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(-b^3-b+a^3+a) \\
&= -\chi_2(2)\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(b^3+b-a^3-a) \\
&= 0. \qquad (2.7)
\end{aligned}
$$

Combining the identities (2.5) and (2.7), for any prime $p > 3$, if $p = 4k+3$, we obtain

$$
\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1) = \sqrt{p}\chi_2(4a^3+6a^2+4a). \qquad (2.8)
$$

Combining the identities (2.6) and (2.8), for any prime $p > 3$, we have the following results:

$$
\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1) = \begin{cases} 2\chi_2(6)p^{\frac{3}{2}} + O(p), & \text{if } p = 4k+1, \\ \sqrt{p}\chi_2(4a^3+6a^2+4a), & \text{if } p = 4k+3. \end{cases}
$$

This proves Lemma 2.2.

Lemma 2.3. *Let $p > 3$ be a prime with $p = 4k+1$, λ be a fourth-order character mod p and χ_2 denotes the Legendre symbol, then we have the asymptotic formula*

$$
\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right) = 3p^2 + O(p^{\frac{3}{2}}).
$$

Proof. Note that λ is a fourth-order character mod p, and χ_2 denotes the Legendre symbol, so, if $p \equiv 1 \bmod 4$, then from the formulas (2.3) and (2.4) we have

$$
\begin{aligned}
& \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right)\\
= & \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{d=0}^{p-1}(1+\lambda(d)+\chi_2(d)+\overline{\lambda}(d))e\left(\frac{d(a^4+b^4-c^4-1)}{p}\right)\\
= & \; p\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{d=0}^{p-1}(\lambda(d)+\chi_2(d)+\overline{\lambda}(d))e\left(\frac{d(a^4+b^4-(a+b-1)^4-1)}{p}\right)\\
= & \; \tau(\lambda)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}(a^4+b^4-(a+b-1)^4-1)+\tau(\overline{\lambda})\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\lambda(a^4+b^4-(a+b-1)^4-1)\\
& +\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1)+p\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1\\
= & \; 2\chi_2(6)p^{\frac{3}{2}}+\tau(\lambda)\overline{\lambda}(2)\chi_2(-3)\sum_{a=1}^{p-1}\overline{\lambda}(a^3-2a^2+a)\sum_{b=0}^{p-1}\lambda(b^3+4b)+p\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1\\
& +\tau(\overline{\lambda})\lambda(2)\chi_2(-3)\sum_{a=1}^{p-1}\lambda(a^3-2a^2+a)\sum_{b=0}^{p-1}\overline{\lambda}(b^3+4b)+O(p)\\
= & \; p\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+O(p^{\frac{3}{2}}), \qquad (2.9)
\end{aligned}
$$

where

$$\sum_{d=0}^{p-1} e\left(\frac{md^4}{p}\right)=\sum_{d=0}^{p-1}(1+\chi_2(d))e\left(\frac{md^2}{p}\right)=\sum_{d=0}^{p-1} e\left(\frac{md^2}{p}\right)=\sum_{d=0}^{p-1}(1+\chi_2(d))e\left(\frac{md}{p}\right).$$

From L. Chen and X. Wang [7] , for any prime $p>3$, if $p=24k+13$ or $p=24k+1$, we have

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a^4+b^4\equiv c^4+1 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1=3p-5-\chi_2(7). \qquad (2.10)$$

For any prime $p > 3$, if $p = 24k + 5$ or $p = 24k + 17$, we have

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 1 - \chi_2(7). \tag{2.11}$$

Thus, for any prime $p > 3$, we have the asymptotic formula

$$\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right) = 3p^2 + O(p^{\frac{3}{2}}).$$

This proves Lemma 2.3.

