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Research article

Bifurcation curves of positive solutions for one-dimensional Minkowski curvature problem

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Abstract: In this paper, we study the shape of the bifurcation curves of positive solutions for the one-dimensional Minkowski-curvature problem. By developing some new time mapping techniques, we find that the bifurcation curve is ⊂-shaped/monotone increasing/S-like shaped on the $(\lambda, \|u\|_\infty)$ plane when the nonlinearity satisfies different assumptions. Finally, two examples are given to illustrate our result.

Keywords: mean curvature equation; positive solutions; bifurcation curve; time mapping; existence
Mathematics Subject Classification: 34C23, 35J60, 34B18

1. Introduction

In this paper, we study the shapes of bifurcation curves of positive solutions for the one-dimensional Minkowski curvature problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda f(u), \quad x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter, $L > 0$ is a constant and $f : [0, \infty) \to \mathbb{R}$ is a continuous function. By a solution of Problem (1.1), we understand that it is a function that belongs to $C^1[0, 1]$ with $\|u'\|_\infty < 1$, such that $u'/\sqrt{1-u'^2}$ is differentiable and Problem (1.1) is satisfied.

Notice that Problem (1.1) is the one-dimensional version of the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

Problem (1.2) comes from the study of spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1}$ with a prescribed mean extrinsic curvature (see [7]), where $\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ is endowed with the Lorentzian metric $\sum_{i=1}^N (dx_i)^2 - (dt)^2$. These kinds of problems originate from classical relativity. To determine the existence and regularity properties of maximal and constant mean curvature hypersurfaces is very important in classical relativity. These are spacelike submanifolds of codimension one in the spacetime manifold, with the property that the trace of the extrinsic curvature is zero and constant respectively. Such surfaces are important because they provide Riemannian submanifolds with properties that reflect those of the spacetime. Recently, a great deal of research has been devoted to the study of these types of problems; see [8–10, 23] for zero or constant curvature, and [2, 3] for variable curvature.

Recently, work led by Huang [15–18] used the time mapping to study the classification and evolution of bifurcation curves of positive solutions for Problem (1.1), where $\lambda > 0$, $f \in C[0,\infty) \cap C^2(0,\infty)$ and $f(u) > 0$ for $u \geq 0$. However, to the authors' best knowledge, most of the results in the above-mentioned references are focused on f having a fixed sign while few works have considered that f may change its sign (see [3, 4, 20]).

On the other hand, the semilinear elliptic boundary value problem

$$\triangle u + \lambda f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

and its special case

$$u'' + \lambda f(u) = 0,\ t \in (0,1),\ \ u(0) = u(1) = 0, \tag{1.4}$$

have been extensively studied since early 1970s (see [1,5,6,19]). In 1979, Brown and Budin [6] applied the quadrature arguments to obtain the following result.

Theorem 1.1. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *satisfies the following:*

(A1) f *has a continuous derivative;*

(A2) $f(0) > 0$*;*

(A3) There exist $a_1, a_2, \cdots a_n \in \mathbb{R}$ *such that* $0 < a_1 < a_2 < \cdots < a_n$ *and* $f(a_i) \leq 0$ *for* $i = 1, 2, \cdots, n$*;*

(A4) If $F(u) := \int_0^u f(s)ds$*, there exist* $b_1, \cdots, b_{n-1}$ *with* $a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n$ *such that*

$$f(b_i) > 0 \text{ and } F(b_i) > F(u) \text{ for } 0 \leq u < b_i,\ i = 1, 2, \cdots, n-1.$$

Then,

(a) For all $\lambda > 0$*, there exists a solution* (λ, u) *of Problem (1.4);*

(b) If $\lambda > \inf\{\lambda(\rho) | \rho \in (\alpha_i, \beta_i)\}$*, there exist at least two solutions* (λ, u) *of Problem (1.4) such that*

$$\alpha_i \leq \|u\|_\infty \leq \beta_i,\ i = 1, 2, \cdots, n-1,$$

where

$$\beta_i = \inf\{u | u > b_i, f(u) = 0\}, \quad \alpha_i = \inf\{u | (u, \beta_i) \subseteq \mathcal{S}\}, \tag{1.5}$$

and $\mathcal{S} := \{u | u > 0, f(u) > 0, F(u) > F(s) \text{ for all } s : 0 \le s < u\}$, $\|u\|_\infty := \max_{t\in[0,1]} u(t)$*;*

(c) If (λ, u) *is any solution of Problem (1.4) such that* $\alpha_i \le \|u\|_\infty \le \beta_i$*, then* $\lambda > 4\alpha_i M_f^{-1}$*, where* $M_f = \sup\{|f(u)| | 0 \le u \le \beta_i\}$.

However, to the best of our knowledge, the one-dimensional Minkowski-curvature problem given by Problem (1.1) wherein f undergoes a sign change, in spite of its simple looking structure, is considered to be a hard problem in the literature. One of the difficulties is related to developing some new time mapping techniques. The other difficulty is how to prove the direction of bifurcation curves.

Motivated by the interesting studies of [3–5, 8, 15, 16, 20, 21] and some earlier works in the literature (see in particular [6] and the references therein), here, we continue the investigations into the bifurcation curves of positive solutions for Problem (1.1) when f may change its sign. To the best of our knowledge, such a scheme is completely new and has not been described before for related problems.

Throughout, we assume the following:

(H1) $f \in C[0,\infty) \cap C^2(0,\infty)$;

(H2) There exist $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that $0 < a_1 < a_2 < \cdots < a_n < L$ and $f(a_i) \le 0$ for $i = 1, 2, \cdots, n$;

(H3) There exist $b_1, \cdots, b_n$ with $a_1 < b_1 < a_2 < b_2 < \cdots < a_{n-1} < b_{n-1} < a_n < b_n < L$ such that

$$f(s) > 0, \ s \in (b_n, L), \ f(b_i) > 0 \text{ and } F(b_i) > F(u) \text{ for } 0 \le u \le b_i, \ i = 1, 2, \cdots, n,$$

where $F(u) = \int_0^u f(s)ds$.

Roughly speaking, the above hypotheses imply that, the graph of f has $n + 1$ positive humps and n negative bumps, with each positive hump having a larger area than the previous negative hump.

The rest of the paper is organized as follows: Section 2 contains statements on the main result. Section 3 contains preparatory lemmas. Section 4 contains the proof of the result. Finally, in Section 5, we give two examples to illustrate the feasibility of our result.

2. Main result

In this section, in order to state our main result, we first give some terminologies related to the shape of the bifurcation curve. Let

$$S_L := \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of Problem (1.1)}\} \tag{2.1}$$

be the bifurcation curve for Problem (1.1) on the $(\lambda, \|u\|_\infty)$ plane.

