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Research article

On weakly bounded well-filtered spaces

Xiaoyuan Zhang[1], Meng Bao[2], Xinpeng Wen[3] and Xiaoquan Xu[4,*]

[1] School of Big Data Science, Hebei Finance University, Baoding 071051, China
[2] College of Mathematics, Sichuan University, Chengdu 610064, China
[3] College of Mathematics and Information, Nanchang Hangkong University, Nanchang 330063, China
[4] Fujian Key Laboratory of Granular Computing and Applications, Minnan Normal University, Zhangzhou 363000, China

* **Correspondence:** E-mail: xiqxu2002@163.com.

Abstract: In [16], using Rudin sets, Miao, Li and Zhao introduced a new concept of weakly well-filtered spaces—k-bounded well-filtered spaces. Now, also using Rudin sets, we introduce another type of T_0 spaces—weakly bounded well-filtered spaces, which are strictly stronger than k-bounded well-filtered spaces. Some basic properties of k-bounded well-filtered spaces and weakly bounded well-filtered spaces are investigated and the relationships among some kinds of weakly sober spaces and weakly well-filtered spaces are posed. It is proved that the category **KBWF** is not reflective in the category $\mathbf{Top}_0$.

Keywords: k-bounded well-filtered space; weakly bounded well-filtered space; Scott topology; Smyth power space; reflection
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1. Introduction

With the development of non-Hausdorff topology, sober spaces, well-filtered spaces and d-spaces form the most important three types of T_0 spaces. During the past few years, a large number of properties of these spaces are investigated (see [1, 3–5, 9, 14, 17, 19, 29, 30, 32, 33]). By the further researches of sober spaces and well-filtered spaces, many classes of weakly sober spaces and weakly well-filtered spaces have been posed and extensively investigated from various different perspectives (see [4, 16, 23, 24, 28, 29, 36–38]). In particular, Xu, Shen, Xi and Zhao introduced and investigated Rudin sets and WD sets for researching well-filtered spaces and gave the characterization of well-filtered spaces by the two kinds of T_0 spaces in [19, 29]. Rudin sets and WD sets play an important role in study of well-filtered spaces and sober spaces (see [19, 27–30, 33]). In [16], using Rudin sets, Miao, Li and Zhao introduced and studied a new kind of weakly well-filtered spaces—*k-bounded well-filtered spaces*, namely, T_0 spaces X in which every nonempty closed Rudin subset A of X that has a sup is the closure of a (unique) point of X.

In this paper, also using Rudin sets, we introduce a new type of weakly well-filtered spaces—weakly bounded well-filtered spaces, which are strictly stronger than k-bounded well-filtered spaces. The main purpose of the paper is to investigate some basic properties of k-bounded well-filtered spaces and weakly bounded well-filtered spaces. It is proved that the category **KBWF** of all k-bounded well-filtered spaces with continuous mappings is not reflective in the category $\mathbf{Top}_0$ of all T_0 spaces with continuous mappings, and also the category $\mathbf{KBWF}_r$ of all k-bounded well-filtered spaces with continuous mappings preserving all existing sups of Rudin sets is not reflective in the category $\mathbf{Top}_r$ of all T_0 spaces with continuous mappings preserving all existing sups of Rudin sets. Moreover, some fundamental properties, such as hereditary, products, retracts, .etc, in k-bounded well-filtered spaces and weakly bounded well-filtered spaces are investigated and the relationships among some weakly sober spaces are posed. It is proved that if the Smyth power space $P_S(X)$ is k-bounded well-filtered (resp., weakly bounded well-filtered), then so is X. Two examples are given to show that the converses do not hold in general.

2. Preliminaries

In this section, we introduce the necessary notations, terminologies and some facts that will be used in the paper. For further details, we refer the reader to [5, 6, 17].

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For arbitrary $x \in P$, let $\downarrow x$ represent $\downarrow\{x\}$ and $\uparrow x$ represent $\uparrow\{x\}$, respectively. We call A a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). For a set X, the cardinality of X is denoted by $|X|$. The set of all natural numbers with the usual order is denoted by $\mathbb{N}$ and $\omega = |\mathbb{N}|$. 2^X denotes the set of all subsets of X. Put $X^{(<\omega)} = \{F \subseteq X : F \text{ is a nonempty finite set}\}$ and $\mathbf{Fin}P = \{\uparrow F : \emptyset \neq F \in P^{(<\omega)}\}$. A subset D of P is *directed* provided that it is nonempty and every finite subset of D has an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. P is said to be a *directed complete poset*, a *dcpo* for short, if every directed subset of P has the least upper bound in P. As in [5], the *upper topology* on a poset P, generated by the complements of the principal ideals of P, is denoted by $\upsilon(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\gamma(P)$. The space $\Gamma P = (P, \gamma(P))$ is called the *Alexandroff space* of P. A subset U of a poset P is called *Scott open* if $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\vee D \in U$ whenever $\vee D$ exists. The topology formed by all Scott open sets of P is called the *Scott topology*, written as $\sigma(P)$. $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. Clearly, ΣP is a T_0 space. For the chain $2 = \{0, 1\}$ (with the order $0 < 1$), we have $\sigma(2) = \{\emptyset, \{1\}, \{0, 1\}\}$. The space $\Sigma 2$ is well-known under the name of *Sierpinski space*.

Given a T_0 space X, we can define a partially order $\leq_X$, called the *specialization order*, which is defined by $x \leq_X y$ iff $x \in \overline{\{y\}}$. Let ΩX denote the poset $(X, \leq_X)$. Clearly, each open set is an upper set and each closed set is a lower set with respect to the partially order $\leq_X$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned in a T_0 space, it is to be interpreted with respect to the specialization order. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X and denote $\mathcal{S}(X) = \{\{x\} : x \in X\}$, $\mathcal{D}(X) = \{D \subseteq X : D \text{ is a directed set of } X\}$. A T_0 space

X is called a d-space (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [5]).

Remark 2.1. Let X be a T_0 space and $C \subseteq X$. Then $\vee C$ exists in X iff $\vee \mathrm{cl}_X(C)$ exists in X. Moreover, If $\vee C$ exists in X, then $\vee \mathrm{cl}_X(C) = \vee C$.

A nonempty subset A of a T_0 space X is said to be *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ always implies $A \subseteq F_1$ or $A \subseteq F_2$. The set of all irreducible (resp., irreducible closed) subsets of X is denoted by $\mathbf{Irr}(X)$ (resp., $\mathbf{Irr}_c(X)$). The space X is said to be *sober*, if for any $A \in \mathbf{Irr}_c(X)$, we can find a unique point $x \in X$ with $A = \overline{\{x\}}$. Put $\mathbf{Top}_0$ the category of all T_0 spaces with continuous mappings and **Sob** the full subcategory of $\mathbf{Top}_0$ containing all sober spaces.

Let X be a topological space, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$. Set $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \cap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. We write $\Diamond A$ and $\Box A$ for $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$, respectively if there is no confusion. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathbf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$.

Remark 2.2. Let X be a T_0 space. The space $X^s = P_H(\mathbf{Irr}_c(X))$ with the canonical mapping $\eta_X : X \longrightarrow X^s$, given by $\eta_X(x) = \overline{\{x\}}$, is the *sobrification* of X (cf. [5, 6]).

A subset K of a topological space X is called *supercompact* if for any family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $K \subseteq \bigcup_{i \in I} U_i$, we can find $i \in I$ satisfying $K \subseteq U_i$. It follows from [8, Fact 2.2] that the supercompact saturated sets of a T_0 space X are exactly the sets $\uparrow x$ with $x \in X$. We call $A \subseteq X$ *saturated* if A equals the intersection of all open sets which contain A (equivalently, A is an upper set in the specialization order). The set of all nonempty compact saturated subsets of X is denoted by $\mathsf{K}(X)$ and is endowed with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. We call a T_0 space X *well-filtered* (WF for short) if for any filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, any open set U and $\bigcap \mathcal{K} \subseteq U$, there exists $K \in \mathcal{K}$ satisfying $K \subseteq U$.

For any topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, the *upper Vietoris topology* on $\mathcal{G}$ is the topology denoted by $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base, and $P_S(\mathcal{G})$ denotes the resulting space. The *Smyth power space* or *upper space* $P_S(\mathsf{K}(X))$ of X is denoted shortly by $P_S(X)$ (cf. [7, 8, 17]). As we all know, the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$).

Rudin's Lemma is a useful and important tool in non-Hausdorff topology and domain theory (see [4–6, 10, 19, 27–29, 35]). Heckmann and Keimel [8] presented the following topological variant of Rudin's Lemma.

Lemma 2.3. *([8, Lemma 3.1]) (Topological Rudin Lemma) Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.*

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \cap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

Definition 2.4. ([29, Definition 4.6]) Let X be a T_0 space. A nonempty subset A of X is said to have *Rudin property*, if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ satisfying $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}(X) = \{A \in \mathcal{C}(X) : A \text{ has Rudin property}\}$. The sets in $\mathsf{RD}(X)$ will also be called *Rudin sets*.

