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Research article

Variational approach to *p*-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses

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Abstract: In this paper, we examine the existence of solutions of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses. New criteria guaranteeing the existence of infinitely many solutions are established for the considered problem. The problem is reduced to an equivalent form such that the weak solutions of the problem are defined as the critical points of an energy functional. The main result of the present work is established by using a variational approach and a mountain pass lemma. Finally, an example is given to illustrate our main result.

Keywords: fractional differential equation; impulsive effects; p-Laplacian operator; variational approach; mountain pass lemma
Mathematics Subject Classification: 34A08, 34B37

1. Introduction

In this paper, we consider the following system of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) + a(t)\Phi_p\left(u(t)\right) = \lambda f_i(t,\ u(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, m, \\ \Delta_t D_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i)\right)\right) = I_i(u(t_i)), & i = 1,\ 2,\ \cdots,\ m, \\ {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) = {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i^+)\right)\right), & t \in (t_i, s_i],\quad i = 1,\ 2,\ \cdots,\ m, \\ {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^-)\right)\right) = {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^+)\right)\right), & i = 1,\ 2,\ \cdots,\ m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0^cD_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left Caputo fractional derivative and the right Riemann-Liouville fractional derivative of order α, respectively. Additionally, $1 < p < +\infty$ and $\frac{1}{p} < \alpha \leq 1$. $\Phi_p(s) = |s|^{p-2}s(s \neq 0)$ is the p-Laplacian operator and $\Phi_p(0) = 0$. Furthermore, $\lambda > 0$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_n <$

$t_{n+1} = T$ and $a \in C([0,T],\mathbb{R})$. There are two positive constants a_1 and a_2 such that $0 < a_1 \leq a(t) \leq a_2$. $I_i \in C(\mathbb{R},\mathbb{R})$ and $f_i \in C([s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$; also

$$\Delta {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i)\right)\right) = {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i^+)\right)\right) - {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i^-)\right)\right),$$

$${}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i^\pm)\right)\right) = \lim_{t\to t_i^\pm} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i)\right)\right),$$

$${}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(s_i^\pm)\right)\right) = \lim_{t\to s_i^\pm} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^\alpha u(t_i)\right)\right).$$

It is generally known that the impulsive effect is a common phenomenon in many evolutionary processes, that it describes the abrupt disturbance phenomenon at certain moments in the evolutionary process of the system. Impulsive differential equations, as a good tool to describe impulsive phenomena in systems, have become popular in recent years; for example the existence of solutions [1, 2, 3], stability of solutions [4, 5, 6], oscillation of solutions [7, 8] and numerical solutions [9, 10] of impulsive differential equations have been extensively studied. In particular, the existence of solutions is the premise of the quantitative and qualitative research on the solutions of differential equations. Since significant applications in various sciences include the population dynamics, pharmacology and optimal control [11, 12], some important methods have been obtained for the study of solutions of fractional differential equations. Fix point theorems [13], upper and lower solutions method [14], coincidence degree theory [15] and variational methods [16] have been viable tools to study the existence of solutions of differential equations. The variational methods are applied to study the existence of solution of fractional differential equations, which can be traced back to Reference [16]. The key to studying the existence of solutions of differential equations via the variational method is to define a suitable functional space and transform the boundary value problem into an energy functional.

The p-Laplacian operator is a non-standard growth operator, which arises from nonlinear electrorheological fluids [17], image restoration [18], elasticity theory [19], etc. Fractional differential equations with a p-Laplacian operator have been widely applied for in many physical phenomena, such as nonlinear diffusion and filtration, non-fluids and flows in porous media [20, 21, 22]. The literature on p-Laplacian fractional differential equations with impulsive conditions include References [23, 24, 25, 26, 27, 28, 29, 30]. We are interested in the existence of infinitely many solutions when the p-Laplacian operator is added to the fractional impulsive differential equations. To the best of our knowledge, to date, the existence of infinitely many solutions of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses has been rarely investigated. The main contributions of this work include the following: (i) new criteria are introduced to verify the existence of infinitely many solutions of fractional order nonlinear systems; (ii) an example is given to ensure the validity of our conclusion.

In a recent paper [31], the authors discussed the following system of non-instantaneous impulsive fractional differential equations:

$$\begin{cases} {}_tD_T^\alpha\left({}_0^CD_t^\alpha u(t)\right) = f_i(t), & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\left(\left({}_0^CD_t^\alpha u(t_i)\right)\right) = c_i\,, & t \in (t_i, s_i],\quad i = 1,\ 2,\ \cdots,\ n, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u(s_i^-)\right) = {}_tD_T^{\alpha-1}\left(\left({}_0^CD_t^\alpha u(s_i^+)\right)\right), & i = 1,\ 2,\ \cdots,\ n, \\ {}_tD_T^{\alpha-1}\left(\left({}_0^CD_t^\alpha u(0)\right)\right) = c_0, & u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

The authors gave sufficient conditions for the existence and uniqueness of weak solutions of the fractional system by using the Lax-Milgram theorem.