Definition 2.1. *[15, 18] Let* S_L *be the bifurcation curve for Problem (1.1) on the* $(\lambda, \|u\|_\infty)$ *plane.*

(i) S-like shaped: The curve S_L *is said to be S-like shaped if* S_L *has at least two turning points at some points* $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ *and* $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ *where* $\lambda_1 < \lambda_2$ *are two positive numbers such that:*

(a) At $(\lambda_1, \|u_{\lambda_1}\|_\infty)$*, the bifurcation curve* S_L *turns to the right;*

(b) $\|u_{\lambda_2}\|_\infty < \|u_{\lambda_1}\|_\infty$*;*

(c) At $(\lambda_2, \|u_{\lambda_2}\|_\infty)$*, the bifurcation curve* S_L *turns to the left.*

(ii) Monotone increasing: The curve S_L is said to be monotone increasing if S_L is a continuous curve and for each pair of points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ and $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ of S_L, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 \le \lambda_2$.

(iii) $\subset$-shaped: The curve S_L is said to be $\subset$-shaped if S_L is a continuous curve that initially continues to the left and eventually continues to the right.

Throughout this paper, assume, in addition to (H1)–(H3), that f satisfies one of the following several possibilities:

(C1) $f_0 = 0$ and $\lim_{u\to 0^+} f''(u) \in (0, \infty]$, where $f_0 := \lim_{s\to 0} \frac{f(s)}{s}$.

(C2) $f_0 = \infty$.

(C3) $f_0 \in (0, \infty)$ and $\lim_{u\to 0^+} f''(u) \in (0, \infty]$.

(C4) $f_0 \in (0, \infty)$ and $\lim_{u\to 0^+} f''(u) \in (-\infty, 0]$.

Theorem 2.1. *Assume that (H1)–(H3) hold. Then*

(i) (See Figure 1) The bifurcation curve S_L starts from $(K, 0)$ and goes to infinity along the horizontal line $\|u\|_\infty = r$ (r be as in (3.8)), where

$$K \equiv \begin{cases} \infty, & \text{if } f_0 = 0, \\ \dfrac{\pi^2}{4 f_0 L^2}, & \text{if } f_0 \in (0, \infty), \\ 0, & \text{if } f_0 = \infty. \end{cases}$$

Furthermore, if either (C1) and (C3) holds, then S_L is $\subset$-shaped; if either (C2) and (C4) holds, then S_L is either monotone increasing or S-like shaped.

(ii) The bifurcation curve S_L is $\subset$-shaped for all $\rho \in (\alpha_i, \beta_i)$, $i = 1, 2, \cdots, n-1$, where

$$\beta_i = \inf\{u | u > b_i, f(u) = 0\}, \quad \alpha_i = \inf\{u | (u, \beta_i) \subseteq \mathcal{S}\},$$

ρ is defined in Problem (3.1) and $\mathcal{S} = \{u | u > 0, f(u) > 0, F(u) > F(s)$ for all $s : 0 \le s < u\}$.

(iii) The bifurcation curve S_L is $\subset$-shaped for all $\rho \in (\alpha_n, L)$, where $\alpha_n = \inf\{u | (u, \beta_n) \subseteq \mathcal{S}\}$.

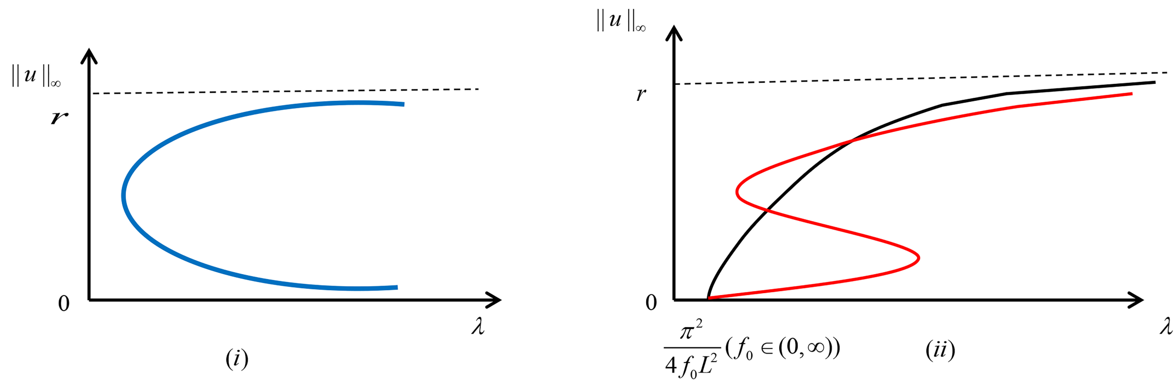

Figure 1. Graph of bifurcation curves S_L of Problem (1.1): (i) $\subset$-shaped. (ii) monotone increasing or S-like shaped.

Corollary 2.1. *(See Figure 2) Assume that (H1)–(H3) hold. Then for all $\rho \in \mathcal{S}$, we have the following:*

(i) If $f_0 \in (0, \infty)$, then Problem (1.1) has a positive solution (λ, u) satisfying $\|u\|_\infty < r$ for all $\lambda > \frac{\pi^2}{4f_0L^2}$; if $f_0 = 0$, then Problem (1.1) has at least two positive solutions (λ, u_1) and (λ, u_2) satisfying $0 < \|u_1\|_\infty < \|u_2\|_\infty < r$ for all $\lambda > \inf\{\lambda(\rho)|\rho \in (0, r)\}$; if $f_0 = \infty$, then for all $\lambda > 0$, Problem (1.1) has at least one positive solution (λ, u) satisfying $0 < \|u\|_\infty < r$.

(ii) If $\lambda > \inf\{\lambda(\rho)|\rho \in (\alpha_i, \beta_i)\}$, then Problem (1.1) has at least two positive solutions (λ, u_{i1}) and (λ, u_{i2}) satisfying $\alpha_i < \|u_{i1}\|_\infty < \|u_{i2}\|_\infty < \beta_i$, $i = 1, \cdots, n-1$.

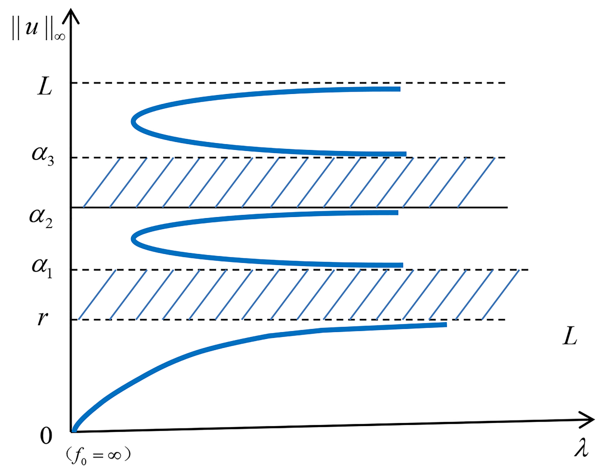

Figure 2. Graph of bifurcation curves S_L of Problem (1.1) when $f_0 = \infty$.