Lemma 2.4. *Let $p > 3$ be a prime with $p = 4k+3$, λ be a fourth-order character mod p and χ_2 denotes the Legendre symbol, then we have the identity*

$$\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right)$$

$$= \begin{cases} 3p^2 - 5p + \chi_2(7)p + i\sqrt{p}\sum_{b=0}^{p-1}\chi_2(4a^3+6a^2+4a), & \text{if } p = 12k+7, \\ 3p^2 - p + \chi_2(7)p + i\sqrt{p}\sum_{b=0}^{p-1}\chi_2(4a^3+6a^2+4a), & \text{if } p = 12k+11. \end{cases}$$

Proof. Note that $\tau(\chi_2) = i\sqrt{p}$ when $p \equiv 3 \bmod 4$, then from the definition of Gauss sums and identity (2.8), we have

$$\begin{aligned}
&\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right) \\
= &\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} (1+\chi_2(d))e\left(\frac{d(a^4+b^4-c^4-1)}{p}\right) \\
= &\ p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{d=0}^{p-1} \chi_2(d) e\left(\frac{d(a^4+b^4-(a+b-1)^4-1)}{p}\right) \\
= &\ i\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2(a^4+b^4-(a+b-1)^4-1) + p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1
\end{aligned}$$

$$= p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 + i\sqrt{p}\sum_{b=0}^{p-1} \chi_2(4a^3+6a^2+4a), \tag{2.12}$$

From L. Chen and X. Wang [7] , for any prime $p > 3$, if $p = 12k + 7$, we have

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 5 + \chi_2(7). \tag{2.13}$$

If $p = 12k + 11$, we have

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 1 + \chi_2(7). \tag{2.14}$$

So for any prime $p > 3$, if $p \equiv 3 \bmod 4$, combining the formulas (2.12)–(2.14), we have

$$\sum_{\substack{a=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right)$$
$$= \begin{cases} 3p^2 - 5p + \chi_2(7)p + i\sqrt{p}\sum_{b=0}^{p-1}\chi_2(4a^3+6a^2+4a), & \text{if } p = 12k+7, \\ 3p^2 - p + \chi_2(7)p + i\sqrt{p}\sum_{b=0}^{p-1}\chi_2(4a^3+6a^2+4a), & \text{if } p = 12k+11. \end{cases}$$

This proves Lemma 2.4.

Lemma 2.5. *Let $p > 3$ be a prime with $p = 4k+1$, λ be a fourth-order character mod p and χ_2 denotes the Legendre symbol, then we have the estimation*

$$\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right) = O(p).$$

Proof. Note that $\tau(\chi_2) = \sqrt{p}$ when $p \equiv 1 \bmod 4$, then from the definition of Gauss sums we have

$$\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)$$
$$= \sum_{\substack{a=0 \\ a+b\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{a^4+b^4}{p}\right) + \sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=1}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)$$

$$= \sum_{a=0}^{p-1} e\left(\frac{2a^4}{p}\right) + \sum_{\substack{a=0\\a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c^4(a^4+b^4-1)}{p}\right)$$

$$= \sum_{a=0}^{p-1}(1+\lambda(a)+\chi_2(a)+\overline{\lambda}(a))e\left(\frac{2a}{p}\right) + \sum_{\substack{a=0\\a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{c^4(a^4+b^4-1)}{p}\right) - \sum_{\substack{a=0\\a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1$$

$$= \tau(\lambda)\overline{\lambda}(2) + \chi(2)\sqrt{p} + \tau(\overline{\lambda})\lambda(2) + \sum_{a=0}^{p-1}\sum_{c=0}^{p-1}(\lambda(c)+\chi_2(c)+\overline{\lambda}(c))e\left(\frac{c(2a^4-4a^3+6a^2-4a)}{p}\right)$$

$$+p \sum_{\substack{a=0\\a^4+b^4\equiv 1 \bmod p\\a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 - p. \tag{2.15}$$

For any prime $p > 3$, note that the properties of Gauss sums, we have

$$\sum_{a=0}^{p-1}\sum_{c=0}^{p-1}(\lambda(c)+\chi_2(c)+\overline{\lambda}(c))e\left(\frac{c(2a^4-4a^3+6a^2-4a)}{p}\right)$$

$$= \sum_{a=0}^{p-1}\tau(\lambda)\overline{\lambda}(2a^4-4a^3+6a^2-4a) + \sum_{a=0}^{p-1}\tau(\overline{\lambda})\lambda(2a^4-4a^3+6a^2-4a)$$

$$+\sqrt{p}\sum_{a=0}^{p-1}\chi_2(2a^4-4a^3+6a^2-4a)$$

$$= \lambda(8)\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}((2a)^4-4(2a)^3+12(2a)^2-16(2a))$$

$$+\overline{\lambda}(8)\tau(\overline{\lambda})\sum_{a=0}^{p-1}\lambda((2a)^4-4(2a)^3+12(2a)^2-16(2a))$$