In [32], the authors discussed the following system of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) + \Phi_p\left(u(t)\right) = \lambda f_i(t,\ u(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, m, \\ \Delta_t D_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i)\right)\right) = I_i(u(t_i)), & i = 1,\ 2,\ \cdots,\ m, \\ {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right) = {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i^+)\right)\right), & t \in (t_i, s_i],\quad i = 1,\ 2,\ \cdots,\ m, \\ {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^-)\right)\right) = {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^+)\right)\right), i = 1,\ 2,\ \cdots,\ m, \\ u(0) = u(T) = 0. \end{cases} \tag{1.3}$$

The authors proved that System (1.3) has infinitely many solutions by applying variational methods and the critical theory. The main result of [32] is as follows.

Theorem 1.1. ([32] *Theorem* 3.1) *Suppose that the following conditions hold:*
(A_1) *For* $i = 1, \cdots, m$, *there exist* $a_i > 0$ *and* $0 \le \tau_i < p - 1$ *such that*

$$\lim_{|u|\to\infty} \sup \frac{|I_i(u)|}{|u|^{\tau_i}} \le a_i, \qquad u \in \mathbb{R}.$$

(A_2)

$$\lim_{\xi\to+\infty} \inf \frac{\sum\limits_{i=0}^{m}\int_{s_i}^{t_{i+1}} \max\limits_{|u|\le\xi} F_i(t,u)dt}{\xi^p} < \frac{1}{T\Lambda^p} \lim_{\xi\to+\infty} \sup \frac{\sum\limits_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,\xi)dt}{\xi^p},$$

where $F_i(t,u) = \int_0^u f_i(t,s)ds$.

Then, for $\lambda \in (\lambda_1, \lambda_2)$, *where* $\lambda_1 = \frac{T}{pB}$ *and* $\lambda_2 = \frac{1}{pA\Lambda^p}$, $A = \lim\limits_{\xi\to+\infty} \inf \frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \max_{|u|\le\xi} F_i(t,u)dt}{\xi^p}$ *and* $B = \lim\limits_{\xi\to+\infty} \sup \frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,\xi)dt}{\xi^p}$, *also, given the definition of* Λ *in Remark* (2.6), *the System* (1.3) *possesses infinitely many solutions.*

Motivated by the fact above, the main aim of this work was to establish infinitely many solutions for System (1.1) under different assumptions and show that our results are totally different from those above. Now, we will state our main result.

Theorem 1.2. *Suppose that the following conditions hold:*
(H_1) *There exists a constant* $\nu > p$ *such that* $I_i(u)u \le \nu \int_0^u I_i(s)ds < 0$ *for* $\forall u \in E_0^{\alpha,p}\backslash\{0\}$, $i = 1, 2, \cdots, m$.
(H_2) *There exists a constant* $\omega \in (p, \nu)$ *such that* $0 < \omega F_i(t,u) \le f_i(t,u)u$ *for* $\forall u \in E_0^{\alpha,p}\backslash\{0\}$, *and given* $t \in [0, T]$, $i = 0, 1, 2, \cdots, m$.
(H_3) *There exist constants* $\delta_i > 0$ *such that* $\int_0^u I_i(s)ds \ge -\delta_i|u|^{\nu}$ *for* $\forall u \in E_0^{\alpha,p}\backslash\{0\}$, $i = 1, 2, \cdots, m$.

Moreover, $f_i(t,u)$ *and* $I_i(u)$ *are odd about* u, $i = 0, 1, 2, 3, \cdots, m$. *Then, problem (1.1) admits infinitely many solutions when* $\lambda \in (0, \frac{1}{(m+1)BT\Lambda^{\omega}p})$.

The main ideas of this work are organized as follows. In Section 2, some preliminaries and results that are applied later in the paper are presented. In Section 3, we focus on the proof of the existence of infinitely many solutions of the p-Laplacian fractional impulsive differential system. In Section 4, an example is given to illustrate our main result.

2. Preliminaries

Definition 2.1. *([33]) (left and right Rimann-Liouville fractional derivatives) Let u be a function defined on $[a,b]$. The left and right Rimann-Liouville fractional derivatives of order $0 \le \alpha < 1$ for a function u denoted by ${}_aD_t^{\alpha}u(t)$ and ${}_tD_b^{\alpha}u(t)$, respectively, are defined by*

$$ {}_aD_t^{\alpha}u(t) = \frac{d}{dt}{}_aD_t^{\alpha-1}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\alpha}u(s)ds\right), $$

$$ {}_tD_b^{\alpha}u(t) = -\frac{d}{dt}{}_tD_b^{\alpha-1}u(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\alpha}u(s)ds\right), $$

where $t \in [a,b]$.