Remark 2.1. *Conditions (H2) and (H3) contain three cases:*

Case 1. *$f(a_i) = 0$, and $b_i \in (a_i, a_{i+1})$, $i = 1, 2, \cdots, n$;*

Case 2. *$f(a_i) < 0$, and b_i, $i = 1, 2, \cdots, n$ satisfies (H3). Roughly speaking, the graph of f has $n + 1$ positive humps and n negative bumps, with each positive hump having a larger area than the previous negative hump (see Figure 3).*

Case 3. *$f(a_i) < 0$ for some $i \in I_0 \subseteq I := \{1, 2, \cdots, n\}$, and $f(a_i) = 0$, for $i \in I \backslash I_0$, where b_i, $i = 1, 2, \cdots, n$, satisfies (H3).*

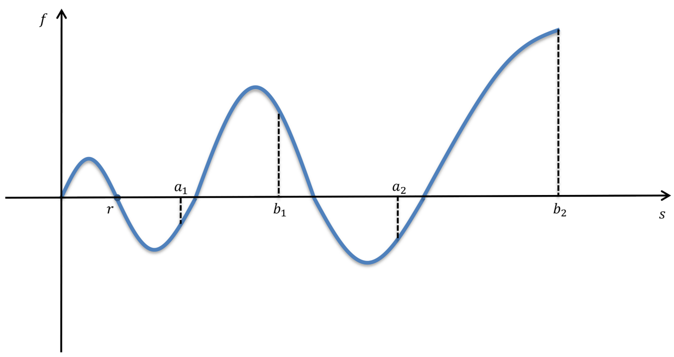

Figure 3. Graph of the nonlinear term f in Problem (1.1).

Remark 2.2. *It is worth remarking that References [8, 11–18, 22, 24] provided some sufficient conditions to determine the shape of the bifurcation curve or multiplicity of positive solutions of Problem (1.1) or (1.2) with the general* $f \in C([0,\infty),[0,\infty))$. *However, these results cannot be applied to our Problem (1.1) because the nonlinearity* f *in this paper is not always positive in* $[0,\infty)$.

Remark 2.3. *Let* (λ, u) *be a solution of Problem (1.1), then, it follows from* $\|u'\|_\infty < 1$ *that*

$$\|u\|_\infty < L.$$

This causes the bifurcation diagrams to be mainly dependent on the behavior of $f = f(s)$ *near* $s = 0$. *In addition, if we compare it with known results for semilinear elliptic problems with convex-concave nonlinearities, the bifurcation diagram is reversed in some sense. This reveals a significant difference between Minkowski curvature problems and corresponding semilinear elliptic problems.*

Remark 2.4. *Note that the condition* $f(0) > 0$ *is a special case of Condition (C2), so Theorem 2.1 improves and generalizes the main result in [20].*

Remark 2.5. *By virtue of quadrature arguments, Ma and Lu [20] presented a full description of the set of positive solutions of Problem (1.1) for certain classes of* f *nonlinearity (that is,* $f(0) > 0$, *or* $f(0) = 0$, $f_0 > 0$*), they provided no information about the directions of a bifurcation. Therefore, our main result generalizes generalize and improves the main result in [20].*

3. Lemmas

In this section, we shall make a detailed analysis of the so-called time map for Problem (1.1). Various properties of the time map will be used to obtain the existence and multiplicity results of the positive solutions for Problem (1.1).

Lemma 3.1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous. Let* (λ, u) *be a positive solution of Problem (1.1) with* $\|u\|_\infty = \rho \in \mathcal{S}$ *and* $\lambda > 0$. *Let* $x_0 \in (-L, L)$ *be such that* $u'(x_0) = 0$. *Then,*

(i) $x_0 = 0$;

(ii) x_0 *is the unique point at which* u *attains its maximum;*

(iii) $u'(x) > 0, x \in (-L, x_0)$ *and* $u'(x) < 0, x \in (x_0, L)$.

Proof. The arguments are quite similar to those from the proof of Lemma 2.4 in [20]. However, for the sake of completeness, we give a sketch of the proof below.

(i) Suppose on the contrary that $x_0 \neq 0$ also without loss of generality, suppose that $x_0 < 0$, then, $u(2x_0 - x)$ is also a solution of Problem (1.1). Moreover,

$$u(x_0) = u(2x_0 - x_0),\ u'(x_0) = u'(2x_0 - x_0) = 0.$$

Therefore, by the uniqueness of the associated Cauchy problem, we have that $u(x) = u(2x_0 - x)$ for all $x \in [-L, L]$. In particular, $0 = u(-L) = u(L + 2x_0)$, which contradicts the fact that $u(x) > 0$ for all $x \in (-L, L)$. Similarly, if $x_0 > 0$, we get a similar contradiction. Therefore, $x_0 = 0$.

(ii) Suppose on the contrary that there exists $x_1 \in (-L, L)$ with $x_1 \neq x_0$ such that $u(x_1) = u(x_0) = \rho$. Without loss of generality, assume that $x_1 < x_0$. Because $x_1 > x_0$ can be treated in a similar way, if $u(x) = u(x_0)$ for $x \in (x_1, x_0)$, then from [20, Lemma 2.3], we have that

$$u(x) \equiv u(x_0) = \rho > 0, x \in (-L, L).$$

This contradicts the boundary conditions $u(-L) = u(L) = 0$. Therefore, $u(x) \not\equiv u(x_0)$ in any subinterval of $(-L, L)$. So, there exists $x_* \in (x_1, x_0)$ such that

$$u(x_*) = \min\{u(x)|x \in (x_1, x_0)\}.$$

Clearly, $0 < u(x_*) < \rho$ and $u'(x_*) = 0$. Multiplying both sides of the equation given by Problem (1.1) by u' and integrating from x to x_0, we get that, for $x \in [-L, 0]$,

$$(u'(x))^2 = \sqrt{1-(u'(x))^2}[1 - \sqrt{1-(u'(x))^2} + \lambda(F(\rho) - F(u(x)))].$$

Subsequently,

$$\begin{aligned} 0 = (u'(x_*)^2 &= \sqrt{1-(u'(x_*))^2}[1 - \sqrt{1-(u'_*(x))^2} + \lambda(F(\rho) - F(u(x_*)))] \\ &= 1 + \lambda(F(\rho) - F(u(x_*))) > 0. \end{aligned}$$

This is a contradiction because of $\rho \in \mathcal{S}$ and $u(x_*) < \rho$. Thus, x_0 is unique and $u(0) > u(t)$, $t \in [-L, 0)$. By a similar argument, we can prove that $u(0) > u(t)$, $t \in (0, L]$. Therefore, $x_0 = 0$ is the unique point on which u attains its maximum.