$$+\sqrt{p}\chi_2(2)\sum_{a=0}^{p-1}\chi_2((2a)^4-4(2a)^3+12(2a)^2-16(2a))$$

$$= \lambda(8)\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}(a^4-4a^3+12a^2-16a) + \overline{\lambda}(8)\tau(\overline{\lambda})\sum_{a=0}^{p-1}\lambda(a^4-4a^3+12a^2-16a)$$

$$+\sqrt{p}\chi_2(2)\sum_{a=0}^{p-1}\chi_2(a^4-4a^3+12a^2-16a)$$

$$= \lambda(8)\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}((a-1)^4+6(a-1)^2-7) + \overline{\lambda}(8)\tau(\overline{\lambda})\sum_{a=0}^{p-1}\lambda((a-1)^4+6(a-1)^2-7)$$

$$+\sqrt{p}\chi_2(2)\sum_{a=0}^{p-1}\chi_2((a-1)^4+6(a-1)^2-7)$$

$$
\begin{aligned}
&= \lambda(8)\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}(a^4+6a^2-7)+\overline{\lambda}(8)\tau(\overline{\lambda})\sum_{a=0}^{p-1}\lambda(a^4+6a^2-7)+\sqrt{p}\chi_2(2)\sum_{a=0}^{p-1}\chi_2(a^4+6a^2-7)\\
&= \overline{\lambda}(2)\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}(a^4+6a^2-7)+\lambda(2)\tau(\overline{\lambda})\sum_{a=0}^{p-1}\lambda(a^4+6a^2-7)+\sqrt{p}\chi_2(2)\sum_{a=0}^{p-1}\chi_2(a^4+6a^2-7)\\
&= O(p). \qquad (2.16)
\end{aligned}
$$

Combining the formulas (2.15) and (2.16), we have the estimation

$$
\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)=O(p).
$$

This proves Lemma 2.5.

Lemma 2.6. *Let p be an odd prime with $p=4k+3$, χ_2 denotes the Legendre symbol. Then we have the identity*

$$
\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)=(2-\chi_2(7))p-i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a).
$$

Proof. For any prime $p>3$, if $p=4k+3$, then $\chi_2(-1)=-1$ and $\tau(\chi_2)=i\sqrt{p}$, we have

$$
\begin{aligned}
&\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)\\
&= \sum_{\substack{a=0 \\ a\equiv c \bmod p}}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^4-c^4}{p}\right)+\sum_{\substack{a=0 \\ a+b\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a^4+b^4-c^4}{p}\right)\\
&= p+\sum_{\substack{a=0 \\ a+1\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^4(a^4-c^4+1)}{p}\right)\\
&= p+\sum_{\substack{a=0 \\ a+1\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{b^4(a^4-c^4+1)}{p}\right)-\sum_{\substack{a=0 \\ a+1\equiv c \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1\\
&= \sum_{\substack{a=0 \\ a+1\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}(1+\chi_2(b))e\left(\frac{b(a^4-c^4+1)}{p}\right)\\
&= i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(-4a^3-6a^2-4a)+p\sum_{\substack{a=0 \\ a^4+1\equiv c^4 \bmod p \\ a+1\equiv c \bmod p}}^{p-1}\sum_{c=0}^{p-1} 1
\end{aligned}
$$

$$
\begin{aligned}
&= i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(-4a^3-6a^2-4a)+p\sum_{\substack{a=0\\2a^3+3a^2+2a\equiv 0 \bmod p}}^{p-1}1\\
&= i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(-4a^3-6a^2-4a)+p+p\sum_{\substack{a=1\\2a^2+3a+2\equiv 0 \bmod p}}^{p-1}1\\
&= i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(-4a^3-6a^2-4a)+p+p\sum_{\substack{a=0\\(4a+3)^2\equiv -7 \bmod p}}^{p-1}1\\
&= i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(-4a^3-6a^2-4a)+(2-\chi_2(7))p. \qquad (2.17)
\end{aligned}
$$

So for any prime $p>3$, if $p\equiv 3 \bmod 4$, we have the identity

$$
\sum_{\substack{a=0\\a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}e\left(\frac{a^4+b^4-c^4}{p}\right)=(2-\chi_2(7))p-i\sqrt{p}\sum_{a=0}^{p-1}\chi_2(4a^3+6a^2+4a).
$$

This proves Lemma 2.6.