Definition 2.2. *([33]) (left and right Caputo fractional derivatives) Let $0 < \alpha < 1$ and $u \in AC([a,b],\mathbb{R}^N)$; then the left and right Caputo fractional derivatives of order α for a function u denoted by ${}_a^CD_t^{\alpha}u(t)$ and ${}_t^CD_b^{\alpha}u(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^CD_t^{\alpha}u(t)$ and ${}_t^CD_b^{\alpha}u(t)$ are respectively represented by*

$$ {}_a^CD_t^{\alpha}u(t) = {}_aD_t^{\alpha-1}u'(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}u'(s)ds\right), $$

$$ {}_t^CD_b^{\alpha}u(t) = -{}_tD_b^{\alpha-1}u'(t) = -\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-t)^{-\alpha}u'(s)ds\right), $$

where $t \in [a,b]$.

Let us recall that, for any fixed $t \in [0,T]$ and $1 < p < \infty$,

$$ \|u\|_{\infty} = \max_{t\in[0,T]}|u(t)|,\ u \in C([0,T],R^N), $$

$$ \|u\|_{L^p} = \left(\int_0^T |u(s)|^p ds\right)^{\frac{1}{p}},\ u \in L^p([0,T],\mathbb{R}^N), $$

$$ \|u\|_{L^p([0,t])} = \left(\int_0^t |u(s)|^p ds\right)^{\frac{1}{p}},\ u \in L^p([0,t],\mathbb{R}^N), $$

$$ \|u\|_{L^p([s_i,t_{i+1}])} = \left(\int_{s_i}^{t_{i+1}} |u(s)|^p ds\right)^{\frac{1}{p}},\ u \in L^p([s_i,t_{i+1}],\mathbb{R}^N), i = 0,1,2,\cdots,m. $$

Definition 2.3. *Let $\alpha \in \left(\frac{1}{p},1\right]$ and $p \in (1,\infty)$, the fractional derivative space $E_0^{\alpha,p}$ is defined as the closure of $C_0^{\infty}([0.T];\mathbb{R})$, that is, $E_0^{\alpha,p} = \overline{C_0^{\infty}([0,T],\mathbb{R})}$ with the norm*

$$ \|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^CD_t^{\alpha}u(t)|^p dt\right)^{\frac{1}{p}},\ \forall u \in E_0^{\alpha,p}. $$

Proposition 2.1. *([34]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Then*

$$E_0^{\alpha,p} = \left\{u : [0,T] \to \mathbb{R} \middle| u,\ {}_0^C D_t^\alpha u(t) \in L^p([0,T],\mathbb{R}), u(0) = u(T) = 0\right\},$$

and the space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Proposition 2.2. *([35]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Then for $\forall u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}};$$

moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}} \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}}.$$

Remark 2.1. *It is easy to check that if $a \in C([0,T],\mathbb{R})$ and $0 < a_1 \leq a(t) \leq a_2$, then an equivalent norm in $E_0^{\alpha,p}$ is as follows*

$$\|u\|_a = \left(\sum_{i=0}^m \int_{s_i}^{t_{i+1}} a(t)|u(t)|^p dt + \int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}}$$

and $\|u\|_\infty \leq \wedge \|u\|_a$, where $\wedge = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}$.

Proposition 2.3. *([16]) Assume that $\frac{1}{p} < \alpha \leq 1$ and $1 < p < \infty$; then, the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e. $u_k \rightharpoonup u$; then, $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e. $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.*

Definition 2.4. *A function*

$$u \in \left\{u \in AC([0,T]) : \int_{s_i}^{t_{i+1}} \left(|u(t)|^p + |{}_0^C D_t^\alpha|^p\right) dt < +\infty, i = 0,1,2,\cdots,m\right\}$$

is a classical solution of Problem (1.1) if u satisfies the following conditions: (i) u satisfies the first equation of Problem (1.1); (ii) the limit ${}_tD_T^{\alpha-1}\Phi_p\left({}_0^C D_t^\alpha u(t_i^\pm)\right)$ and limit ${}_tD_T^{\alpha-1}\Phi_p\left({}_0^C D_t^\alpha u(s_i^\pm)\right)$ exist; (iii) u satisfies the impulsive conditions of Problem (1.1); (iv) u satisfies the boundary value condition $u(0) = u(T) = 0$.

Lemma 2.1. *([36]) Let X be an infinite dimensional Banach space, $I \in C^1(X,\mathbb{R})$ be even, thereby satisfying the Palais-Smale condition, and let $I(0) = 0$. Suppose that $X = Y \bigoplus Z$, where Y is finite dimensional and I satisfies the following conditions:*
(i) there exists constants ρ, $\gamma > 0$ such that $I\big|_{\partial B_\rho \cap Z} \geq \gamma$;
(ii) for each finite dimensional subspace $W \subset X$, there is an $\rho' = \rho'(W)$ such that $I(u) \leq 0$ on $E_1 \backslash B_{\rho'(W)}$; then, I has an unbounded sequence of critical values.