(iii) Suppose on the contrary that there exists $\hat{x} \in (-L, 0)$ with $u'(\hat{x}) = 0$. Then $u(\hat{x}) < \rho$. Therefore,

$$0 = (u'(\hat{x}))^2 = 1 + \lambda(F(\rho) - F(u(\hat{x}))).$$

This contradicts the fact that $\rho \in \mathcal{S}$ and $u(\hat{x}) < \rho$. Thus, $u'(t) > 0$, $t \in (-L, 0)$. By a similar argument, it follows that $u'(t) < 0$, $t \in (0, L)$. □

By Lemma 3.1, we know that $u(x)$ takes its maximum at 0 and $u(x)$ is symmetric with respect to 0, $u'(x) > 0$ for $-L \le x < 0$ and $u'(x) < 0$ for $0 < x \le L$. Hence, Problem (1.1) is equivalent to the following problem defined on $[0, L]$:

$$\begin{cases} \Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' + \lambda f(u) = 0, \quad x \in (0, L), \\ u'(0) = u(L) = 0, \ u(0) = \rho \in \mathcal{S}. \end{cases} \tag{3.1}$$

To prove Theorem 2.1, we shall first define the time-map formula for Problem (3.1) as

$$T_\lambda(\rho) = \int_0^\rho \frac{1 + \lambda F(\rho) - \lambda F(u)}{\sqrt{[1 + \lambda F(\rho) - \lambda F(u)]^2 - 1}} du \text{ with } \rho \in \mathcal{S}. \tag{3.2}$$

Notice that the function $T_\lambda(\rho)$ is well-defined and continuous on $\rho \in \mathcal{S}$ (see [20]). Therefore, the positive solutions $u_\lambda \in C^2(-L, L) \cap C[-L, L]$ of Problem (3.1) correspond to curves for which

$$||u_\lambda||_\infty = \rho \in \mathcal{S} \ \text{ and } \ T_\lambda(\rho) = L.$$

So, by the definition of S_L in Eq (2.1), we may see that

$$S_L = \{(\lambda, \rho) : T_\lambda(\rho) = L \text{ for } \lambda > 0, \rho \in \mathcal{S}\} = \{(\lambda_L(\rho), \rho) : \rho \in \mathcal{S}\}. \tag{3.3}$$

This leads us to investigate the shape of $T_\lambda(\rho)$.

By an argument similar to proving [20, Lemma 3.1] with obvious changes, we may obtain the following result.

Lemma 3.2. *If $\rho \in \mathcal{S}$, then there exists a unique $\lambda > 0$ such that (λ, u) is a positive solution of Problem (3.1) satisfying $\|u\|_\infty = \rho$. Moreover, the bifurcation curve $\{(\lambda_L(\rho), \rho) : \rho \in \mathcal{S}\}$ is continuous on the $(\lambda, \|u\|_\infty)$-plane.*

Lemma 3.3. *Consider Problem (3.1). Then*

$$\lim_{\rho\to 0^+} T_\lambda(\rho) = \begin{cases} 0, & \text{if } f_0 = \infty, \\ \dfrac{\pi}{2\sqrt{\lambda f_0}}, & \text{if } f_0 \in (0,\infty), \quad \text{for } \lambda > 0, \\ \infty, & \text{if } f_0 = 0. \end{cases}$$

Proof. Assume that $f_0 = 0$ or $f_0 \in (0, \infty)$. By L'Hopital's rule, we observe that, for $0 < t < 1$,

$$\lim_{\rho\to 0^+} \frac{F(\rho) - F(\rho t)}{\rho^2} = \lim_{\rho\to 0^+} \frac{f(\rho) - tf(\rho t)}{2\rho} = \frac{(1-t^2)f_0}{2}. \tag{3.4}$$

Assume that $f_0 = \infty$. If $f(0) > 0$, we apply L'Hopital's rule to get

$$\lim_{\rho\to 0^+} \frac{F(\rho) - F(\rho t)}{\rho^2} = \lim_{\rho\to 0^+} \frac{f(\rho) - tf(\rho t)}{2\rho} = \infty.$$

If $f(0) = 0$, and by L'Hopital's rule, we observe that $\lim_{u\to 0^+} f'(u) = \infty$; for $0 < t < 1$,

$$\lim_{\rho\to 0^+} \frac{F(\rho) - F(\rho t)}{\rho^2} = \lim_{\rho\to 0^+} \frac{f(\rho) - tf(\rho t)}{2\rho} \geq \lim_{\rho\to 0^+} \frac{f(\rho t) - tf(\rho t)}{2\rho} = (1-t)t \lim_{\rho\to 0^+} \frac{f(\rho t)}{2\rho t} = \infty.$$

Therefore,

$$\lim_{\rho\to 0^+} \frac{F(\rho) - F(\rho t)}{\rho^2} = \begin{cases} \infty, & \text{if } f_0 = \infty, \\ \dfrac{1-t^2}{2} f_0, & \text{if } f_0 \in (0,\infty), \\ 0, & \text{if } f_0 = 0. \end{cases} \tag{3.5}$$

By Eq (3.5), we have

$$\begin{aligned} \lim_{\rho\to 0^+} T_\lambda(\rho) &= \lim_{\rho\to 0^+} \rho \int_0^1 \frac{1 + \lambda(F(\rho) - F(\rho t))}{\sqrt{\lambda^2[F(\rho) - F(\rho t)]^2 + 2\lambda[F(\rho) - F(\rho t)]}} dt \\ &= \lim_{\rho\to 0^+} \int_0^1 \frac{1 + \lambda(F(\rho) - F(\rho t))}{\sqrt{\{\lambda^2[F(\rho) - F(\rho t)] + 2\lambda\}[\frac{F(\rho)-F(\rho t)}{\rho^2}]}} dt \\ &= \begin{cases} 0, & \text{if } f_0 = \infty, \\ \dfrac{\pi}{2\sqrt{2\lambda f_0}}, & \text{if } f_0 \in (0,\infty), \\ \infty, & \text{if } f_0 = 0. \end{cases} \end{aligned}$$

The proof of Lemma 3.3 is complete. □

Using an argument similar to proving [16, Lemma 3.2] with obvious changes, we have the following lemma.

