



Research article

Symmetry analysis of an acid-mediated cancer invasion model

Juya Cui^{1,*} and Ben Gao²

¹ Department of Lymphoma, Shanxi Bethune Hospital, Shanxi Academy of Medical Sciences, Taiyuan, Shanxi 030032, China

² College of Mathematics, Taiyuan University of Technology, Taiyuan, Shanxi 030024, China

* **Correspondence:** Email: cuijuya@163.com.

Abstract: Under investigation in this paper is a reaction-diffusion system, which describes acid-mediated tumor growth. First, in view of Lie group analysis, infinitesimal generators of the considered system are presented. At the same time, some group invariant solutions are computed using reduced equations. In particular, we construct explicit solutions by applying the power-series method. Furthermore, the convergence of the solutions of the power-series is certificated. Finally, the stability behavior of the model can be understood by analyzing the solutions of different parameters.

Keywords: tumor growth; symmetries; power-series solutions

Mathematics Subject Classification: 35E05, 35J05, 43A80

1. Introduction

It is known to us all that tumor growth has attracted considerable attention over the past few decades. In order to study the importance of tumor growth and better understand the disease itself, it is critical to find a model that may help to treat the tumor. For this purpose, some basic relations between mathematical modeling and tumor model have been presented in [1]. With the diffusion process about cell and nutrient proliferation as a basis, many mathematical models [2–8] have been used to discuss these growth phenomena, such as the fractional mathematical model of tumor invasion and metastasis [2], tumor spheroid models [3, 4], the androgen-deprivation prostate cancer treatment model [5], the foundations of cancer modeling [6] and hepatitis C evolution models [7, 8].

In this paper, we will discuss the following reaction-diffusion model for cancer invasion [9, 10]:

$$\begin{aligned}u_t &= u(1 - u) - auw, \\v_t &= d[(1 - u)v_x]_x + bv(1 - v), \\w_t &= w_{xx} + c(v - w),\end{aligned}\tag{1.1}$$

where $u(x, t)$, $v(x, t)$ and $w(x, t)$ stand for dimensionless and rescaled versions of healthy tissue, tumor tissue and excess H^+ ions, respectively. The subscripts represent the partial derivatives relative to the corresponding variables. a, b, c and d are all constant functions. a indicates the destructive effect of H^+ ions on the healthy tissue, b is the productivity of neoplastic tissue which pumps H^+ ions at a rate c and d is displayed in the form $d = D_2/D_3$ where D_2 and D_3 are the diffusion coefficients of malignant tissue and H^+ ions, respectively. The phenomenon in many instances of tumor propagation of an interstitial gap has been discussed in [9]. Some wave propagation dynamics were considered in [11].

As everyone knows, the Lie group method plays an important role in studying the exact significant solutions of nonlinear partial differential equations [12–15]. The main aims of the symmetry method are to construct invariance conditions and obtain reductions to differential equations [16–18]. Once the reduced equations are given, a large number of corresponding exact solutions can be obtained. Utilizing Lie group analysis, we are going to get some fascinating special solutions of Eq (1.1) and identify the analysis stability behaviors of the model.

The remainder of the paper is arranged as follows. Symmetries of the acid-mediated cancer invasion model are analyzed in Section 2; Section 3 considers the symmetry reductions through the use of similar variables; in Section 4, some new explicit solutions are provided with help of the power-series method, and the convergence of the solutions of the power-series is presented; also, we will investigate the properties of different solutions via imaging analysis; the last section summarizes the results of the study.

2. Lie point symmetry

In this paper, we demonstrate the Lie symmetry technique for Eq (1.1). First of all, let us think about a vector field of infinitesimal transformations of Eq (1.1) with the form

$$X = \xi \partial_x + \tau \partial_t + \phi \partial_u + \varphi \partial_v + \eta \partial_w, \quad (2.1)$$

where ξ, τ, ϕ, φ and η are functions of x, t, u, v, w respectively and are called infinitesimals of the symmetry group.