Proposition 2.4. *([35])*

(i) *Let* $\alpha > 0$*,* $p \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha (p \neq 1, q \neq 1$*; in the case when* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha)$*, if* $u \in L^p(a,b)$ *and* $v \in L^q(a,b)$*, then*

$$\int_a^b \left({}_aD_t^{-\alpha}u(t)\right)v(t)dt = \int_a^b u(t)\left({}_tD_b^{-\alpha}v(t)\right)dt.$$

(ii) *Let* $0 < \alpha < 1$*,* $u \in AC([a,b])$ *and* $v \in L^p(a,b)(1 \leq p < \infty)$*; then,*

$$\int_a^b u(t)\left({}_a^CD_t^{\alpha}v(t)\right)dt = \left({}_tD_b^{\alpha-1}u(t)\right)v(t)\Big|_{t=a}^{t=b} + \int_a^b \left({}_tD_b^{\alpha}u(t)\right)v(t)dt.$$

For $\forall v \in E_0^{\alpha,p}$, by Proposition 2.10, we have

$$\begin{aligned}
&\int_0^T {}_tD_T^{\alpha}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v(t)dt\\
=&\sum_{i=0}^m \int_{s_i}^{t_{i+1}} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v(t)dt + \sum_{i=1}^m \int_{t_i}^{s_i} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v(t)dt\\
=&-\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \frac{d}{dt}\left({}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)\right)v(t)dt - \sum_{i=1}^m \int_{t_i}^{s_i} \frac{d}{dt}\left({}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)\right)v(t)dt\\
=&-\sum_{i=0}^m {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v(t)\Big|_{s_i^+}^{t_{i+1}^-} + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v'(t)dt\\
&-\sum_{i=1}^m {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v(t)\Big|_{t_i^+}^{s_i^-} + \sum_{i=1}^m \int_{t_i}^{s_i} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v'(t)dt\\
=&\sum_{i=1}^m \left[{}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i^+)\right)\right)v(t_i^+) - {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t_i^-)\right)\right)v(t_i^-)\right]\\
&+\sum_{i=1}^m \left[{}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^+)\right)\right)v(s_i^+) - {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(s_i^-)\right)\right)v(s_i^-)\right]\\
&+\sum_{i=0}^m \int_{s_i}^{t_{i+1}} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v'(t)dt + \sum_{i=1}^m \int_{t_i}^{s_i} {}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)v'(t)dt\\
=&\sum_{i=1}^m I_i\left(u(t_i)\right)v(t_i) + \int_0^T \Phi_p\left({}_0^CD_t^{\alpha}u(t)\right){}_0^CD_t^{\alpha}v(t)dt.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^T {}_tD_T^{\alpha}\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)v(t)dt\\
=&\lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} a(t)|u(t)|^{p-2}u(t)v(t)dt\\
&-\sum_{i=1}^m \int_{t_i}^{s_i} \frac{d}{dt}\left({}_tD_T^{\alpha-1}\left(\Phi_p\left({}_0^CD_t^{\alpha}u(t)\right)\right)\right)v(t)dt
\end{aligned}$$

$$=\lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt - \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} a(t)|u(t)|^{p-2}u(t)v(t)dt.$$

Hence, we have

$$\begin{aligned}
&\int_0^T \Phi_p\left({}_0^C D_t^{\alpha} u(t)\right) {}_0^C D_t^{\alpha} v(t)dt + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} a(t)|u(t)|^{p-2}u(t)v(t)dt \\
=&\lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt - \sum_{i=1}^{m} I_i(u(t_i))v(t_i).
\end{aligned} \tag{2.1}$$

Definition 2.5. *A function $u \in E_0^{\alpha,p}$ is a weak solution of problem (1.1), if Eq (2.1) holds for $\forall v \in E_0^{\alpha,p}$.*

Remark 2.2. *([30]) $u \in E_0^{\alpha,p}$ is a weak solution of Problem* (1.1) *if and only if u is a classical solution of Problem* (1.1).

In order to study the existence of infinitely many solutions of problem (1.1), we consider the function $\varphi : E_0^{\alpha,p} \to \mathbb{R}$ as follows

$$\varphi(u) = \frac{1}{p}\|u\|_a^p + \sum_{i=1}^{m} \int_0^{u(t_i)} I_i(s)ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,u(t))dt, \tag{2.2}$$

where $F_i(t,u) = \int_0^u f_i(t,s)ds$. Now let

$$\psi(u) = \frac{1}{p}\|u\|_a^p, \ \phi(u) = \sum_{i=1}^{m} \int_0^{u(t_i)} I_i(s)ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,u(t))dt;$$

then, $\varphi = \psi + \phi$. It is easy to get that φ ,ψ and ϕ are Fr*é*chet differentials at any point $u \in E_0^{\alpha,p}$ and that

$$\begin{aligned}
\varphi'(u)(v) = &\int_0^T \Phi_p\left({}_0^C D_t^{\alpha} u(t)\right) {}_0^C D_t^{\alpha} v(t)dt + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} a(t)|u(t)|^{p-2}u(t)v(t)dt \\
&+ \sum_{i=1}^{m} I_i u(t_i)v(t_i) - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt, \quad for\ \forall v \in E_0^{\alpha,p}.
\end{aligned} \tag{2.3}$$

3. Main results

Now, we will illustrate Theorem 1.1.