Lemma 2.5. *([33, Lemma 2.7]) Suppose that X, Y are both T_0 spaces and $f : X \to Y$ is a continuous mapping, if $A \in \mathsf{RD}(X)$, then $\overline{f(A)} \in \mathsf{RD}(Y)$.*

3. Some kinds of weakly sober spaces and weakly well-filtered spaces

In this section, the concept of weakly bounded well-filtered spaces is posed and we investigate relationships among some kinds of weakly sober spaces and weakly well-filtered spaces.

Definition 3.1. ([14, Definition 2.1]) Let X be a T_0 space. We call X *bounded well-filtered* (b-WF for short) if, whenever a nonempty open set U contains a filtered intersection $\bigcap_{i\in I} K_i$ of compact saturated subsets, then U contains K_i for some $i \in I$.

Bounded well-filtered spaces were called *weak well-filtered* in [14]. It was shown in [14, Proposition 3.3] that a T_0 space X is bounded well-filtered if and only if for any nonempty open set U and every filtered family $\{K_i\}_{i\in I}$ of nonempty compact saturated subsets with $\bigcap_{i\in I} K_i \neq \emptyset$, $\bigcap_{i\in I} K_i \subseteq U$ implies $K_i \subseteq U$ for some $i \in I$. In order to correspond to the concept of bounded sober spaces, we call it bounded well-filtered here (note that $\{K_i\}_{i\in I} \subseteq \mathsf{K}(X)$ is upper bounded in $\mathsf{K}(X)$ iff $\bigcap_{i\in I} K_i \neq \emptyset$).

Proposition 3.2. *A bounded well-filtered space is saturated-hereditary.*

Proof. Assume that X is a bounded well-filtered space and U is a nonempty saturated subspace of X. Suppose that $\mathcal{K}_U \subseteq \mathsf{K}(U)$ is filtered, $V \in \mathcal{O}(U) \setminus \{\emptyset\}$, and $\bigcap \mathcal{K}_U \subseteq V$. Then we can find $W \in \mathcal{O}(X) \setminus \{\emptyset\}$ with $V = W \cap U$. As $U = \uparrow_X U$, $\mathcal{K}_U \subseteq \mathsf{K}(X)$ and $\bigcap \mathcal{K}_U \subseteq W$. It follows from the bounded well-filteredness of X that there exists $K \in \mathcal{K}_U$ with $K \subseteq W$, and hence $K \subseteq W \cap U = V$. Therefore, U is bounded well-filtered. □

Definition 3.3. ([38, Definition 4.1]) Let X be a T_0 space. We call X *k-bounded sober* (k-b-sober for short) if for any irreducible closed set A with $\vee A$ existing, we can find a unique point $x \in X$ with $A = \mathrm{cl}(\{x\})$.

Definition 3.4. ([16, Definition 4.2]) Let X be a T_0 space. We call X *k-bounded well-filtered* (k-b-WF for short) if for any non-empty closed Rudin subset A with $\vee A$ existing, we can find a unique point $x \in X$ such that $A = \mathrm{cl}(\{x\})$.

Throughout this paper, **KBWF** denotes the category of all k-bounded well-filtered spaces with continuous mappings, $\mathbf{Top}_r$ denotes the category of all T_0 spaces with continuous mappings preserving all existing sups of Rudin sets and $\mathbf{KBWF}_r$ denotes the full subcategory of $\mathbf{Top}_r$ containing all k-bounded well-filtered spaces.

By [33], we know that for a T_0 space X, $\mathcal{S}_c(X) \subseteq \mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathbf{Irr}_c(X)$. So if X is a k-bounded sober space, it is obvious that X is k-bounded well-filtered. Moreover, it was proved in [16] that if a T_0 space X is a k-bounded well-filtered space and X is locally compact, then X is a k-bounded sober space. Clearly, each well-filtered space is k-bounded well-filtered, but the converse is not valid, see [16, Example 4.5].

Definition 3.5. ([37, Definition 2]) A T_0 space is called *bounded sober* (b-sober for short) if every upper bounded closed irreducible set is the closure of a unique singleton.

Proposition 3.6. *Let X be a T_0 space and Y a bounded sober space. Then the function space $\mathbf{Top}_0(X, Y)$ of all continuous functions $f : X \to Y$ equipped with the topology of pointwise convergence is bounded sober.*

Proof. Let A be an upper bounded irreducible subset of $\mathbf{Top}_0(X, Y)$ with the topology induced by the product topology on Y^X. Then $A_x = \pi_x(A) = \{a(x) : a \in A\}$ is irreducible for each $x \in X$ and has an upper bound in Y. Since Y is bounded sober, we can find a unique $a_x \in Y$ with $\mathrm{cl}_Y(A_x) = \mathrm{cl}_Y(\{a_x\})$. Define $f : X \to Y$ by $f(x) = a_x$. We show that the function f is continuous. Let $x \in X$ and $V \in \mathcal{O}(Y)$ with $f(x) = a_x \in V$. It follows from $\overline{\pi_x(A)} = \overline{\{a_x\}}$ that $V \bigcap \pi_x(A) \neq \emptyset$, hence there exists $a \in A$ such that $a(x) \in V$. By the continuity of $a : X \longrightarrow Y$, we can find a $U \in \mathcal{O}(X)$ with $x \in U$ such that $a(z) \in V$ for each $z \in U$. From $a \in A$, it follows that $a(z) \in \pi_z(A) \subseteq \overline{\pi_z(A)} = \overline{\{a_z\}} = \overline{\{f(z)\}}$, and hence $f(z) \in V$ for all $z \in U$, which means that f is continuous. Finally, we show that $\overline{A} = \overline{\{f\}}$ in $\mathbf{Top}_0(X, Y)$ (with the topology induced by the product topology on Y^X). Let $\pi_x^{-1}(U_x)$ ($x \in X$ and $U_x \in \mathcal{O}(Y)$) be arbitrary subbasic open set such that $f \in \pi_x^{-1}(U_x)$. By $f(x) = a_x \in \overline{\pi_x(A)}$, we have $f \in \pi_x^{-1}(\overline{\pi_x(A)}) \bigcap \pi_x^{-1}(U_x) = \pi_x^{-1}(\overline{\pi_x(A)} \bigcap U_x)$. Therefore, $\overline{\pi_x(A)} \bigcap U_x \neq \emptyset$, and hence $\pi_x(A) \bigcap U_x \neq \emptyset$ or, equivalently, $A \bigcap \pi_x^{-1}(U_x) \neq \emptyset$. Since A is irreducible, we deduce that all basic open sets containing f must meet A. It follows that $\overline{A} = \overline{\{f\}}$. We conclude that the space $\mathbf{Top}_0(X, Y)$ is bounded sober. □

Definition 3.7. Let X be a T_0 space. We call X *bounded d-space* (*b*-*d*-space for short) if for every upper bounded $D \in \mathcal{D}(X)$, we can find $x \in X$ such that $\overline{D} = \mathrm{cl}(\{x\})$.

Proposition 3.8. *A bounded d-space is closed-hereditary and saturated-hereditary.*

Proof. Suppose that X is a bounded d-space and A is a closed subspace of X. For each upper bounded directed set D in A, it is clear that D is directed and has an upper bound in X. As X is a bounded d-space, there exists a unique $x \in X$ such that $\mathrm{cl}_X(D) = \mathrm{cl}_X(\{x\})$. As the directed set D has upper bound in A and $A = \downarrow A$, we have $x \in A$. Then $\mathrm{cl}_A(D) = \mathrm{cl}_X(D) \cap A = \mathrm{cl}_X(\{x\}) \cap A = \mathrm{cl}_A(\{x\})$. Hence, A is a bounded d-space.

Let U be a nonempty saturated subspace of X. For each upper bounded directed set D_U in U, it is clear that D_U is directed and has upper bound in X. From X being a bounded d-space, it follows that we can find a unique $x \in X$ such that $\mathrm{cl}_X(D_U) = \mathrm{cl}_X(\{x\})$. We know that $x \in U$ by $U = \uparrow U$. Then $\mathrm{cl}_U(D_U) = \mathrm{cl}_X(D_U) \cap U = \mathrm{cl}_X(\{x\}) \cap U = \mathrm{cl}_U(\{x\})$. Hence U is a bounded d-space. □

Definition 3.9. Let X be a T_0 space. We call X *weakly bounded well-filtered* (*w*-*b*-WF for short) provided that for arbitrary upper bounded Rudin set $A \subseteq X$, there exists $x \in X$ with $A = \mathrm{cl}(\{x\})$.