Based on the transformation (2.1), applying the invariance conditions to Eq (1.1), we get [16, 19]

$$\begin{aligned} \text{pr}^{(1)}X(u_t - u(1 - u) + auw) &= 0, \\ \text{pr}^{(2)}X(v_t - d((1 - u)v_x)_x - bv(1 - v)) &= 0, \\ \text{pr}^{(2)}X(w_t - w_{xx} - c(v - w)) &= 0, \end{aligned}$$

where $\text{pr}^{(i)}X$, $i = 1, 2$ is the i th-order prolongation of X [16, 19]. For Eq (1.1),

$$\begin{aligned} \text{pr}^{(1)}X &= X + \phi_t^{(1)} \frac{\partial}{\partial u_t}, \\ \text{pr}^{(2)}X &= X + \phi_x^{(1)} \frac{\partial}{\partial u_x} + \varphi_x^{(1)} \frac{\partial}{\partial v_x} + \varphi_t^{(1)} \frac{\partial}{\partial v_t} + \eta_t^{(1)} \frac{\partial}{\partial w_t} + \varphi_{xx}^{(2)} \frac{\partial}{\partial v_{xx}} + \eta_{xx}^{(2)} \frac{\partial}{\partial w_{xx}}, \end{aligned}$$

where

$$\phi_x^{(1)} = D_x \phi - u_x D_x \xi - u_t D_x \tau,$$

$$\begin{aligned}
\phi_t^{(1)} &= D_t\phi - u_x D_t\xi - u_t D_t\tau, \\
\varphi_x^{(1)} &= D_x\varphi - v_x D_x\xi - v_t D_x\tau, \\
\varphi_t^{(1)} &= D_t\varphi - v_x D_t\xi - v_t D_t\tau, \\
\eta_t^{(1)} &= D_t\eta - w_x D_t\xi - w_t D_t\tau, \\
\varphi_{xx}^{(2)} &= D_x^2(\varphi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt}, \\
\eta_{xx}^{(2)} &= D_x^2(\eta - \xi w_x - \tau w_t) + \xi w_{xxx} + \tau w_{xxt},
\end{aligned}$$

and D_x and D_t represent the total differential operators; for example,

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + w_t \frac{\partial}{\partial w} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + u_{tt} \frac{\partial}{\partial u_t} + w_{tt} \frac{\partial}{\partial w_t} + \dots$$

Next, we get an overdetermined system of equations for ξ, τ, ϕ, φ and η

$$\begin{aligned}
\xi_x = \xi_t = \xi_u = \xi_v = \xi_w &= 0, \\
\tau_x = \tau_t = \tau_u = \tau_v = \tau_w &= 0, \\
\phi = \varphi = \eta &= 0.
\end{aligned}$$

Solving the above equations, one gets

$$\xi = c_1, \tau = c_2, \phi = \varphi = \eta = 0,$$

where c_1 and c_2 are arbitrary constants. Therefore, Lie algebra L_2 of the transformations of Eq (1.1) is spanned by the following vector fields

$$X_1 = \partial_x, X_2 = \partial_t.$$

To obtain the symmetry groups, we solve the initial problems of the following ordinary differential equations

$$\begin{aligned}
\frac{d\tilde{x}}{d\epsilon} &= \xi(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}), \quad \tilde{x}|_{\epsilon=0} = x, \\
\frac{d\tilde{t}}{d\epsilon} &= \tau(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}), \quad \tilde{t}|_{\epsilon=0} = t, \\
\frac{d\tilde{u}}{d\epsilon} &= \phi(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}), \quad \tilde{u}|_{\epsilon=0} = u, \\
\frac{d\tilde{v}}{d\epsilon} &= \varphi(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}), \quad \tilde{v}|_{\epsilon=0} = v, \\
\frac{d\tilde{w}}{d\epsilon} &= \eta(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w}), \quad \tilde{w}|_{\epsilon=0} = w;
\end{aligned}$$

then we get the one-parameter symmetry groups $G_i : (x, t, u, v, w) \rightarrow (\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{w})$ of the infinitesimal generators $X_i (i = 1, 2)$ as follows:

$$\begin{aligned}
G_1 &: (x, t, u, v, w) \rightarrow (x + \epsilon, t, u, v, w), \\
G_2 &: (x, t, u, v, w) \rightarrow (x, t + \epsilon, u, v, w).
\end{aligned}$$

Based on the above discussion, we obtain the following theorem.