It is easy to know that $\varphi(0) = 0$ and $\varphi(u)$ about u is even. Next we shall illustrate the other conditions of Lemma 2.9 by describing the following three steps.

Step 1 We illustrate that φ satisfies the Palais-Smale condition. Assume that $u_k \subset E_0^{\alpha,p}$ is a sequence such that $\varphi(u_k)$ is bounded, i.e., there exists a constant $c > 0$ such that $|\varphi(u_k)| \leq c$ and $\lim_{k\to\infty} \varphi'(u_k) = 0$. We first prove that $\{u_k\}, k \in N$ is bounded in $E_0^{\alpha,p}$. By Condition (H_1), we have

$$\omega \int_0^{u_k(t_i)} I_i(s)ds - I_i(u_k(t_i))u_k(t_i) \geq (\omega - \nu) \int_0^{u_k(t_i)} I_i(s)ds \geq 0, \ (\nu > \omega). \tag{3.1}$$

From Condition (H_2), we have that

$$f_i(t,u(t))u(t) - \omega F_i(t,u(t)) \geq 0. \tag{3.2}$$

Thus, together (2.2), (2.3), (3.1) and (3.2), we have

$$\begin{aligned}
\omega\varphi(u_k) - \varphi'(u_k)u_k =& \frac{\omega}{p}\|u_k\|_a^p + \omega \sum_{i=1}^{m} \int_0^{u_k(t_i)} I_i(s)ds - \omega\lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,u_k(t))dt \\
&- \|u_k\|_a^p - \sum_{i=1}^{m} I_i(u_k(t_i))u_k(t_i) + \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i(t,u_k(t))u_k(t)dt \\
=& \left(\frac{\omega}{p} - 1\right)\|u_k\|_a^p + \omega \sum_{i=1}^{m} \left[\int_0^{u_k(t_i)} I_i(s)ds - I_i(u_k(t_i))u_k(t_i) \right] \\
&+ \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i(t,u_k(t))u_k(t) - \omega F_i(t,u_k(t))dt \\
\geq& \left(\frac{\omega}{p} - 1\right)\|u_k\|_a^p,
\end{aligned}$$

which implies that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$.

Since $E_0^{\alpha,p}$ is a reflexive Banach space, going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$ and that $u_k \to u$ uniformly in $([0,T],R)$; hence,

$$\begin{cases} \displaystyle\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} (f_i(t,u_k(t)) - f_i(t,u(t)))(u_k(t) - u(t))dt \to 0, \\ \displaystyle\sum_{i=1}^{m} (I_i(u_k(t_i)) - I_i(u(t_i)))(u_k(t_i) - u(t_i)) \to 0 \end{cases}$$

as $k \to \infty$. Moreover, by $\varphi'(u_k) \to 0$ as $k \to \infty$, we have

$$< \varphi'(u_k) - \varphi'(u), u_k - u > \leq \|u_k - u\|_{(E_0^{\alpha,p})^*}\|u_k - u\|_{E_0^{\alpha,p}} - < \varphi'(u), u_k - u > \to 0,$$

as $k \to \infty$. On the other hand, for $\forall u, v \in E_0^{\alpha,p}$, we have

$$\begin{aligned}
&< \psi'(u_k) - \psi'(u), u_k - u > \\
=& \int_0^T \left(\phi_p\left({}_0^C D_t^\alpha(u_k(t)\right) - \phi_p\left({}_0^C D_t^\alpha(u(t))\right)\right)\left({}_0^C D_t^\alpha(u_k(t)) - {}_0^C D_t^\alpha(u(t))\right) \\
&+ \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_k(t)) - \phi_p(u(t))\right)(u_k(t) - u(t))dt.
\end{aligned}$$

From the well-known inequality

$$|x-y|^p \leq \begin{cases} \left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y), & if\ p \geq 2, \\ \left(\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y)\right)^{\frac{p}{2}} (|x|^p + |y|^p)^{\frac{2-p}{2}}, & if\ 1 < p < 2, \end{cases}$$

for all $x, y \in R$, we can show that there exist constants $C_i > 0 (i = 1, 2)$ such that
$< \psi^{'}(u_k) - \psi^{'}(u), u_k - u >$

$$\geq \begin{cases} C_1\left(\int_0^T \left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^p dt + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} a(t)\left|u_k(t) - u(t)\right|^p dt\right), if\ p \geq 2, \\ C_2\left(\int_0^T \frac{\left|{}_0^C D_\alpha^t u_k(t) -{}_0^C D_\alpha^t u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}}dt + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \frac{a(t)|u_k(t) - u(t)|^2}{(|u_k(t)| + |u(t)|)^{2-p}}dt\right), if\ 1 < p < 2. \end{cases} \tag{3.3}$$