Lemma 3.4. *Consider Problem (3.1). Then*

$$\lim_{\rho\to 0^+} T'_\lambda(\rho) = \begin{cases} -\infty, & \text{if } \lim\limits_{u\to 0^+} f''(u) = \infty, \\ -\dfrac{1}{3f_0\sqrt{\lambda f_0}} \lim\limits_{u\to 0^+} f''(u), & \text{if } \lim\limits_{u\to 0^+} f''(u) \text{ exists}, \ \text{for } \lambda > 0, \\ \infty, & \text{if } \lim\limits_{u\to 0^+} f''(u) = -\infty. \end{cases}$$

Lemma 3.5. *Consider Problem (3.1). Then for any $\rho \in \mathcal{S}$, the following statements (i) and (ii) hold:*

(i) $\partial T_\lambda(\rho)/\partial\lambda < 0$ for $\lambda > 0$ and $\rho \in \mathcal{S}$;

(ii) Assume $3f(u) + uf'(u) > 0$ for $0 < u < r$. Then

$$\frac{\partial}{\partial\lambda}[\sqrt{\lambda}T'_\lambda(\rho)] = \sqrt{\lambda}\frac{\partial}{\partial\lambda}T'_\lambda(\rho) + \frac{1}{2\sqrt{\lambda}}T'_\lambda(\rho) > 0 \text{ for } \lambda > 0,\ \rho \in \mathcal{S}.$$

Proof. Let $B = B(\rho, u) \equiv F(\rho) - F(u)$, combining this with the fact that $f(s) > 0$ for $s \in \mathcal{S}$ gives

$$B(\rho, u) = F(\rho) - F(u) = \int_u^\rho f(t)dt > 0 \ \text{ for } \rho \in \mathcal{S} \text{ and } 0 < u < r. \tag{3.6}$$

By Eqs (3.2) and (3.6), we see that

$$\frac{\partial}{\partial\lambda}T_\lambda(\rho) = \int_0^\rho \frac{-B(\rho,u)}{[\lambda^2 B^2(\rho,u) + 2\lambda B(\rho,u)]^{3/2}}du < 0 \text{ for } \lambda > 0,\ \rho \in \mathcal{S}.$$

So Statement (i) holds.

Let

$$A(\rho, u) \equiv \rho f(\rho) - uf(u).$$

It is easy to check that

$$T'_\lambda(\rho) = \frac{1}{\rho}\int_0^\rho \frac{\lambda^3 B^3 + 3\lambda^2 B^2 + \lambda(2B - A)}{(\lambda^2 B^2 + 2\lambda B)^{3/2}}du \text{ for } \lambda > 0,\ \rho \in \mathcal{S}.$$

Since $A(\rho,\rho) + 2B(\rho,\rho) = 0$ and

$$\frac{\partial}{\partial u}[A(\rho,u) + 2B(\rho,u)] = -[3f(u) + uf'(u)] < 0 \text{ for } 0 < u < r,$$

we observe that

$$A(\rho,u) + 2B(\rho,u) > 0 \text{ for } 0 < u < r. \tag{3.7}$$

Then by Eqs (3.6) and (3.7), we see that, for $\lambda > 0$ and $\rho \in \mathcal{S}$,

$$\begin{aligned} \frac{\partial}{\partial\lambda}[\sqrt{\lambda}T'_\lambda(\rho)] &= \sqrt{\lambda}\frac{\partial}{\partial\lambda}T'_\lambda(\rho) + \frac{1}{2\sqrt{\lambda}}T'_\lambda(\rho) \\ &= \frac{1}{2\rho\sqrt{\lambda}}\int_0^\rho \frac{\lambda^3 B^2(B^3\lambda^2 + 5B^2\lambda + 3A + 6B)}{\lambda^2 B^2 + 2\lambda B^{5/2}}du > 0. \end{aligned}$$

So Statement (ii) holds. The proof is complete. □

Lemma 3.6. *$sgn(\lambda_L'(\rho)) = sgn(T'_{\lambda_L(\rho)}(\rho))$ for $\rho \in \mathcal{S}$, where $sgn(u)$ is the signum function.*

Proof. Since $\lim_{\lambda\to 0^+} T_\lambda(\rho) = \infty$ and $\lim_{\lambda\to\infty} T_\lambda(\rho) = \rho$ for $\rho \in \mathcal{S}$. By Lemma 3.2, there exists a unique $\lambda_L(\rho) > 0$ such that $T_{\lambda_L(\rho)}(\rho) = L$ for $\rho \in \mathcal{S}$. Since

$$0 = \frac{\partial}{\partial\rho} T_{\lambda_L(\rho)}(\rho) = T'_{\lambda_L(\rho)}(\rho) + \frac{\partial}{\partial\lambda} T_\lambda(\rho)|_{\lambda=\lambda_L(\rho)}\lambda_L'(\rho) \text{ for } \rho \in \mathcal{S}.$$

So by Lemma 3.5(i), we can obtain the desired result. □

Let

$$r := \inf\{u > 0 : f(u) = 0\}. \tag{3.8}$$

Since $f(u) > 0$, $0 < u < r$. Then $(0, r) \subseteq \mathcal{S}$.

By the definitions of β_i and α_i, we have

$$a_i \le \alpha_i < b_i < \beta_i \le a_{i+1} \le \alpha_{i+1} < \cdots$$

and $(\alpha_i, \beta_i) \subseteq \mathcal{S}$ for $i = 1, 2, \cdots, n-1$.

Lemma 3.7. *Consider Problem (3.1). Then we have $\lim_{\rho\to 0^+} \lambda_L(\rho) = K$, where K is defined in Theorem 2.1. Moreover, the bifurcation curve S_L starts from the point $(K, 0)$ and goes to infinity along the horizontal line $\|u\|_\infty = r$.*

Proof. We divide the proof into the following four steps.

Step 1. We prove that $\lim_{\rho\to 0^+} \lambda_L(\rho) = 0$ if $f_0 = \infty$. Assume that $f_0 = \infty$. If $\limsup_{\rho\to 0^+} \lambda_L(\rho) > 0$, there exist $M_1 > 0$ and $\{\rho_n\}_{n\in\mathbb{N}} \subset (0, r)$ such that

$$\lim_{n\to\infty} \rho_n = 0 \text{ and } \lambda_L(\rho_n) > M_1 \text{ for } n \in \mathbb{N}. \tag{3.9}$$

Since $T_{\lambda_L(\rho)}(\rho) = L$, and by Lemmas 3.3 and 3.5 and Eq (3.9), we see that

$$L = \lim_{n\to\infty} T_{\lambda_L(\rho_n)}(\rho_n) \le \lim_{n\to\infty} T_{M_1}(\rho_n) = 0,$$

which is a contradiction. Thus, $\lim_{\rho\to 0^+} \lambda_L(\rho) = 0$.

Step 2. We prove that $\lim_{\rho\to 0^+} \lambda_L(\rho) = \infty$ if $f_0 = 0$. Assume that $f_0 = 0$. If $\liminf_{\rho\to 0^+} \lambda_L(\rho) < \infty$, there exist $M_2 > 0$ and $\{\rho_n\}_{n\in\mathbb{N}} \subset (0, r)$ such that

$$\lim_{n\to\infty} \rho_n = 0 \text{ and } \lambda_L(\rho_n) < M_2 \text{ for } n \in \mathbb{N}. \tag{3.10}$$

Since $T_{\lambda_L(\rho)}(\rho) = L$, and by Lemmas 3.3 and 3.5 and Eq (3.10), we see that

$$L = \lim_{n\to\infty} T_{\lambda_L(\rho_n)}(\rho_n) \ge \lim_{n\to\infty} T_{M_2}(\rho_n) = \infty,$$

which is a contradiction. Thus $\lim_{\rho\to 0^+} \lambda_L(\rho) \ge \liminf_{\rho\to 0^+} \lambda_L(\rho) = \infty$.