Theorem 2.1. If $u = f(x, t)$, $v = g(x, t)$ and $w = h(x, t)$, constitute a solution of Eq (1.1), then by applying the above-mentioned groups $G_i (i = 1, 2)$, the corresponding new solutions $u_i, v_i, w_i (i = 1, 2)$ can be presented respectively as follows:

$$\begin{aligned}u_1 &= f(x - \epsilon, t), \quad v_1 = g(x - \epsilon, t), \quad w_1 = h(x - \epsilon, t), \\u_2 &= f(x, t - \epsilon), \quad v_2 = g(x, t - \epsilon), \quad w_2 = h(x, t - \epsilon).\end{aligned}$$

3. Similarity reductions

In this section, we are going to cope with the similarity reductions of Eq (1.1).

Case 3.1. For the generator $X_1 + X_2$, the invariants are $z = x - t$, $u = f(z)$, $v = g(z)$ and $w = h(z)$, Eq (1.1) becomes

$$\begin{aligned}-f' &= f - f^2 - afh, \\-g' &= d(-f'g' + g'' - fg'') + bg - bg^2, \\-h' &= h'' + c(g - h),\end{aligned}\tag{3.1}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$ and $h' = \frac{dh}{dz}$.

Case 3.2. For the generator X_1 , the invariants are $z = t$, $u = f(z)$, $v = g(z)$ and $w = h(z)$, Eq (1.1) can be reduced to

$$\begin{aligned}f' &= f - f^2 - afh, \\g' &= bg - bg^2, \\h' &= c(g - h),\end{aligned}\tag{3.2}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$ and $h' = \frac{dh}{dz}$. The invariant solution of Eq (1.1) is as follows: $u(x, t) = f(t)$, $v(x, t) = g(t)$, $w(x, t) = h(t)$. Obviously, in this case, the variable x has no effect on the solution of Eq (1.1).

Case 3.3. For the generator X_2 , analogously, we have $z = x$, $u = f(z)$, $v = g(z)$ and $w = h(z)$. The reduction of Eq (1.1) is

$$\begin{aligned}f - f^2 - afh &= 0, \\d(-f'g' + g'' - fg'') + bg - bg^2 &= 0, \\h'' + c(g - h) &= 0,\end{aligned}\tag{3.3}$$

where $f' = \frac{df}{dz}$, $g' = \frac{dg}{dz}$ and $h' = \frac{dh}{dz}$. The invariant solution of Eq (1.1) is as follows: $u(x, t) = f(x)$, $v(x, t) = g(x)$, $w(x, t) = h(x)$. In this case, the variable t has no effect on the solution of Eq (1.1).

4. Power-series solutions

Next, by way of the power-series method which is a very useful technique for treating partial differential equations [20], we will discuss cases 3.1–3.3.

4.1. Power-series solutions of Eq (3.1)

For Case 3.1, we assume that the power-series solution to Eq (3.1) is as follows

$$f(z) = \sum_{n=0}^{\infty} p_n z^n, \quad g(z) = \sum_{n=0}^{\infty} q_n z^n, \quad h(z) = \sum_{n=0}^{\infty} l_n z^n, \quad (4.1)$$

where the coefficients p_n , q_n and l_n are constants to be resolved.