When $1 < p < 2$, using the Hölder inequality, we have

$$\begin{aligned}
&\int_0^T \left(\left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^p\right) dt \\
=&\int_0^T \frac{\left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^p}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{\frac{p(2-p)}{2}}}\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{\frac{p(2-p)}{2}} dt \\
\leq&\left\{\left[\int_0^T \frac{\left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^p}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{\frac{p(2-p)}{2}}}\right]^{\frac{2}{p}} dt\right\}^{\frac{p}{2}}\left[\int_0^T \left(\left|{}_0^C D_t^\alpha u_k(t)dt\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{\frac{p(2-p)}{2}\frac{2}{2-p}} dt\right]^{\frac{2-p}{2}} \\
=&\left[\int_0^T \frac{\left|{}_0^C D_t^\alpha u_k(t) -{}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{(2-p)}}dt\right]^{\frac{p}{2}}\left[\int_0^T \left(\left|{}_0^C D_t^\alpha u_k(t)dt\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^p dt\right]^{\frac{2-p}{2}} \\
\leq&\left[\int_0^T \frac{\left|{}_0^C D_t^\alpha u_k(t) -{}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{(2-p)}}dt\right]^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}\left(\left\|{}_0^C D_t^\alpha u_k(t)\right\|_{L^p}^p + \left\|{}_0^C D_t^\alpha u(t)\right\|_{L^p}^p\right)^{\frac{2-p}{2}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_0^T \frac{\left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_k(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}}dt \\
\geq& 2^{\frac{(p-2)(p-1)}{p}}\left(\left\|{}_0^C D_t^\alpha u_k(t)\right\|_{L^p}^p + \left\|{}_0^C D_t^\alpha u(t)\right\|_{L^p}^p\right)^{\frac{p-2}{p}}\left(\int_0^T \left|{}_0^C D_t^\alpha u_k(t) - {}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{2}{p}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{s_i}^{t_{i+1}} \left|u_k(t) - u(t)\right|^p dt \\
\leq&\left(\int_{s_i}^{t_{i+1}} \frac{\left|u_k(t) - u(t)\right|^2}{\left|u_k(t) + u(t)\right|^{(2-p)}}dt\right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}\left(\left\|u_k(t)\right\|_{L^p[s_i,t_{i+1}]}^p + \left\|u(t)\right\|_{L^p[s_i,t_{i+1}]}^p\right)^{\frac{2-p}{2}}
\end{aligned}$$

and

$$\int_{s_i}^{t_{i+1}} \frac{\left|u_k(t)-u(t)\right|^2}{\left(\left|u_k(t)+u(t)\right|\right)^{2-p}}dt$$
$$\geq 2^{\frac{(p-2)(p-1)}{p}}\left(\left\|u_k(t)\right\|^p_{L^p[s_i,t_{i+1}]}+\left\|u(t)\right\|^p_{L^p[s_i,t_{i+1}]}\right)^{\frac{p-2}{p}}\left(\int_{s_i}^{t_{i+1}}\left|u_k(t)-u(t)\right|^p dt\right)^{\frac{2}{p}}.$$

Further, when $1 < p < 2$, we have

$$< \psi^{'}(u_k(t)) - \psi^{'}u(t), u_k(t)-u(t) >$$
$$\geq C_2\Bigg[2^{\frac{(p-2)(p-1)}{p}}\left(\left\|{}_0^CD_t^\alpha u_k(t)\right\|^p_{L^p}+\left\|{}_0^CD_t^\alpha u(t)\right\|^p_{L^p}\right)^{\frac{p-2}{p}}\left(\int_0^T\left|{}_0^CD_t^\alpha u_k(t)-{}_0^CD_t^\alpha u(t)\right|^p dt\right)^{\frac{2}{p}}$$
$$+\,2^{\frac{(p-2)(p-1)}{p}}\sum_{i=0}^{m}\left(\left\|u_k(t)\right\|^p_{L^p[s_i,t_{i+1}]}+\left\|u(t)\right\|^p_{L^p[s_i,t_{i+1}]}\right)^{\frac{p-2}{p}}\left(\int_{s_i}^{t_{i+1}}a(t)\left|u_k(t)-u(t)\right|^p dt\right)^{\frac{2}{p}}\Bigg].$$