Since $S_c(X) \subseteq \mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathbf{Irr}_c(X)$ for a T_0 space X by [33, Proposition 2.6], it is obvious that each bounded sober space is weakly bounded well-filtered and each weakly bounded well-filtered space is a bounded d-space. By the following example, the well-known Johnstone's dcpo $\mathbb{J}$ equipped with Scott topology is a weakly bounded well-filtered space but not a well-filtered space.

Example 3.10. Let $\mathbb{J}$ be the well-known dcpo constructed by Johnstone in [11], that is, $\mathbb{J} = \mathbb{N}\times(\mathbb{N}\cup\{\omega\})$ with the order defined as follows: $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. It is well-known that $\Sigma\mathbb{J}$ is a dcpo and a non-sober space. Moreover, it was proved in [14, Example 3.1] that $\Sigma\mathbb{J}$ is not well-filtered. Now we show that $\Sigma\mathbb{J}$ is weakly bounded well-filtered.

In fact, it is straightforward to verify that $\mathbf{Irr}_c(\Sigma\mathbb{J}) = \{\downarrow x : x \in \mathbb{J}\} \cup \{\mathbb{J}\}$. Since $\mathbb{J}$ does not have an upper bound in $\Sigma\mathbb{J}$, we have that $\Sigma\mathbb{J}$ is bounded sober, and hence it is weakly bounded well-filtered.

Theorem 3.11. *Let X be a T_0 space. X is weakly bounded well-filtered if and only if for an upper bounded closed set B and any filtered family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ such that $K_i \cap B \neq \emptyset$ for all $i \in I$, $\bigcap_{i\in I} K_i \cap B \neq \emptyset$.*

Proof. Let X be a weakly bounded well-filtered space, B an upper bounded closed set and $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ a filtered family with $K_i \cap B \neq \emptyset$ for all $i \in I$. By Topological Rudin Lemma, we can find a minimal upper bounded closed subset $C \subseteq B$ with $K_i \cap C \neq \emptyset$ for all $i \in I$. Since X is weakly bounded well-filtered, it is natural to find a unique $x \in X$ with $C = \downarrow x$, then $x \in K_i$ for each $i \in I$, which implies that $x \in \bigcap_{i\in I} K_i$. Therefore, $C \cap (\bigcap_{i\in I} K_i) \neq \emptyset$, and hence $B \cap (\bigcap_{i\in I} K_i) \neq \emptyset$.

Conversely, let A be an upper bounded Rudin set in X. By the definition of Rudin sets, we can find a filtered family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ with $A \in m(\{K_i : i \in I\})$. Therefore, $\bigcap_{i\in I} K_i \cap A \neq \emptyset$. Select a point $x \in \bigcap_{i\in I} K_i \cap A$. Then $\downarrow x \subseteq A$ and $(\downarrow x) \cap K_i \neq \emptyset$. It follows from the minimality of A that $A = \downarrow x$. □

Proposition 3.12. *Every bounded well-filtered space is weakly bounded well-filtered.*

Proof. Assume that X is bounded well-filtered, B is an upper bounded closed set and $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is a filtered family with $K_i \cap B \neq \emptyset$ for all $i \in I$. If $\bigcap_{i\in I} K_i \cap B = \emptyset$, then $\bigcap_{i\in I} K_i \subseteq X \setminus B$. Since B has upper bound, there exists a $c \in X$ with $B \subseteq \downarrow c$. Clearly, $c \in K_i$ for each $i \in I$, which means that $\bigcap_{i\in I} K_i \neq \emptyset$. As X is bounded well-filtered, it is natural that $K_i \subseteq X \setminus B$ for some $i \in I$, which contradicts with $K_i \cap B \neq \emptyset$ for each $i \in I$. Therefore, $\bigcap_{i\in I} K_i \cap B \neq \emptyset$. Thus X is weakly bounded well-filtered by Theorem 3.11. □

The converse of Proposition 3.12 is not valid, see the following example.

Example 3.13. Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ be the Johnstone's dcpo (see Example 3.10). Put $P = \mathbb{J} \cup \{a\}$ and a is incomparable with all elements of $\mathbb{J}$. Then $\{a\} \in \sigma(P)$ and $\mathbf{Irr}_c(\Sigma P) = \{\downarrow x : x \in P\} \cup \{\mathbb{J}\}$. Since $\mathbb{J}$ does not have upper bound in P, ΣP is weakly bounded well-filtered. Set $K_i = \{(j, \omega) : j \geq i\} \cup \{a\}$. Then $\{K_i : i \in \mathbb{N}\}$ is a filtered family and $\{K_i : i \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma P)$. However, $\bigcap_{i\in\mathbb{N}} K_i = \{a\} \in \sigma(P)$ and $K_i \nsubseteq \{a\}$ for each $i \in I$. Therefore, ΣP is not bounded well-filtered.

As every weakly bounded well-filtered space is k-bounded well-filtered, one can directly get the following result by Proposition 3.12.

Corollary 3.14. *Every bounded well-filtered space is k-bounded well-filtered.*

By Example 3.13, we know that the converse of Corollary 3.14 is not valid in general.

Figure 1 shows the relationships among some types of weakly sober spaces.

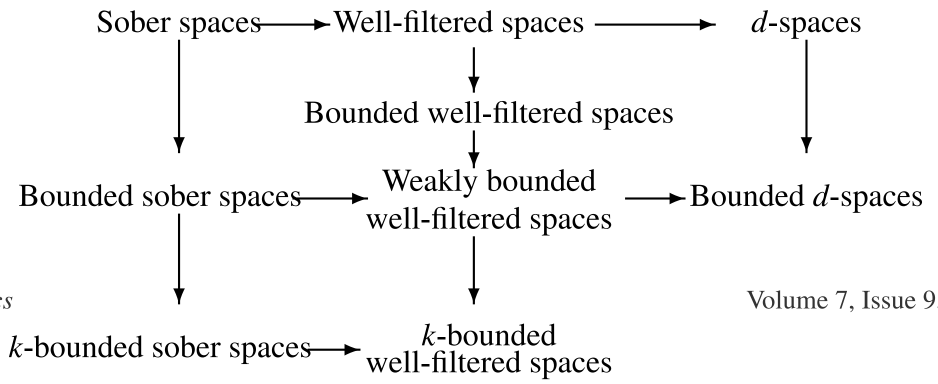


Figure 1. The relationships among weakly sober spaces.

4. Some properties of k-bounded well-filtered spaces and weakly bounded well-filtered spaces

In this section, it is shown that k-bounded well-filteredness and weakly bounded well-filteredness both are saturated-hereditary, but k-bounded well-filteredness is not closed-hereditary. Moreover, weakly bounded well-filtered spaces are closed under retracts and products.

Proposition 4.1. *Suppose that X is k-bounded well-filtered and U is a nonempty saturated subspace of X. Then U is a k-bounded well-filtered space.*

Proof. Let C be a Rudin set in U and $\vee_U C$ exist. Let $u = \vee_U C$. Then u is an upper bound of C in X. For any upper bound v of C in X, since U is a saturated subspace of X and $C \subseteq U$, we have that $v \in U$, and hence $u \leq v$ in U or, equivalently, in X. It follows that $u = \vee_X C$, whence $u = \vee_X \mathrm{cl}_X(C)$ by Remark 2.1. Since C is a Rudin set in U, by Lemma 2.5, $\mathrm{cl}_X(C)$ is a Rudin set in X. From k-bounded well-filteredness of X, it follows that there exists $x \in X$ with $\mathrm{cl}_X(C) = \mathrm{cl}_X(\{x\})$. Since $C \subseteq U = \uparrow U$ and $C \subseteq \downarrow x$, we obtain that $x \in U$ and $C = \mathrm{cl}_U(C) = (\mathrm{cl}_X(C)) \cap U = (\mathrm{cl}_X(\{x\})) \cap U = \mathrm{cl}_U(\{x\})$. Thus as a saturated subspace, U is k-bounded well-filtered. □

The following example shows that k-bounded well-filteredness is not closed-hereditary.

Example 4.2. Suppose that $P = \mathbb{N} \cup \{a, b\}$ and the partial order $\leq$ on P is given as follows (see Figure 2):

(i) $n < n + 1$ for each $n \in \mathbb{N}$,
(ii) $n < a$ and $n < b$ for all $n \in \mathbb{N}$,
(iii) a and b are incomparable.

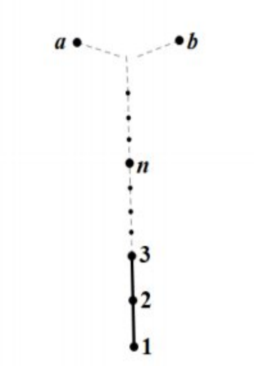


Figure 2. A closed subspace of a k-bounded well-filtered space is not k-bounded well-filtered.