Step 3. Inspired by the idea presented in [20], we prove that $\lim_{\rho\to 0^+}\lambda_L(\rho)=\frac{\pi^2}{4f_0L^2}$ if $f_0\in(0,\infty)$. Since $f_0\in(0,\infty)$, for any $\epsilon>0$, there exists $0<\delta<r$ such that

$$|\frac{f(u)}{u}-f_0|<\epsilon,\ \forall\ 0<u<\delta.$$

Thus, if $\rho<\delta$, then

$$(f_0-\epsilon)(\frac{\rho^2-u^2}{2})\leq F(\rho)-F(u)=\int_u^\rho f(s)ds\leq(f_0+\epsilon)(\frac{\rho^2-u^2}{2}).$$

This together with Eq (3.2) yields that

$$\begin{aligned}
L&=\int_0^\rho\frac{du}{\sqrt{1-[1+\lambda(F(\rho)-F(u))]^{-2}}}\\
&\leq\rho\int_0^1\frac{dt}{\sqrt{1-[1+\frac{\lambda(f_0-\epsilon)\rho^2(1-t^2)}{2}]^{-2}}}\\
&=\rho\int_0^1\frac{1+\frac{\lambda(f_0-\epsilon)}{2}\rho^2(1-t^2)}{\sqrt{[\frac{\lambda(f_0-\epsilon)\rho^2(1-t^2)}{2}]^2+\lambda(f_0-\epsilon)\rho^2(1-t^2)}}dt\\
&=\frac{1}{\sqrt{\lambda(f_0-\epsilon)}}\int_0^1\frac{1+\frac{\lambda(f_0-\epsilon)}{2}\rho^2(1-t^2)}{\sqrt{\lambda\frac{\rho^2}{4}(1-t^2)^2(f_0-\epsilon)+(1-t^2)}}dt,
\end{aligned}$$

and

$$\begin{aligned}
L&=\int_0^\rho\frac{du}{\sqrt{1-[1+\lambda(F(\rho)-F(u))]^{-2}}}\\
&\geq\rho\int_0^1\frac{dt}{\sqrt{1-[1+\frac{\lambda(f_0+\epsilon)\rho^2(1-t^2)}{2}]^{-2}}}\\
&=\frac{1}{\sqrt{\lambda(f_0+\epsilon)}}\int_0^1\frac{1+\frac{\lambda(f_0+\epsilon)}{2}\rho^2(1-t^2)}{\sqrt{\lambda\frac{\rho^2}{4}(1-t^2)^2(f_0+\epsilon)+(1-t^2)}}dt,
\end{aligned}$$

which imply that

$$\sqrt{\lambda}\leq\frac{1}{\sqrt{(f_0-\epsilon)}L}\int_0^1\frac{1+\frac{\lambda(f_0-\epsilon)}{2}\rho^2(1-t^2)}{\sqrt{\lambda\frac{\rho^2}{4}(1-t^2)^2(f_0-\epsilon)+(1-t^2)}}dt,$$

and

$$\sqrt{\lambda}\geq\frac{1}{\sqrt{(f_0+\epsilon)}L}\int_0^1\frac{1+\frac{\lambda(f_0+\epsilon)}{2}\rho^2(1-t^2)}{\sqrt{\lambda\frac{\rho^2}{4}(1-t^2)^2(f_0+\epsilon)+(1-t^2)}}dt,$$

and subsequently,

$$\begin{aligned}\frac{\pi}{2\sqrt{f_0}L} &= \lim_{\rho\to 0}\frac{1}{\sqrt{(f_0+\epsilon)L}}\cdot\frac{\pi}{2} = \lim_{\rho\to 0}\frac{1}{\sqrt{(f_0+\epsilon)L}}\int_0^1\frac{dt}{\sqrt{1-t^2}}\\ &\le \lim_{\rho\to 0}\sqrt{\lambda_L(\rho)} \le \lim_{\rho\to 0}\frac{1}{\sqrt{(f_0-\epsilon)L}}\int_0^1\frac{dt}{\sqrt{1-t^2}}\\ &= \lim_{\rho\to 0}\frac{1}{\sqrt{(f_0-\epsilon)L}}\cdot\frac{\pi}{2} = \frac{\pi}{2\sqrt{f_0}L}\quad\text{for } \epsilon>0 \text{ small enough as } \rho\to 0.\end{aligned}$$

Therefore, we have that

$$\lim_{\rho\to 0}\lambda_L(\rho) = \frac{\pi^2}{4f_0L^2}.$$

Step 4. We prove that $\lim\limits_{\rho\to r^-}\lambda_L(\rho)=\infty$.

From the definition of r, we have that

$$f(r)=0 \text{ and } \lim_{u\to r^-}\frac{f(u)}{u}=0.$$

Hence, for any $\epsilon>0$, there exists $\delta\in(0,r)$ such that

$$f(u)\le\epsilon u,\ \forall r-\delta<u<r,$$

and for $\rho<r$, it follows that

$$F(\rho)-F(u)=\int_u^\rho f(s)ds\le\epsilon\int_u^\rho sds=\frac{\epsilon(\rho^2-u^2)}{2}.$$

This together with Eq (3.2) implies that

$$\begin{aligned}L &= \int_0^\rho\frac{du}{\sqrt{1-[1+\lambda(F(\rho)-F(u))]^{-2}}}\\ &\ge\int_0^\rho\frac{du}{\sqrt{1-[1+\lambda\epsilon(\rho^2-u^2)/2]^{-2}}}\\ &=\int_0^1\frac{2+\lambda\epsilon\rho^2(1-t^2)}{\sqrt{\lambda^2\epsilon^2\rho^4(1-t^2)^2+4\lambda\epsilon\rho^2(1-t^2)}}dt,\end{aligned}$$

which implies that

$$\sqrt{\lambda_L(\rho)}\ge\frac{1}{\sqrt{\epsilon}L}\int_0^1\frac{2+\lambda\epsilon\rho^2(1-t^2)}{\sqrt{\lambda\epsilon\rho^4(1-t^2)^2+4\rho^2(1-t^2)}}dt.$$

So, we get that

$$\lim_{\rho\to r^-}\sqrt{\lambda_L(\rho)}\ge\lim_{\rho\to r^-}\frac{1}{\sqrt{\epsilon}Lr}\int_0^1\frac{1}{\sqrt{1-t^2}}dt=\infty,\ \text{for } \epsilon>0 \text{ small enough as } \rho\to r^-,$$

that is

$$\lim_{\rho\to r^-}\lambda_L(\rho)=\infty.$$

□

4. Proof of Theorem 2.1

Proof of Theorem 2.1. (i) Let $L > 0$ be given. By Lemmas 3.2 and 3.7, the bifurcation curve S_L of Problem (3.1) is continuous in the $(\lambda, \|u\|_\infty)$ plane, starts from the point $(K, 0)$ and goes to infinity along the horizontal line $\|u\|_\infty = r$. Next, we divide the remainder proofs of Theorem 2.1(i) into the following two steps.