Let

$$m_1 = \min\left\{\left(\left\|u_k(t)\right\|^p_{L^p_{[s_i,t_{i+1}]}}+\left\|u(t)\right\|^p_{L^p_{[s_i,t_{i+1}]}}\right)^{\frac{p-2}{p}}\middle| i=0,1,2,\cdots,m\right\},$$

$$M = \min\left\{2^{\frac{(p-2)(p-1)}{p}}\left(\left\|{}_0^CD_t^\alpha u_k(t)\right\|^p_{L_p}+\left\|{}_0^CD_t^\alpha u(t)\right\|^p_{L_p}\right)^{\frac{p-2}{p}},\ 2^{\frac{(p-2)(p-1)}{p}}(m+1)m_1\right\}.$$

Thus, we have

$$< \psi^{'}(u_k) - \psi^{'}(u), u_k - u > \geq C_2M\left\|u_k-u\right\|^2_a.$$

This means that $\left\|u_k-u\right\|_a \to 0\ \ as\ k \to +\infty$.
Moreover, in view of Eq (3.3), when $p \geq 2$ we have

$$< \psi^{'}(u_k) - \psi^{'}(u), u_k - u > \geq C_1\left\|u_k-u\right\|^p_a.$$

Thus $\left\|u_k-u\right\|_a \to 0\ \ as\ k \to +\infty$. In summary, $\varphi(u)$ satisfies the Palais-Smale condition.
Step 2 We illustrate that φ satisfies Condition (i) of Lemma 2.9. Applying condition (H_2), we have that for $\forall R_1 > 0$

$$F_i(t,u(t)) \leq \frac{u}{\omega}f_i(t,u(t)),\ \ u(t) \geq R_1,$$
$$F_i(t,u(t)) \leq \frac{u}{\omega}f_i(t,u(t)),\ \ u(t) \leq -R_1.$$

Further, we have the following

$$\frac{\omega}{u} \leq \frac{f_i(t,u(t))}{F_i(t,u(t))},\ \ u(t) \geq R_1, \tag{3.4}$$

$$\frac{\omega}{u} \leq \frac{f_i(t,u(t))}{F_i(t,u(t))}, \quad u(t) \leq -R_1. \tag{3.5}$$

Then, by integrating Eqs (3.4) and (3.5) from R_1 to $u(t)$ and $u(t)$ to $-R_1$, respectively, we obtain the following:

$$\int_{R_1}^{u(t)} \frac{\omega}{s} ds \leq \int_{R_1}^{u(t)} \frac{f_i(t,s)}{F_i(t,s)} ds,$$

$$\int_{u(t)}^{-R_1} \frac{\omega}{s} ds \leq \int_{u(t)}^{-R_1} \frac{f_i(t,s)}{F_i(t,s)} du.$$

Moreover, we have

$$\int_{u(t)}^{R_1} \frac{\omega}{s} ds \geq \int_{u(t)}^{R_1} \frac{f_i(t,s)}{F_i(t,s)} ds,$$

$$\int_{-R_1}^{u(t)} \frac{\omega}{s} ds \geq \int_{-R_1}^{u(t)} \frac{f_i(t,s)}{F_i(t,s)} ds.$$

By calculation, we have

$$F_i(t,u(t)) \geq A_i|u(t)|^{\omega}, \quad |u(t)| \geq R_1, \quad F_i(t,u(t)) \leq B_i|u(t)|^{\omega}, \quad 0 < |u(t)| \leq R_1,$$

where

$$A_i = R_1^{-\omega} \min_{t\in[s_i,t_{i+1}]} \{F_i(t,R_1), F_i(t,-R_1)\} > 0$$
$$B_i = R_1^{-\omega} \max_{t\in[s_i,t_{i+1}]} \{F_i(t,R_1), F_i(t,-R_1)\} > 0$$

Since $F_i(t,u(t))$ is continuous, there exist constants $D_i > 0$ such that

$$F_i(t,u(t)) \geq A_i|u(t)|^{\omega} - D_i, \quad u(t) \in E_0^{\alpha,p}.$$

By applying the inequalities $F_i(t,u(t)) \leq B_i|u|^{\omega}$ and $0 < |u| \leq R_1$, we have that

$$\int_{s_i}^{t_{i+1}} F_i(t,u(t)) \leq \int_{s_i}^{t_{i+1}} B_i|u(t)|^{\omega} \leq B_i T \Lambda^{\omega} \|u\|_a^{\omega}.$$

Since $E_0^{\alpha,p}$ is a reflexive and separable Banach space, there exists $e_i \in E_0^{\alpha,p}$, $i = 1,2,3,\cdots$, such that $E_0^{\alpha,p} = \overline{span\{e_i : i = 1,2,\cdots\}}$. For $k = 1,2,\cdots$, denote

$$X_i := span\{e_i\}, \quad Y_k £ = \bigoplus_{i=1}^{k} X_i, \quad Z_k := \overline{\bigoplus_{i=k}^{\infty} X_i}.$$