Clearly, $\mathbf{Irr_c}(\Sigma P) = \{\downarrow x : x \in P\} \cup \{\mathbb{N}\}$. The irreducible closed set $\mathbb{N}$ does not have supremum in ΣP, and thus ΣP is k-bounded sober, hence k-bounded well-filtered.

Put $A = \mathbb{N} \cup \{a\}$. Then $A = \mathbb{N} \cup \{a\} = \downarrow a$ is a closed subspace of ΣP and $\mathbb{N}$ is a Rudin set in the subspace A. Indeed, set $\mathcal{K} = \{\uparrow_A n : n \in \mathbb{N}\}$. It is clear that $\mathcal{K} \subseteq \mathsf{K}(A)$ and $\mathbb{N} \in m(\mathcal{K})$. Moreover, $\bigvee_A \mathbb{N} = a$ and $\mathbb{N} \neq \downarrow a = \overline{\{a\}}$. Therefore, A (as a closed subspace of ΣP) is not k-bounded well-filtered.

Definition 4.3. ([5]) A topological space X is a *retract* of a topological space Y provided that there exist two continuous maps $s : X \to Y$ and $r : Y \to X$ with $r \circ s = id_X$.

Next, we give an example to show that the class of k-bounded well-filtered spaces is not closed under retracts.

Example 4.4. Let $X = \Sigma P$ and $Y = (A, \sigma(P)|_A)$ be the two spaces in Example 4.2. $f : X \to Y$ is defined as follows:

$$f(x) = \begin{cases} x, & x \in \mathbb{N}, \\ a, & x \in \{a, b\}, \end{cases}$$

and define $g : Y \to X$ by $g(y) = y$ for each $y \in Y$, that is, g is the identical embedding of Y in X. Clearly, f is continuous and $f \circ g(y) = y$ for each $y \in Y$, that is, $f \circ g = id_Y$. Therefore, Y is a retract of X. It has been shown in Example 4.2 that X is k-bounded well-filtered, but Y is not.

Proposition 4.5. *Suppose that X is a k-bounded well-filtered space and Y is a topological space. If there exist two continuous mappings $f : X \to Y$ and $g : Y \to X$ satisfying $f \circ g = id_Y$ and $g \circ f \leq id_X$, then Y is k-bounded well-filtered.*

Proof. Firstly, it is trivial that Y is a T_0 space.

Secondly, let F be a Rudin set in Y and $\vee_Y F$ exist. Let $a = \vee_Y F$. Since g is continuous, by Lemma 2.5, $\mathrm{cl}_X(g(F))$ is a closed Rudin set in X. We show that $\vee \mathrm{cl}_X(g(F)) = g(a)$. For each $x \in F$, by $a = \vee_Y F$, we have that $x \leq_Y a$ and hence $g(x) \leq_X g(a)$. Therefore, $g(a)$ is an upper bound of $g(F)$. On the other hand, for an arbitrary upper bound b of $g(F)$, and for each $x \in F$, $g(x) \leq_X b$, whence $x = f \circ g(x) \leq_Y f(b)$. Since $g \circ f \leq id_X$, we have $g \circ f(b) \leq b$. Since $\vee_Y F = a \leq_Y f(b)$, we have $g(a) \leq g \circ f(b) \leq b$. Thus $g(a) = \vee \mathrm{cl}_X(g(F))$.

Finally, since X is k-bounded well-filtered, we can find a unique $x \in X$ satisfying $\mathrm{cl}_X(g(F)) = \mathrm{cl}_X(\{x\})$. Then we show that $F = \mathrm{cl}_Y(\{f(x)\})$. On the one hand, $F = f \circ g(F) \subseteq f \circ \mathrm{cl}_X(g(F)) = f(\mathrm{cl}_X(\{x\})) \subseteq \mathrm{cl}_Y(\{f(x)\})$. On the other hand, $f(x) \in f(\mathrm{cl}_X(g(F))) \subseteq \mathrm{cl}_Y(f \circ g(F)) = \mathrm{cl}_Y(\{F\}) = F$, so $\mathrm{cl}_Y(\{f(x)\}) \subseteq F$. Therefore, $F = \mathrm{cl}_Y(\{f(x)\})$, proving that Y is k-bounded well-filtered. □

Example 4.6. Consider the space $X = \Sigma P$ in Example 4.2. We know that X is k-bounded sober and hence k-bounded well-filtered.

Let $f = id_X : X \to X$ be the identity mapping. $g : X \to X$ is defined as follows:

$$g(x) = \begin{cases} x, & x \in \mathbb{N}, \\ a, & x = a, \\ a, & x = b. \end{cases}$$

Clearly, g is continuous and $Z = \{x \in X : f(x) = g(x)\} = \mathbb{N} \cup \{a\}$. It has been shown in Example 4.2 that subspace Z of X is not k-bounded well-filtered.

Theorem 4.7. *Let X_i $(i \in I)$ be T_0 spaces and $X = \prod_{i \in I} X_i$. Then the followings are equivalent:*

(1) *X is k-bounded well-filtered.*
(2) *For all $i \in I$, X_i is k-bounded well-filtered.*

Proof. (1) $\Rightarrow$ (2): For $j \in I$, suppose that F_j is a Rudin set and $\vee_{X_j} F_j$ exists, denoted by $\vee_{X_j} F_j = x_j$. For each $i \in I$, choose $s_i \in X_i$. Put $B = \prod_{i \in I} B_i$, where $B_j = F_j$ and $B_i = \downarrow s_i$ for $i \neq j$. Then by [19, Theorem 2.10], B is a Rudin set in X and $\vee_X B = (a_i)_{i \in I}$, where $a_j = x_j$ and $a_i = s_i$ for $i \neq j$. From the k-bounded well-filteredness of X, it follows that we can find a unique $x = (x_i)_{i \in I} \in X$ with $B = \mathrm{cl}_X(\{x\}) = \prod_{i \in I} \mathrm{cl}_{X_i}(\{x_i\})$. Then $F_j = p_j(B) = \overline{\{x_j\}}$. Thus X_j is k-bounded well-filtered.

(2) ⇒ (1): By [2, Theorem 2.3.11], X is a T_0 space. Let F be a Rudin set in X and $\vee_X F$ exist, denoted by $\vee_X F = b = (b_i)_{i\in I}$. For all $i \in I$, set $p_i : X \to X_i$ to be the ith projection. By Lemma 2.5, $\mathrm{cl}_{X_i}(p_i(F))$ is a Rudin set in X_i.

Claim 1. For each $j \in I$, b_j is the supremum of $p_j(F)$ in X_j.

For $m \in p_j(F)$, there is $x = (x_i)_{i\in I} \in F$ with $m = x_j = p_j(x) \in p_j(F)$. Since $(b_i)_{i\in I} = \vee_X F$, we have $x \leq (b_i)_{i\in I}$. Then $m = x_j = p_j(x) \leq p_j(b) = b_j$, and hence b_j is an upper bound of $p_j(F)$ in X_j. Let s be an upper bound of $p_j(F)$ in X_j and $c = (c_i)_{i\in I}$, where $c_j = s$ and $c_i = b_i$ if $i \neq j$. Then c is an upper bound of F, whence $b \leq c$ and $b_j \leq c_j = s$. Thus $b_j = \vee_{X_j} p_j(F)$.

Claim 2. There is $x \in X$ satisfying $F = \overline{\{x\}}$.

For each $i \in I$, since X_i is k-bounded well-filtered, there is $x_i \in X_i$ with $\mathrm{cl}_{X_i}(p_i(F)) = \mathrm{cl}_{X_i}(\{x_i\})$. Let $x = (x_i)_{i\in I}$. Since F is a Rudin set, and hence a closed irreducible subset, then $F = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(F)) = \prod_{i\in I} \mathrm{cl}_{X_i}(\{x_i\}) = \mathrm{cl}_X(\{x\})$.

Therefore, X is k-bounded well-filtered. □

By the following example, a k-bounded well-filtered space X for which the function space $[X \to X]$ equipped with the topology of pointwise convergence may not be k-bounded well-filtered.

Example 4.8. Let $X = \Sigma P$ be the space in Example 4.2. It was shown in Example 4.2 that X is k-bounded well-filtered. Clearly, $f : X \to X$ is continuous iff $f : P \to P$ is order preserving. For each $i \in \mathbb{N}$, define a mapping $f_i : X \to X$ by

$$f_i(x) = \begin{cases} b, & x \in P \setminus \{1\}, \\ i, & x = 1. \end{cases}$$

Put $\mathcal{D} = \{f_i : i \in \mathbb{N}\}$. It was proved in [24, Example 3.12] that $\mathcal{D}$ is a directed subset in $[X \to X]$ under the pointwise ordering and $\downarrow_{[X\to X]}\mathcal{D}$ is a closed subset in $[X \to X]$ with respect to the topology of pointwise convergence. So $\mathrm{cl}_{[X\to X]}(\mathcal{D}) = \downarrow_{[X\to X]}\mathcal{D}$. By [33, Proposition 2.6], we know that $\mathcal{D}_c(X) \subseteq \mathsf{RD}(X)$ for each T_0 space X. Then $\downarrow_{[X\to X]}\mathcal{D}$ is a Rudin set in $[X \to X]$. Moreover, we see that g is the only upper bound of $\mathcal{D}$ in $[X \to X]$, where $g(x) = b$ for each $x \in P$, and hence $g = \bigvee_{[X\to X]} \mathcal{D}$. However, $\downarrow_{[X\to X]}\mathcal{D} \neq \downarrow_{[X\to X]}\{g\}$. Therefore, $[X \to X]$ of all continuous functions with the topology of pointwise convergence is not k-bounded well-filtered.