Putting Eq (4.1) into Eq (3.1), we obtain

$$\begin{aligned} & - \sum_{n=0}^{\infty} (n+1)p_{n+1}z^n = \sum_{n=0}^{\infty} p_n z^n - \sum_{n=0}^{\infty} \sum_{k=0}^n p_k p_{n-k} z^n - a \sum_{n=0}^{\infty} \sum_{k=0}^n p_k l_{n-k} z^n, \\ & - \sum_{n=0}^{\infty} (n+1)q_{n+1}z^n = d \left[- \sum_{n=0}^{\infty} \sum_{k=0}^n (k+1)(n-k+1)p_{k+1}q_{n-k+1}z^n \right. \\ & \left. + \sum_{n=0}^{\infty} (n+1)(n+2)q_{n+2}z^n - \sum_{n=0}^{\infty} \sum_{k=0}^n (n-k+1)(n-k+2)p_k q_{n-k+2}z^n \right] \\ & + b \sum_{n=0}^{\infty} q_n z^n - b \sum_{n=0}^{\infty} \sum_{k=0}^n q_k q_{n-k} z^n, \\ & - \sum_{n=0}^{\infty} (n+1)l_{n+1}z^n = \sum_{n=0}^{\infty} (n+1)(n+2)l_{n+2}z^n + c \left(\sum_{n=0}^{\infty} q_n z^n - \sum_{n=0}^{\infty} l_n z^n \right). \end{aligned} \quad (4.2)$$

Comparing the coefficients for Eq (4.2), we get

$$\begin{aligned} p_1 &= p_0(-1 + p_0 + al_0), \\ q_2 &= \frac{bq_0(q_0 - 1) + q_1(dp_1 - 1)}{2d(1 - p_0)}, \\ l_2 &= \frac{1}{2}[c(l_0 - q_0) - l_1]. \end{aligned} \quad (4.3)$$

Generally, for $n \geq 1$, we have

$$\begin{aligned} p_{n+1} &= -\frac{1}{n+1} \left[p_n - \sum_{k=0}^n p_k (p_{n-k} + al_{n-k}) \right], \\ q_{n+2} &= \frac{1}{(n+1)(n+2)d(1-p_0)} \left\{ (dp_1 - 1)(n+1)q_{n+1} - bq_n + b \sum_{k=0}^n q_k q_{n-k} \right. \\ & \left. + \sum_{k=1}^n d(n-k+1)[(k+1)p_{k+1}q_{n-k+1} + (n-k+2)p_k q_{n-k+2}] \right\}, \\ l_{n+2} &= \frac{1}{(n+1)(n+2)} [c(l_n - q_n) - (n+1)l_{n+1}]. \end{aligned} \quad (4.4)$$

Given Eq (4.4), the coefficients p_i ($i \geq 2$), q_j and l_j ($j \geq 3$) of (4.1) can be obtained, i.e.,

$$p_2 = -\frac{1}{2} [p_1 - 2p_0 p_1 - a(p_0 l_1 + p_1 l_0)],$$

$$q_3 = \frac{1}{6d(1-p_0)}(4dp_1q_2 + 2bq_0q_1 + 2dp_2q_1 - 2q_2 - bq_1),$$

$$l_3 = \frac{1}{6}[c(l_1 - q_1) - 2l_2].$$

Therefore, for the arbitrary constants $p_0 \neq 1, q_0, l_0, q_1$ and l_1 , the other terms of the sequences $\{p_n, q_n, l_n\}_{n=0}^{\infty}$, according to Eqs (4.3) and (4.4), can be determined. This implies that there is a power-series solution, i.e., Eq (4.1) which has coefficients that are composed of Eqs (4.3) and (4.4).