Then $E_0^{\alpha,p} = Y_k \bigoplus Z_k$. For $\forall u \in Z_k$ with $\|u\|_a < 1$, and together with Condition (H_3), we have

$$\varphi(u) = \frac{1}{p}\|u\|_a^p + \sum_{i=1}^{m} \int_0^{u(t_i)} I_i(s) ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i t, u(t) dt$$

$$\geq \frac{1}{p}\|u\|_a^p - \delta_i \sum_{i=1}^{m} |u(t)|^\nu - \lambda \sum_{i=0}^{m} B_i T \Lambda^\omega \|u\|_a^\omega$$
$$\geq \frac{1}{p}\|u\|_a^p - m\delta\Lambda^\nu \|u\|_a^\nu - \lambda(m+1)BT\Lambda^\omega \|u\|_a^\omega$$
$$\geq \left(\frac{1}{p} - \lambda(m+1)BT\Lambda^\omega\right)\|u\|_a^p - m\delta\Lambda^\nu\|u\|_a^\nu,$$

where $\delta = \max\{\delta_1, \delta_2, \cdots, \delta_m\}$ and $B = \max\{B_1, B_2, \cdots, B_m\}$. Since $\nu > p$, when $\lambda \in (0, \frac{1}{(m+1)BT\Lambda^\omega p})$, there exists a constant $\rho > 0$ and such that when $\|u\|_a = \rho$, $\varphi(u) = \eta > 0$.

Step 3 We illustrate that φ satisfies Condition (ii) of Lemma 2.9. From Condition (H_1) we have

$$\sum_{i=1}^{m} \int_0^{u(t_i)} I_i(s)ds < 0.$$

Thus

$$\begin{aligned}
\varphi(\xi u) &= \frac{1}{p}\|\xi u\|_a^p + \sum_{i=1}^{m} \int_0^{\xi u(t_i)} I_i(s)ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i=1}} F_i(t, \xi u(t))dt \\
&\leq \frac{1}{p}\|\xi u\|_a^p - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} A_i|\xi u(t)|^\gamma - D_i dt \\
&\leq \frac{1}{p}\|\xi u\|_a^p + \lambda(m+1)DT - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} A_i|\xi u(t)|^\gamma dt \\
&= \frac{1}{p}\|\xi u\|_a^p + \lambda(m+1)DT - \lambda \sum_{i=0}^{m} A_i \xi^\gamma \|u\|^\gamma_{L^\gamma_{[s_i,t_{i+1}]}},
\end{aligned}$$

where $D = \max\{D_0,\ D_1,\ D_2,\ \cdots,\ D_m\}$. Since $\gamma > p$, the above inequality implies that for any finite dimensional space $E_1 \subset E_0^{\alpha,p}$ for $\forall u(t) \in E_1$ there exists a sufficiently large constant ξ such that $\|\xi u\|_a > \rho^{'}$ and $\varphi(\xi u) < 0$. □

4. Example

In this section, we give a simple example to prove our main result. Consider the following equation:

$$\begin{cases}
{}_tD_T^\alpha\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(t)\right)\right) + a(t)\phi_{\frac{3}{2}}(u(t)) = \lambda(2t^2 + \cos t)u^3(t), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\
\Delta_t D_T^{\alpha-1}\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(t_i)\right)\right) = -2u^5(t_i), & i = 1,\ 2,\ \cdots,\ m, \\
{}_tD_T^{\alpha-1}\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(t)\right)\right) = {}_tD_T^{\alpha-1}\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(t_i^+)\right)\right), & t \in (t_i, s_i],\quad i = 1,\ 2,\ \cdots,\ m, \\
{}_tD_T^{\alpha-1}\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(s_i^-)\right)\right) = {}_tD_T^{\alpha-1}\left(\phi_{\frac{3}{2}}\left({}_0^CD_t^\alpha u(s_i^+)\right)\right), & i = 1,\ 2,\ \cdots,\ m, \\
u(0) = u(T) = 0,
\end{cases} \tag{4.1}$$

where $p = 1.5$ and $f_i(t, u) = (2t^2+\cos t)u^3(t)$ and $I_i(u) = -2u^5$ are odd about u. Take $\nu = 6$, $\omega = 4$, $\delta_i = \frac{1}{3}$ and $R_1 = 1$. By simple computation, Conditions $(H_1) - (H_3)$ are satisfied and $B \leq \frac{1}{4}(2T^2 + 1)$. Thus, from Theorem 1.1, when $\lambda \in \left(0, \frac{4}{(m+1)(2T^2+1)T\Lambda^4 p}\right)$ (Λ is defined in Remark 2.6), Problem (4.1) admits infinitely many solutions.

5. Conclusions

In this paper, we explored the existence of solutions of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses; the corresponding result has been presented. Based on the variational method and a mountain pass lemma, sufficient conditions for the existence of infinitely many solutions were obtained. The uniqueness and stability of solutions of the initial value problem of the systems should be considered, this may be our future work.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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