Proposition 4.9. *Weakly bounded well-filteredness is closed-hereditary and saturated-hereditary.*

Proof. Suppose that X is a weakly bounded well-filtered space and A is a closed subspace of X. Then A is T_0. Assume that F is a Rudin set of A and has an upper bound in A. Then F is a closed subset of X. By Lemma 2.5, F is a Rudin set in X and also upper bounded in X. By the weakly bounded well-filteredness of X, we can find a unique $x \in X$ with $F = \mathrm{cl}_X(\{x\})$. Since $F \subseteq A$, we have that $x \in A$ and $F = \mathrm{cl}_X(\{x\}) \cap A = \mathrm{cl}_A(\{x\})$. Thus A is weakly bounded well-filtered.

Then assume that $U \subseteq X$ is non-empty saturated and F is a Rudin set of U which has an upper bound in U. Then U is T_0 and by Lemma 2.5, $\mathrm{cl}_X(F)$ is a Rudin set in X. As F has an upper bound in U, we can find $a \in U$ satisfying $F \subseteq \downarrow_U a = \mathrm{cl}_X(\{a\}) \cap U = \downarrow_X a \cap U \subseteq \downarrow_X a$, and hence $\mathrm{cl}_X(F)$ is also upper bounded in X. It follows from the weakly bounded well-filteredness of X that we can choose a unique $x \in X$ with $\mathrm{cl}_X(F) = \mathrm{cl}_X(\{x\})$. Then $F \subseteq \mathrm{cl}_X(\{x\})$. Since U is a saturated subset of X and $F \subseteq U$, we have $x \in U$. Therefore, $F = \mathrm{cl}_U(F) = (\mathrm{cl}_X(F)) \cap U = (\mathrm{cl}_X(\{x\})) \cap U = \mathrm{cl}_U(\{x\})$. Therefore, U is weakly bounded well-filtered. □

Proposition 4.10. *Suppose that X is a weakly bounded well-filtered space and Y is a T_0 space, $Z = \{x \in X : f(x) = g(x)\}$ is weakly bounded well-filtered, where $f, g : X \to Y$ are both continuous mappings.*

Proof. Suppose $Z \neq \emptyset$. Then since Z is a subspace of X, Z is T_0. Assume that $F \in \mathsf{RD}(Z)$ which has an upper bound in Z. By Lemma 2.5, $\mathrm{cl}_X(F)$ is a Rudin set in X. According to the hypothesis, there exists $a \in Z$ satisfying $F \subseteq \downarrow_Z\{a\} = \mathrm{cl}_X(\{a\}) \cap Z = \downarrow_X\{a\} \cap Z \subseteq \downarrow_X\{a\}$, which means that a is an upper bound of F in X. Then the Rudin set $\mathrm{cl}_X(F)$ is upper bounded in X. Since X is weakly bounded well-filtered, we can choose a unique $x \in X$ satisfying $\mathrm{cl}_X(F) = \mathrm{cl}_X(\{x\})$. Then $\mathrm{cl}_Y(\{f(x)\}) = \mathrm{cl}_Y(f(\mathrm{cl}_X(\{x\}))) = \mathrm{cl}_Y(f(\mathrm{cl}_X(F))) = \mathrm{cl}_Y(f(F)) = \mathrm{cl}_Y(g(F)) = \mathrm{cl}_Y(g(\mathrm{cl}_X(F))) = \mathrm{cl}_Y(g(\mathrm{cl}_X(\{x\}))) = \mathrm{cl}_Y(\{g(x)\})$. As Y is T_0, it is clear that $f(x) = g(x)$, and hence $x \in Z$. Thus we obtain that $F = \mathrm{cl}_Z(F) = (\mathrm{cl}_X(F)) \cap Z = (\mathrm{cl}_X(\{x\})) \cap Z = \mathrm{cl}_Z(\{x\})$. In conclusion, Z is weakly bounded well-filtered. □

Proposition 4.11. *The class of weakly bounded well-filtered spaces is closed under retraction.*

Proof. Suppose that X is weakly bounded well-filtered and suppose further that Y is a retract of X. Then we can choose two continuous mappings $f : X \to Y$ and $g : Y \to X$ with $f \circ g = id_Y$. Firstly, as a retract of T_0 space X, Y is a T_0 space. Then let F be an upper bounded Rudin set in Y. It follows from Lemma 2.5 that $\mathrm{cl}_X(g(F))$ is a Rudin set in X. Since F is upper bounded in Y, there exists $b \in Y$ with $F \subseteq \downarrow_Y b$, and hence $g(F) \subseteq g(\downarrow_Y b) \subseteq \downarrow_X g(b)$. Thus $g(F)$ is upper bounded in X. Since X is weakly bounded well-filtered, we can choose a unique $c \in X$ with $\mathrm{cl}_X(g(F)) = \mathrm{cl}_X(\{c\})$. We show that $F = \mathrm{cl}_Y(\{f(c)\})$.

On the one hand, $F = f \circ g(F) \subseteq f(\mathrm{cl}_X(g(F))) = f(\mathrm{cl}_X(\{c\})) \subseteq \mathrm{cl}_Y(\{f(c)\})$. On the other hand, we know that $f(c) \in f(\mathrm{cl}_X(\{c\})) = f(\mathrm{cl}_X(g(F))) \subseteq \mathrm{cl}_Y(f(g(F))) = \mathrm{cl}_Y(F) = F$, whence $\mathrm{cl}_Y(\{f(c)\}) \subseteq F$. Therefore, $F = \mathrm{cl}_Y(\{f(c)\})$. So Y is weakly bounded well-filtered. □

Theorem 4.12. *Let $X_i (i \in I)$ be T_0 spaces and $X = \prod_{i \in I} X_i$. Then the followings are equivalent:*

(1) *X is weakly bounded well-filtered;*
(2) *X_i is weakly bounded well-filtered for all $i \in I$.*

Proof. (1) ⇒ (2): It is trivial by Proposition 4.11.

(2) ⇒ (1): Suppose $A \in \mathsf{RD}(X)$ which has an upper bound $a = (a_i)_{i \in I}$ in X. Then for all $i \in I$, by [29, Lemma 4.13], $p_i(A) \in \mathsf{RD}(X_i)$ and $p_i(A) \subseteq p_i(\downarrow a) \subseteq \downarrow p_i(a) = \downarrow_{X_i} a_i$. For each $i \in I$, since X_i is weakly bounded well-filtered, we can choose a unique $x_i \in X_i$ satisfying $p_i(A) = \mathrm{cl}_{X_i}(\{x_i\})$. Let $x = (x_i)_{i \in I}$. Then it follows from [29, Corollary 2.7] that $A = \prod_{i \in I} p_i(A) = \prod_{i \in I} \mathrm{cl}_{X_i}(\{x_i\}) = \mathrm{cl}_X(\{x\})$. Thus X is weakly bounded well-filtered. □

5. Smyth power spaces of k-bounded well-filtered spaces and weakly bounded well-filtered spaces

The Smyth power spaces are very important structures in domain theory and non-Hausdorff topology, which can describe a demonic view of bounded non-determinism (see [5, 7, 17, 21, 22]). Now we prove that for a T_0 space X, if $P_S(X)$ is k-bounded well-filtered (resp., weakly bounded well-filtered), then so is X. Two examples are given to show that the converses do not hold.

Proposition 5.1. *Let X be a T_0 space. If $P_S(X)$ is k-bounded well-filtered, so is X.*

Proof. Assume that $P_S(X)$ is k-bounded well-filtered and F is a Rudin set in X and $\vee_X F$ exists, denoted by $\vee_X F = a$. Let $\eta_X : X \to P_S(X)$ be defined by $\eta_X(x) = \uparrow x$ for each $x \in X$. It follows from [29, Lemma 4.13] that $\overline{\eta_X(F)} = \overline{\{\uparrow x : x \in F\}}$ is a Rudin set in $P_S(X)$. Then we claim that $\vee\overline{\eta_X(F)}$ exists in $P_S(X)$.