Step 1. Assume that (C1) holds. By Lemma 3.7, we see that

$$\lim_{\rho \to 0^+} \lambda_L(\rho) = \lim_{\rho \to r^-} \lambda_L(\rho) = \infty.$$

So S_L is $\subset$-shaped. Assume that (C3) holds. By Lemma 3.4, we see that

$$\lim_{\rho \to 0^+} T'_{\frac{\pi^2}{4f_0L^2}}(\rho) < 0.$$

Then there exists $\delta_1 \in (0, r)$ such that

$$T'_{\frac{\pi^2}{4f_0L^2}}(\rho) < 0 \text{ for } 0 < \rho < \delta_1. \tag{4.1}$$

Since $f_0 \in (0, \infty)$, and by Lemmas 3.2 and 3.3 and Eq (4.1), we observe that

$$T_{\frac{\pi^2}{4f_0L^2}}(\rho) < \lim_{\rho \to 0^+} T_{\frac{\pi^2}{4f_0L^2}}(\rho) = \frac{\pi}{2\sqrt{f_0(\frac{\pi^2}{4f_0L^2})}} = L = T_{\lambda_L(\rho)}(\rho) \text{ for } 0 < \rho < \delta_1.$$

It follows that $\lambda_L(\rho) < \frac{\pi^2}{4f_0L^2}$ for $0 < \rho < \delta_1$ by Lemma 3.5(i). Then by Lemma 3.5(ii) and Eq (4.1), we see that $T'_{\lambda_L(\rho)}(\rho) < 0$ for $0 < \rho < \delta_1$. By Lemmas 3.6 and 3.7, we further see that

$$\lim_{\rho \to r^-} \lambda_L(\rho) = \infty \text{ and } \lambda'_L(\rho) < 0 \text{ for } 0 < \rho < \delta_1.$$

So S_L is $\subset$-shaped by Lemma 3.2.

Step 2. Assume that (C2) holds. By Lemma 3.7, we see that

$$\lim_{\rho \to 0^+} \lambda_L(\rho) = 0 \text{ and } \lim_{\rho \to r^-} \lambda_L(\rho) = \infty.$$

So S_L is S-like shaped or monotone increasing by Lemma 3.2. Assume that (C4) holds. By Lemma 3.4, we have

$$\lim_{\rho \to 0^+} T'_{\frac{\pi^2}{4f_0L^2}}(\rho) > 0.$$

By a similar proof in Step 1, there exists $\delta_2 > 0$ such that

$$\lim_{\rho \to r^-} \lambda_L(\rho) = \infty \text{ and } \lambda'_L(\rho) > 0 \text{ for } 0 < \rho < \delta_2.$$

So S_L is S-like shaped or monotone increasing by Lemma 3.2.

(ii) The arguments are quite similar to those from the proof of Lemma 3.3 in [20]. However, for the sake of completeness, we give a sketch of the proof below. We considered two cases:

Case 1. Suppose that $f(\alpha_i) > 0$. Since $\mathcal{S}$ is open, $\alpha_i \notin \mathcal{S}$, so, there exists $p : 0 < p < \alpha_i$ such that $F(\alpha_i) = F(p)$. Clearly, p must be a local maximum for F, so $f(p) = 0$.

Let $M_3 = \max\{|f'(u)||0 \leq u \leq b_i\}$, then, $f(u) \leq M_3|u - p|$, $0 \leq u \leq b_i$. Let $M_4 = \max\{|f(u)||0 \leq u \leq b_i\}$, then, for $\alpha_i < \rho < b_i$,

$$\begin{aligned}
F(\rho) - F(u) &= F(\rho) - F(\alpha_i) + F(p) - F(u) \\
&= (\rho - \alpha_i)f(\zeta) + (p - u)f(\xi), \quad \exists \zeta \in (\alpha_i, \rho), \xi \in (p, u) \\
&\leq M_4(\rho - \alpha_i) + M_3(p - u)^2 =: \kappa(\rho, \alpha_i, u).
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
L &= \int_0^\rho \frac{du}{\sqrt{1 - [1 + \lambda(F(\rho) - F(u))]^{-2}}} \\
&\geq \int_0^\rho \frac{du}{\sqrt{1 - [1 + \lambda\kappa(\rho, \alpha_i, u)]^{-2}}} \\
&= \frac{1}{\sqrt{\lambda}} \int_0^\rho \frac{1 + \lambda[\kappa(\rho, \alpha_i, u)]}{\sqrt{2\kappa(\rho, \alpha_i, u) + \lambda\kappa(\rho, \alpha_i, u)^{-2}}} du,
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{\lambda} &\geq \frac{1}{L} \int_0^\rho \frac{1 + \lambda[\kappa(\rho, \alpha_i, u)]}{\sqrt{2\kappa(\rho, \alpha_i, u) + \lambda\kappa(\rho, \alpha_i, u)^{-2}}} du \\
&\geq \frac{1}{L} \int_0^{\alpha_i} \frac{1 + \lambda[\kappa(\rho, \alpha_i, u)]}{\sqrt{2\kappa(\rho, \alpha_i, u) + \lambda\kappa(\rho, \alpha_i, u)^{-2}}} du \\
&= \frac{1}{L} \int_0^{\alpha_i} H_\rho(u) du.
\end{aligned}$$

As $\rho \to \alpha_i^+$, it is easy to verify that

$$H_\rho(u) = \int_0^{\alpha_i} \frac{1 + \lambda[\kappa(\rho, \alpha_i, u)]}{\sqrt{2\kappa(\rho, \alpha_i, u) + \lambda\kappa(\rho, \alpha_i, u)^{-2}}} du$$

is a nondecreasing sequence of measurable functions. Therefore, by the monotone convergence theorem, it follows that

$$\begin{aligned}
\lim_{\rho \to \alpha_i^+} \sqrt{\lambda_L(\rho)} &\geq \lim_{\rho \to \alpha_i^+} \frac{1}{L} \int_0^{\alpha_i} H_\rho(u) du \\
&= \lim_{\rho \to \alpha_i^+} \frac{1}{L} \int_0^{\alpha_i} \frac{1 + \lambda M_3(p - u)^2}{|p - u| \sqrt{2M_3 + \lambda M_3^2(p - u)^2}} du \\
&= \infty, \ \text{since } p \in (0, \alpha_i).
\end{aligned}$$