3. Proofs of the theorems

In this section, we will complete the proofs of our theorems. Applying several basic lemmas in Section 2, we can easily deduce our theorems.

$$
\begin{aligned}
\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{a^4+ma}{p}\right)\right|^4 &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{a^4+b^4-c^4-d^4+m(a+b-c-d)}{p}\right)\\
&= p\sum_{\substack{a=0\\a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}e\left(\frac{a^4+b^4-c^4-d^4}{p}\right)\\
&= p\sum_{\substack{a=0\\a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}e\left(\frac{a^4+b^4-c^4}{p}\right)+p\sum_{\substack{a=0\\a+b\equiv c+d \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{a^4+b^4-c^4-d^4}{p}\right)\\
&= p\sum_{\substack{a=0\\a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}e\left(\frac{a^4+b^4-c^4}{p}\right)+p\sum_{\substack{a=0\\a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right)\\
&= p\sum_{\substack{a=0\\a+b\equiv c \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}e\left(\frac{a^4+b^4-c^4}{p}\right)+p\sum_{\substack{a=0\\a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=0}^{p-1}e\left(\frac{d^4(a^4+b^4-c^4-1)}{p}\right)\\
&\quad -p\sum_{\substack{a=0\\a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}1. \qquad (3.1)
\end{aligned}
$$

For any prime p, if $p = 4k + 1$, then from Lemma 2.3 we know that

$$p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4 + b^4 - c^4 - 1)}{p}\right) = 3p^3 + O(p^{\frac{5}{2}}). \tag{3.2}$$

From Lemma 2.5 we know that

$$p \sum_{a=0}^{p-1} \sum_{\substack{b=0 \\ a+b\equiv c \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^4 + b^4 - c^4}{p}\right) = O(p^2). \tag{3.3}$$

Therefore, from the formulas (3.1)–(3.3), we may immediately deduce

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4 + ma}{p}\right) \right|^4 = 2p^3 + O(p^{\frac{5}{2}}),$$

when $p = 4k + 1$. This proves Theorem 1.1.

For any prime $p > 3$, if $p = 4k + 3$, then from Lemma 2.4 we know that

$$p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{\substack{c=0 \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{d=0}^{p-1} e\left(\frac{d^4(a^4 + b^4 - c^4 - 1)}{p}\right)$$

$$= \begin{cases} 3p^3 - 5p^2 + \chi_2(7)p^2 + ip^{\frac{3}{2}} \sum_{b=0}^{p-1} \chi_2(4a^3 + 6a^2 + 4a), & \text{if } p = 12k + 7, \\ 3p^3 - p^2 + \chi_2(7)p^2 + ip^{\frac{3}{2}} \sum_{b=0}^{p-1} \chi_2(4a^3 + 6a^2 + 4a), & \text{if } p = 12k + 11. \end{cases} \tag{3.4}$$

From Lemma 2.6 we know that

$$p \sum_{a=0}^{p-1} \sum_{\substack{b=0 \\ a+b\equiv c \bmod p}}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a^4 + b^4 - c^4}{p}\right) = (2 - \chi_2(7))p^2 - ip^{\frac{3}{2}} \sum_{a=0}^{p-1} \chi_2(4a^3 + 6a^2 + 4a). \tag{3.5}$$

Therefore, from the formulas (3.1), (3.4) and (3.5), for any prime $p > 3$, we may immediately deduce

$$\sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^4 + ma}{p}\right) \right|^4 = \begin{cases} 2p^3 - 3p^2, & \text{if } p = 12k + 7, \\ 2p^3 + p^2, & \text{if } p = 12k + 11, \end{cases}$$

when $p = 4k + 3$. This completes the proof of our main results.

4. Conclusions

This paper mainly proposed two theorems, which are all closely related to the fourth power mean of the two-term exponential sums. Theorem 1.1 obtained an asymptotic formula of $p \equiv 1 \bmod 4$ and an exact formula of $p \equiv 3 \bmod 4$. The skill of this paper is to solve a class of quartic congruence equations in the process of calculation. In general, this work not only calculates a new kind of two-term exponential sum but also lays a foundation for the follow-up study of related problems.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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