For each $x \in F$, since $\vee_X F = a$, we have $x \leq a$, that is, $\uparrow x \sqsubseteq \uparrow a$. So $\uparrow a$ is an upper bound of $\eta_X(F)$ in $P_S(X)$. For an arbitrary upper bound $G \in \mathsf{K}(X)$ of $\eta_X(F)$, and for each $x \in F$, $\uparrow x \sqsubseteq G$, that is, $G \subseteq \uparrow x$. Therefore, $G \subseteq \bigcap_{x\in F} \uparrow x = \uparrow \vee_X F = \uparrow a$, i.e., $\uparrow a \sqsubseteq G$. So $\vee\eta_X(F) = \uparrow a$, and hence $\vee\overline{\eta_X(F)} = \uparrow a$ in $P_S(X)$ by Remark 2.1. Since $P_S(X)$ is k-bounded well-filtered, we can choose $K \in \mathsf{K}(X)$ satisfying $\overline{\eta_X(F)} = \mathrm{cl}(\{K\})$. Therefore, $\uparrow a = \vee\overline{\eta_X(F)} = \vee\mathrm{cl}(\{K\}) = K$ and hence $\overline{\eta_X(F)} = \mathrm{cl}(\{\uparrow a\})$. Let $U \in \mathcal{O}(X)$, then

$$\begin{aligned} F \cap U \neq \emptyset &\Leftrightarrow \eta_X(F) \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \{\uparrow a\} \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \uparrow a \in \Diamond U \\ &\Leftrightarrow \{a\} \cap U \neq \emptyset. \end{aligned}$$

Thus, $F = \mathrm{cl}_X(F) = \mathrm{cl}_X(\{a\})$, that is, X is k-bounded well-filtered. □

Example 5.2. Suppose that $X = \Sigma P$ is the space in Example 4.2. It is shown in Example 4.2 that X is k-bounded well-filtered. Clearly, $\mathsf{K}(X) = \{\uparrow x : x \in P\} \cup \{\{a, b\}\}$.

Put $Q = \mathbb{N} \cup \{a, b, c\}$ and the order $\leq$ on Q is defined by the following (see Figure 3):

(a) a and b are incomparable,
(b) $c < a$ and $c < b$,
(c) $n < c$ for each $n \in \mathbb{N}$,
(d) $n < n + 1$ for all $n \in \mathbb{N}$.

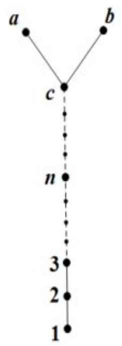


Figure 3. The dcpo Q.

Then Q is a dcpo. Let $f : P_S(\Sigma P) \to \Gamma Q$ be defined as the following

$$f(x) = \begin{cases} n & x = \uparrow n \ (n \in \mathbb{N}), \\ a & x = \{a\}, \\ b & x = \{b\}, \\ c & x = \{a, b\}. \end{cases}$$

Clearly, f is a homeomorphism. In ΓQ, it is clear that $\mathbb{N}$ is a Rudin set (indeed, $\mathbb{N} \in m(\{\uparrow_Q n : n \in \mathbb{N}\})$) and $c = \bigvee_Q^{\uparrow} \mathbb{N}$. However, for any $x \in Q$, $\mathbb{N} \neq \downarrow_Q x = \mathrm{cl}_{\Gamma Q}(\{x\})$. Therefore, ΓQ is not k-bounded well-filtered and hence $P_S(\mathsf{K}(X))$ is not k-bounded well-filtered.

Theorem 5.3. *For a T_0 space X, if $P_S(X)$ is weakly bounded well-filtered, then so is X.*

Proof. Suppose that $A \in \mathsf{RD}(X)$ which has an upper bound a in X. Then by Lemma 2.5, $\overline{\eta_X(A)} = \overline{\{\uparrow a : a \in A\}}$ is a Rudin set in $P_S(X)$. It is clear that $\uparrow_X a$ is an upper bound of $\overline{\eta_X(A)}$ in $P_S(X)$. Since $P_S(X)$ is weakly bounded well-filtered, we can find a unique $K \in \mathsf{K}(X)$ satisfying $\mathrm{cl}_{P_S(X)}(\eta_X(A)) = \mathrm{cl}_{P_S(X)}(\{K\})$.

Now we show that K is supercompact. Let $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ with $K \subseteq \bigcup_{i\in I} U_i$, i.e., $K \in \Box \bigcup_{i\in I} U_i$. By $\mathrm{cl}_{P_S(X)}(\eta_X(A)) = \mathrm{cl}_{P_S(X)}(\{K\})$, we have that $\{\uparrow a : a \in A\} \cap \Box \bigcup_{i\in I} U_i \neq \emptyset$. Then there exists $a_0 \in A$ and $i_0 \in I$ such that $a_0 \in U_{i_0}$, that is, $\uparrow a_0 \in \Box U_{i_0}$. By $\uparrow a_0 \in \mathrm{cl}_{P_S(X)}(\eta_X(A)) = \mathrm{cl}_{P_S(X)}(\{K\})$, $\{K\} \cap \Box U_{i_0} \neq \emptyset$ or, equivalently, $K \subseteq U_{i_0}$. Thus K is supercompact, and hence, by [8, Fact 2.2], $K = \uparrow x$ for some $x \in X$, whence $\mathrm{cl}_{P_S(X)}(\eta_X(A)) = \mathrm{cl}_{P_S(X)}(\{\uparrow x\})$.

Finally, we verify that $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$. Let $U \in \mathcal{O}(X)$. By $\mathrm{cl}_{P_S(X)}(\eta_X(A)) = \mathrm{cl}_{P_S(X)}(\{\uparrow x\})$, we have that

$$\begin{aligned} A \cap U \neq \emptyset &\Leftrightarrow \eta_X(A) \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \{\uparrow x\} \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \uparrow x \in \Diamond U \\ &\Leftrightarrow \{x\} \cap U \neq \emptyset. \end{aligned}$$

Thus, $A = \mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$ and X is weakly bounded well-filtered. □

Example 5.4. Let $Y = \{a_1, a_2, \cdots, a_n, \cdots\}$ and $X = \mathbb{N} \cup Y$. The order $\leq$ on X is defined as the following (see Figure 4):

(i) $n < n + 1$ for all $n \in \mathbb{N}$,
(ii) for any $n, m \in \mathbb{N}$ with $n \neq m$, a_n and a_m are incomparable,
(iii) for any $n, m \in \mathbb{N}$, n and a_m are incomparable.

Let X be endowed with the topology $\tau = \upsilon(X) \bigvee 2^Y$. It is straightforward to verify that the specialization order of (X, τ) agrees with the original order of X and $\mathbf{Irr}_c((X, \tau)) = \{\downarrow x : x \in X\} \cup \mathbb{N}$. Since $\mathbb{N}$ is not upper bounded in X, it is clear that (X, τ) is bounded sober and hence weakly bounded well-filtered. Now we show that $P_S((X, \tau))$ is not weakly bounded well-filtered.

Let $\mathcal{K} = \{\uparrow n \cup Y : n \in \mathbb{N}\}$. We first verify that $\mathcal{K} \subseteq \mathsf{K}((X, \tau))$. For each $n \in \mathbb{N}$, if $\uparrow n \cup Y \subseteq \bigcup_{i\in I}(U_i \cup V_i) = \bigcup_{i\in I} U_i \cup \bigcup_{i\in I} V_i$, where $U_i \in \upsilon(X)$ and $V_i \in 2^Y$, then $\uparrow n \subseteq \bigcup_{i\in I} U_i$. We can choose $i_0 \in I$ such that $\uparrow n \subseteq U_{i_0}$. Then there exists a finite set F in X with $n \in X \setminus \downarrow F \subseteq U_{i_0}$. Then there is $J \in I^{(<\omega)}$ with $\uparrow n \cup Y \subseteq \bigcup_{i\in J}(U_i \cup V_i)$, which means that $\uparrow n \cup Y$ is compact, whence a compact saturated set.

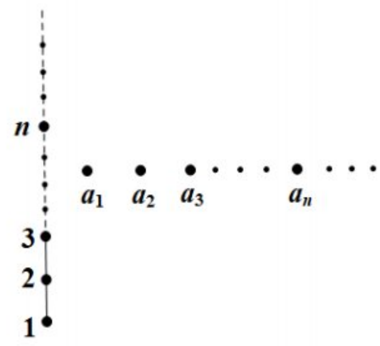


Figure 4. X is bounded well-filtered but $P_S(X)$ is not weakly bounded well-filtered.