Case 2. Suppose that $f(\alpha_i) = 0$, then, $F'(\alpha) = 0$. Since $F(\alpha_i) - F(u) = f(\xi)(\alpha_i - u)$ with $\xi \in (u, \alpha_i)$, and $|f(u)| = |f(u) - f(\alpha_i)| \leq M_3|u - \alpha_i|$, we can obtain that

$$F(\alpha_i) - F(u) \leq M_3(u - \alpha_i)^2$$

and

$$\begin{aligned}L &= \int_0^\rho \frac{du}{\sqrt{1-[1+\lambda(F(\rho)-F(u))]^{-2}}}\\ &\geq \int_0^\rho \frac{du}{\sqrt{1-[1+\lambda M_3(u-\alpha_i)^2]^{-2}}}\\ &= \frac{1}{\sqrt{\lambda}}\int_0^\rho \frac{1+\lambda M_3(u-\alpha_i)^2}{\sqrt{2M_3(u-\alpha_i)^2+\lambda M_3^2(u-\alpha_i)^4}}du,\end{aligned}$$

which implies that

$$\begin{aligned}\sqrt{\lambda} &\geq \frac{1}{L}\int_0^\rho \frac{1+\lambda M_3(\alpha_i-u)^2}{\sqrt{2M_3^2(\alpha_i-u)^2+\lambda M_3(\alpha_i-u)^4}}du\\ &\geq \frac{1}{L}\int_0^{\alpha_i} \frac{1+\lambda M_3(\alpha_i-u)^2}{\sqrt{2M_3^2(\alpha_i-u)^2+\lambda M_3(\alpha_i-u)^4}}du.\end{aligned}$$

So,

$$\lim_{\rho\to\alpha_i^+}\sqrt{\lambda_L(\rho)} \geq \lim_{\rho\to\alpha_i^+}\frac{1}{L}\int_0^{\alpha_i}\frac{1+\lambda M_3(\alpha_i-u)^2}{|\alpha_i-u|\sqrt{2M_3+\lambda M_3^2(\alpha_i-u)^2}}du.$$

Therefore, $\lim_{\rho\to\alpha_i^+}\lambda_L(\rho)=\infty$.

Following similar arguments, we can prove $\lim_{\rho\to\beta_i^-}\lambda_L(\rho)=\infty$. Therefore, the bifurcation curve $T_\lambda(\rho)$ goes to infinity along the horizontal lines $\|u\|_\infty=\alpha_i$, $\|u\|_\infty=\beta_i$, for $i\in\{1,2,\cdots,n-1\}$.

(iii) Following similar arguments in the proof of Theorem 2.1(ii), we have $\lim\limits_{\rho\to\alpha_n^+}\lambda_L(\rho)=\infty$. Next, we prove $\lim\limits_{\rho\to L^-}\lambda_L(\rho)=\infty$.

Assume that $\liminf\limits_{\rho\to L^-}\lambda_L(\rho)<\infty$. Then there exist $M_5>0$ and $\{\rho_n\}_{n\in\mathbb{N}}\subseteq\mathcal{S}$ such that

$$\lim_{n\to\infty}\rho_n = L \text{ and } \lambda_L(\rho_n)<M_5 \text{ for } n\in\mathbb{N}.$$

By Lemmas 3.2 and 3.5(i), we see that

$$L=\lim_{n\to\infty}T_{\lambda_L(\rho_n)}(\rho_n)\geq\lim_{n\to\infty}T_{M_5}(\rho_n)=T_{M_5}(L).$$

By Eq (3.2), it is easy to see that

$$\lim_{n\to\infty}T_{M_5}(\rho_n)>T_{M_5}(L)>L,$$

which is a contradiction. So $\lim_{\rho\to L^-}\lambda_L(\rho)\geq\liminf_{\rho\to L^-}\lambda_L(\rho)=\infty$. Therefore, the bifurcation curve S_L goes to infinity along the horizontal line $\|u\|_\infty=L$ for $L>0$. □

5. Application

In this section, we give two examples to demonstrate the feasibility of our result.

Example 5.1. Let us consider the following boundary value problem:

$$\begin{cases}-\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)'=\lambda\sin u, & \text{in } (0,3\pi),\\ u'(0)=u(3\pi)=0.\end{cases}\tag{5.1}$$

Obviously, $f(0) = 0$, $f_0 = 1$, $r = \pi$ and $\alpha_1 = 2\pi$. Let $a_1 = 4$ and $b_1 = 8$. Then it is easy to verify that $a_1 < \alpha_1 < b_1 < 8$ and $F(b_1) > F(u)$ for $0 \le u \le b_1$.

Therefore, from Theorem 2.1, Problem (5.1) has a positive solution (λ, u) satisfying $\|u\|_\infty < \pi$ for all $\lambda > \frac{1}{36}$; if $\lambda > \inf\{\lambda(\rho)|\rho \in (2\pi, 3\pi)\}$, then Problem (5.1) has at least two positive solutions (λ, u_1) and (λ, u_2) satisfying

$$2\pi < \|u_1\|_\infty < \|u_2\|_\infty < 3\pi.$$

Example 5.2. Let us consider the following boundary value problem:

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1-u'^2}}\Big)' = \lambda(u^3 - 4u^2 + 3u), \quad \text{in } (0, 5), \\ u'(0) = u(5) = 0. \end{cases} \tag{5.2}$$

Obviously, $f(0) = 0$, $f_0 = 3$, $r = 1$ and $\alpha_1 = \frac{8+\sqrt{10}}{3}$. Let $a_1 = 3$ and $b_1 = 4$. Then it is easy to verify that $a_1 < \alpha_1 < b_1 < 5$ and $F(b_1) > F(u)$ for $0 \le u \le b_1$. Therefore, from Theorem 2.1, Problem (5.2) has a positive solution (λ, u) satisfying $\|u\|_\infty < 1$ for all $\lambda > \frac{\pi^2}{300}$; if $\lambda > \inf\{\lambda(\rho)|\rho \in (3, 5)\}$, then Problem (5.2) has at least two positive solutions (λ, u_1) and (λ, u_2) satisfying

$$\frac{8+\sqrt{10}}{3} < \|u_1\|_\infty < \|u_2\|_\infty < 5.$$

6. Conclusions

By using time mapping techniques, we study the shape of the bifurcation curves of positive solutions for the one-dimensional Minkowski curvature problem. In particular, we prove that the bifurcation curve is $\subset$-shaped/monotone increasing/S-like shaped on the $(\lambda, \|u\|_\infty)$ plane when the nonlinear term f satisfies some appropriate assumptions. By figuring the shape of bifurcation curves of positive solutions, we show the existence and multiplicity of positive solutions with respect to the parameter λ.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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