Since $X = \uparrow 1 \cup Y \sqsubseteq \uparrow 2 \cup Y \sqsubseteq \cdots \uparrow n \cup Y \sqsubseteq \cdots$, we know that $\mathcal{K}$ is directed. Then we claim that $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \Diamond\{a_n\}$. Clearly, $K \subseteq \bigcap_{n\in\mathbb{N}} \Diamond\{a_n\}$. On the other hand, if $K \in \mathsf{K}((X,\tau))$ and $K \in \bigcap_{n\in\mathbb{N}} \Diamond\{a_n\}$, then for each $n \in \mathbb{N}$, $a_n \in K$. So $Y \subseteq K$. Since $K \in \mathsf{K}((X,\tau))$, $K \neq Y$ (note $\{a_n\} \in \tau$ for each $n \in \mathbb{N}$ and $Y = \bigcup_{n\in\mathbb{N}}\{a_n\}$), whence there exists $m \in \mathbb{N}$ with $m \in K$. Let $m_0 = min\{n \in \mathbb{N} : n \in K\}$. Then $K = \uparrow m_0 \cup Y \in \mathcal{K}$. Hence, $\bigcap_{n\in\mathbb{N}} \Diamond\{a_n\} \subseteq \mathcal{K}$. Thus $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \Diamond\{a_n\}$.

For each $n \in \mathbb{N}$, $\{a_n\} = \downarrow_X a_n$ is τ-closed. So $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \Diamond\{a_n\}$ is closed in $P_S((X,\tau))$. Moreover, since $\bigcap_{n\in\mathbb{N}}(\uparrow n \cup Y) = Y \in \tau$, we know that $\mathcal{K}$ is upper bounded in $P_S(X)$ (for each $n \in \mathbb{N}$, $\{a_n\} = \uparrow_X a_n$ is an upper bound of $\mathcal{K}$), hence $\mathcal{K}$ is an upper bounded Rudin set in $P_S(X)$. Suppose that there exists $G \in \mathsf{K}((X,\tau))$ with $\mathcal{K} = \mathrm{cl}_{P_S(X)}(\{G\})$, then $G \subseteq \bigcap_{n\in\mathbb{N}}(\uparrow n \cup Y) = Y$, i.e., $G \in \Box Y$, and $\Box Y$ is open in $P_S(X)$, but $\uparrow n \cup Y \nsubseteq Y$ for all $n \in \mathbb{N}$, contradicting with $\mathcal{K} = \mathrm{cl}_{P_S(X)}(\{G\})$. Therefore, $P_S((X,\tau))$ is not weakly bounded well-filtered.

6. Non-reflectivity of the category of k-bounded well-filtered spaces

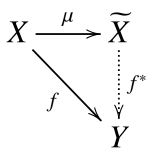
In this section, we mainly consider the following two questions.

Question 6.1. Is **KBWF** reflective in $\mathbf{Top}_0$?

Question 6.2. Is $\mathbf{KBWF}_r$ reflective in $\mathbf{Top}_r$?

Then, we give negative answers to Questions 6.1 and 6.2, respectively. In this section, **K** always denotes a full subcategory of $\mathbf{Top}_0$, and we call the objects of **K** **K**-*spaces*. Moreover, if homeomorphic copies of **K**-spaces are still **K**-spaces, it will be called *closed with respect to homeomorphisms*.

Definition 6.3. ([27, Definition 4.1]) A **K**-*reflection* of a T_0 space X is a pair $\langle \tilde{X}, \mu \rangle$ consisted by a **K**-space $\tilde{X}$ and a continuous mapping $\mu : X \to \tilde{X}$ which satisfy that for any continuous mapping $f : X \to Y$ to a **K**-space, we can choose a unique continuous mapping $f^* : \tilde{X} \to Y$ with $f^* \circ \mu = f$.



If **K**-reflections exist, are unique up to homeomorphism and denoted by X^k.

Definition 6.4. [31, Definition 4.3] Suppose that a T_0 space X has a greatest element $\top_X$ and $X \setminus \{\top_X\} \in \mathbf{Irr}_c(X)$. Let $\top$ satisfy $\top \notin X$. Then $(C(X) \setminus \{X \setminus \{\top_X\}\}) \bigcup \{X \cup \{\top\}\}$ is a topology on $X \cup \{\top\}$. $X^\natural_\top$ denotes the resulting space. For each $x \in X \setminus \{\top_X\}$, let the mapping $\eta^\natural_X : X \to X^\natural_\top$ be such that $\eta^\natural_X(x) = x$ and $\eta^\natural_X(\top_X) = \top$. It is straightforward to prove that $\eta^\natural_X$ is a topological embedding.

Definition 6.5. [31, Definition 4.4] We call a T_0 space X a $\mathbf{K}^\neg$-*space* if it satisfies the following four conditions:

(1) X is not a $\mathbf{K}$-space.
(2) X (with the specialization order) has a greatest element $\top_X$.
(3) $X \setminus \{\top_X\} \in \mathbf{Irr}_c(X)$.
(4) $\overline{\{x\}} \neq X \setminus \{\top_X\}$ for each $x \in X$, or equivalently, $X \setminus \{\top_X\}$ has no greatest element.

Since $\overline{\{\top_X\}} = \downarrow \top_X = X \neq X \setminus \{\top_X\}$, condition (4) in Definition 6.5 is equivalent to the condition: $\overline{\{x\}} \neq X \setminus \{\top_X\}$ for each $x \in X \setminus \{\top_X\}$.

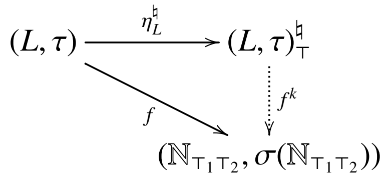
Theorem 6.6. *The* **KBWF***-reflection of* $(\mathbb{N} \cup \{\top_\mathbb{N}\}, \sigma(\mathbb{N} \cup \{\top_\mathbb{N}\}) \cup \{\top_\mathbb{N}\})$ *does not exist.*

Proof. Put $L = \mathbb{N} \cup \{\top_\mathbb{N}\}$. The order on L is defined by $1 < 2 < 3 < \cdots < n < n+1 < \cdots < \top_\mathbb{N}$ and endow L with the topology (as the set of all closed sets) $\tau = \{\downarrow n : n \in \mathbb{N}\} \cup \{\emptyset, L\} \cup \{\mathbb{N}\}$ (clearly, the set of all open subsets is $\upsilon(L) \cup \{\{\top_\mathbb{N}\}\} = \sigma(L) \cup \{\{\top_\mathbb{N}\}\}$). Then

(a) (L, τ) is not k-bounded well-filtered. Indeed, it is straightforward to verify that $\mathcal{O}((L, \tau)) = \{\uparrow_L x : x \in L\} \cup \{\emptyset\}$ and $\mathsf{K}(X) = \{\uparrow_L x : x \in L\}$. Since $\mathbb{N} \in m(\{\uparrow_L n : n \in \mathbb{N}\})$, $\mathbb{N}$ is a Rudin set in (L, τ) and $\bigvee \mathbb{N} = \top_\mathbb{N}$. As $\mathbb{N} \neq \downarrow x$ for all $x \in L$, (L, τ) is not k-bounded well-filtered.
(b) (L, τ) has a greatest element $\top_{(L,\tau)}$.
Clearly, the specialization order of (L, τ) agrees with the original order on L. Whence (L, τ) has a greatest element $\top_{(L,\tau)}$, namely, the element $\top_\mathbb{N}$.
(c) $\mathbf{Irr}_c((L, \tau)) = \{\downarrow x : x \in L\} \cup \{\mathbb{N}\} = \{\downarrow x : x \in L\} \cup \{L \setminus \{\top_{(L,\tau)}\}$ (note that $L = \downarrow \top_\mathbb{N}$). Whence $L \setminus \{\top_\mathbb{N}\} = \mathbb{N} \in \mathbf{Irr}_c((L, \tau))$.
(d) $\overline{\{x\}} = \downarrow x \neq \mathbb{N} = L \setminus \{\top_\mathbb{N}\}$ for each $x \in L$.
(e) (L, τ) is a $\mathbf{KBWF}^\neg$-space and $\mathbf{Irr}_c((L, \tau)) = \{\downarrow x : x \in L\} \cup \{\mathbb{N}\} = \{\downarrow x : x \in L\} \cup \{L \setminus \{\top_{(L,\tau)}\}$.
By (a)-(d), (L, τ) is a $\mathbf{KBWF}^\neg$-space and $\mathbf{Irr}_c((L, \tau)) = \{\downarrow x : x \in L\} \cup \{\mathbb{N}\} = \{\downarrow x : x \in L\} \cup \{L \setminus \{\top_{(L,\tau)}\}$.
(f) $\langle (L, \tau)^\natural_\top, \eta^\natural_L \rangle$ is a sobrification of (L, τ), where $\eta^\natural_L : (L, \tau) \to (L, \tau)^\natural_\top$ is defined by $\eta^\natural_L(x) = x$ for each $x \in \mathbb{N}$ and $\eta^\natural_L(\top_\mathbb{N}) = \top$.
By (e) and [31, Corollary 4.15], $\langle (L, \tau)^\natural_\top, \eta^\natural_L \rangle$ is a sobrification of (L, τ).
(g) $\langle (L, \tau)^\natural_\top, \eta^\natural_L \rangle$ is not a $\mathbf{KBWF}$-reflection of (L, τ).
Assume, on the contrary, that $\langle (L, \tau)^\natural_\top, \eta^\natural_L \rangle$ is a $\mathbf{KBWF}$-reflection of (L, τ). Let $\mathbb{N}_{\top_1\top_2} = \mathbb{N} \cup \{\top_1, \top_2\}$. Define an order on $\mathbb{N}_{\top_1\top_2}$ by $n < n+1$ and $n < \top_1, n < \top_2$ for any $n \in \mathbb{N}$. Endow $\mathbb{N}_{\top_1\top_2}$ with the Scott topology $\sigma(\mathbb{N}_{\top_1\top_2})$. It is proved in Example 4.2 that $(\mathbb{N}_{\top_1\top_2}, \sigma(\mathbb{N}_{\top_1\top_2}))$ is k-bounded well-filtered.
Define a mapping $f : (L, \tau) \to (\mathbb{N}_{\top_1\top_2}, \sigma(\mathbb{N}_{\top_1\top_2}))$ by

$$f(x) = \begin{cases} n & x = n \in \mathbb{N}, \\ \top_1 & x = \top_\mathbb{N}. \end{cases}$$

It is easy to see that f is a topological embedding. Since $\langle (L,\tau)^{\natural}_{\top}, \eta^{\natural}_{L}\rangle$ is a **KBWF**-reflection of (L,τ), there is a unique $f^k : (L,\tau)^{\natural}_{\top} \to ((\mathbb{N}_{\top_1\top_2}, \sigma(\mathbb{N}_{\top_1\top_2}))$ such that $f = f^k \circ \eta^{\natural}_{L}$, that is, the following diagram commutes.



Then $f^k(n) = f^k(\eta^{\natural}_{L}(n)) = f(n) = n$ for each $n \in \mathbb{N}$ and $f^k(\top) = f^k(\eta^{\natural}_{L}(\top_{\mathbb{N}})) = f(\top_{\mathbb{N}}) = \top_1$. For each $n \in \mathbb{N}$, since $n < \top_{\mathbb{N}} < \top$ in $(L,\tau)^{\natural}_{\top}$, we have that $n = f^k(n) \leq f^k(\top_{\mathbb{N}}) \leq f^k(\top) = \top_1$ in $(\mathbb{N}_{\top_1\top_2}, \sigma(\mathbb{N}_{\top_1\top_2}))$. Whence $f^k(\top_{\mathbb{N}})$ is an upper bound of $\mathbb{N}$ in $\mathbb{N}_{\top_1\top_2}$ and $f^k(\top_{\mathbb{N}}) \leq \top_1$. So $f^k(\top_{\mathbb{N}}) = \top_1$ and hence $(f^k)^{-1}(\mathbb{N}) = \mathbb{N} \notin C((L,\tau)^{\natural}_{\top})$ (note that $\mathbb{N}$ is closed in $(\mathbb{N}_{\top_1\top_2}, \sigma(\mathbb{N}_{\top_1\top_2}))$), which contradicts the continuity of f^k.

Thus $\langle (L,\tau)^{\natural}_{\top}, \eta^{\natural}_{L}\rangle$ is not a **KBWF**-reflection of (L,τ).

(h) The **KBWF**-reflection of (L,τ) does not exist.

By (e), (f), (g) and [31, Corollary 4.15], the **KBWF**-reflection of (L,τ) does not exist.

□

Corollary 6.7. **KBWF** *is not reflective in* $\mathbf{Top}_0$.

Remark 6.8. Corollary 6.7 can be obtained by [20, Theorem 2.14, Example 3.7] in a different way. In fact, let X and X_n be the spaces in [20, Example 3.7]. It was proved in [20, Example 3.7] that each X_n is a k-bounded sober subspace of X and hence a k-bounded well-filtered subspace of X. Let $Y = \bigcap_{n\geq 2} X_n = [0,1) \cup \{2\}$, then $\bigvee_Y F = 2$ and $F = [0,1)$ is a directed closed set in Y, but $\forall x \in Y, F \neq \mathrm{cl}_Y(\{x\})$. Therefore, $Y = \bigcap_{n\geq 2} X_n$ is not a k-bounded well-filtered space. That is, the category **KBWF** does not satisfies (K3). It follows from [20, Theorem 2.14] that **KBWF** is not reflective in $\mathbf{Top}_0$.

Then, in [16, Theorem 3.3], it was proved that **KBSob** is not reflective in $\mathbf{Top}_k$ (continuous mappings preserving all existing sups of irreducible sets). In this paper, $\mathbf{Top}_r$ denotes the category of all T_0 spaces with continuous mappings preserving all existing sups of Rudin sets. Similarly, we show that $\mathbf{KBWF}_r$ is not reflective in $\mathbf{Top}_r$, which gives a negative answer to Question 6.2.

Example 6.9. Let $P = \mathbb{J} \cup \{\top_1\}$ be the subset of $L = \mathbb{J} \cup \{\top_1, \top_2\}$ in [16, Lemma 3.1], where $\mathbb{J}$ is the well-known Johnstone's dcpo. In $\Gamma P = (P, \gamma(P))$, $B = \{(1,n) : n \in \mathbb{N}\}$ is an irreducible closed set with $\vee B = (1,\omega)$. In ΓP, a subset is irreducible if and only if it is directed. Then, we know that B is a directed closed set. Therefore, B is a Rudin set and $\vee B = (1,\omega)$. However, for all $x \in P$, $B \neq \downarrow x$. Therefore, ΓP is not k-bounded well-filtered.

Theorem 6.10. *The* $\mathbf{KBWF}_r$*-reflection of* $(P, \gamma(P))$ *does not exist.*

Proof. Let $X = \Gamma P$ in Example 6.9 and $Y = \Sigma L$ in [16, Lemma 3.1]. By [16, Lemma 3.1], Y is k-bounded well-filtered. We assume that the $\mathbf{KBWF}_r$-reflection of ΓP exists, then let $\alpha_X : X \to X^k$ be the reflection of X in $\mathbf{KBWF}_r$. Let $f : X \to Y$ be such that $f(x) = x$ for each $x \in X$. Then f is continuous and $f(\vee_X D) = \vee_X D = \vee_Y D = \vee_Y f(D)$, where D is a directed subset of P. So, f is a homomorphism in the category $\mathbf{Top}_r$. We can find a unique $f^* : X^k \to Y$ with $f = f^* \circ \alpha_X$. Moreover, it was proved in [16, Theorem 3.3] that Y and X^k are homeomorphic.

Suppose that $Z = \Sigma 2$ is the Sierpinski space, f_0 is the constant mapping which maps x to 0 for each $x \in X$ and f^* is the constant mapping mapping y to 0 for each $y \in Y$. Let $g : Y \to Z$ be such that

$$g(x) = \begin{cases} 0, & x \in P; \\ 1, & x = \top_2. \end{cases}$$

Then $f_0 = f_0^* \circ f = g \circ f$, which is a contradiction since f_0^* is unique. In conclusion, the $\mathbf{KBWF}_r$-reflection of ΓP does not exist. □

Corollary 6.11. $\mathbf{KBWF}_r$ *is not reflective in* $\mathbf{Top}_r$.

7. Conclusions

In this paper, based on Rudin sets, we investigate the relationships among some weakly sober spaces and mainly study some properties of k-bounded well-filtered spaces and weakly bounded well-filtered spaces. Combining some previous papers, such as [5, 11–13, 15, 18, 19, 23, 25, 26, 32, 34–38], we obtain the following Table 1. In this table, " + " means that the property is preserved, and " − " denotes that the property is not preserved in spaces. Moreover, "?" denotes that we do not know the related results and it need to study for further. Moreover, in this paper, we do not consider the property of reflection of weakly bounded well-filtered spaces and lead it to be researched in the future.

Table 1. Some properties of kinds of spaces.

Item	Closed heredity	Saturated heredity	Retract	Product	Functional space	Smyth power construction	$P_S(X)$ property $T \Rightarrow X$ property T	Reflection
Sobriety	+	+	+	+	+	+	+	+
b-sobriety	+	+	+	+	+	+	−	+
k-b-sobriety	−	+	−	+	−	+	−	−
WF spaces	+	+	+	+	+	+	+	+
b-WF spaces	?	+	+	?	?	+	+	?
w-b-WF spaces	+	+	+	+	?	+	−	?
k-b-WF spaces	−	+	−	+	−	+	−	−
d-spaces	+	+	+	+	+	+	−	+
b-d-spaces	+	+	+	+	?	+	−	?
Almost sobriety	−	+	+	+	−	−	−	−

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Conflict of interest

All authors declare that there is no conflict of interest.

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