Furthermore, for Eq (3.1), we confirm the convergence of Eq (4.1). In fact, from Eq (4.4), we get

$$|p_{n+1}| \leq M[|p_n| + \sum_{k=0}^n |p_k|(|p_{n-k}| + |l_{n-k}|)],$$

$$|q_{n+2}| \leq N[|q_{n+1}| + |q_n| + \sum_{k=0}^n |q_k||q_{n-k}| + \sum_{k=1}^n (|p_{k+1}||q_{n-k+1}| + |p_k||q_{n-k+2}|)],$$

$$|l_{n+2}| \leq L(|l_n| + |q_n| + |l_{n+1}|),$$

where $M = \max\{1, a\}$, $N = \max\{|\frac{dp_1-1}{d(1-p_0)}|, |\frac{b}{d(1-p_0)}|, |\frac{1}{1-p_0}|\}$ and $L = \max\{1, c\}$.

Next, we construct three power-series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$, $S = S(z) = \sum_{n=0}^{\infty} s_n z^n$ and $T = T(z) = \sum_{n=0}^{\infty} t_n z^n$ by using

$$r_i = |p_i|, \quad i = 0, 1,$$

$$s_j = |q_j|, \quad t_j = |l_j|, \quad i = 0, 1, 2,$$

and

$$r_{n+1} = M[r_n + \sum_{k=0}^n r_k(r_{n-k} + t_{n-k})],$$

$$s_{n+2} = N[s_{n+1} + s_n + \sum_{k=0}^n s_k s_{n-k} + \sum_{k=1}^n (r_{k+1} s_{n-k+1} + r_k s_{n-k+2})],$$

$$t_{n+2} = L(t_n + s_n + t_{n+1}),$$

where $n = 1, 2, \dots$. It is easily seen that

$$|p_n| \leq r_n, \quad |q_n| \leq s_n, \quad |l_n| \leq t_n, \quad n = 0, 1, 2, \dots$$

Therefore, $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$, $S = S(z) = \sum_{n=0}^{\infty} s_n z^n$ and $T = T(z) = \sum_{n=0}^{\infty} t_n z^n$ are majorant series of Eq (4.1) respectively. Next, we prove that $R = R(z)$, $S = S(z)$ and $T = T(z)$ have a positive radius of convergence.

$$R(z) = r_0 + r_1 z + \sum_{n=1}^{\infty} r_{n+1} z^{n+1} = r_0 + r_1 z + M[\sum_{n=1}^{\infty} r_n z^{n+1} + \sum_{n=1}^{\infty} \sum_{k=0}^n r_k(r_{n-k} + t_{n-k}) z^{n+1}]$$

$$= r_0 + r_1 z + M[(R - r_0) + (R^2 - r_0^2) + r_0(T - t_0) + T(R - r_0)]z,$$

$$\begin{aligned}
S(z) &= s_0 + s_1z + s_2z^2 + \sum_{n=1}^{\infty} s_{n+2}z^{n+2} = s_0 + s_1z + s_2z^2 + N\left[\sum_{n=1}^{\infty} s_{n+1}z^{n+2} + \sum_{n=1}^{\infty} s_nz^{n+2}\right] \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=0}^n s_k s_{n-k} z^{n+2} + \sum_{n=1}^{\infty} \sum_{k=1}^n r_{k+1} s_{n-k+1} z^{n+2} + \sum_{n=1}^{\infty} \sum_{k=1}^n r_k s_{n-k+2} z^{n+2}] \\
&= s_0 + s_1z + s_2z^2 + N[z(S - s_0 - s_1z) + z^2(S - s_0) + z^2(S^2 - s_0^2) \\
&\quad + (S - s_0)(R - r_0 - r_1z) + (R - r_0)(S - s_0 - s_1z)],
\end{aligned}$$

and

$$\begin{aligned}
T(z) &= t_0 + t_1z + t_2z^2 + \sum_{n=1}^{\infty} t_{n+2}z^{n+2} \\
&= t_0 + t_1z + t_2z^2 + L\left[\sum_{n=1}^{\infty} t_nz^{n+2} + \sum_{n=1}^{\infty} s_nz^{n+2} + \sum_{n=1}^{\infty} t_{n+1}z^{n+2}\right] \\
&= t_0 + t_1z + t_2z^2 + L[z^2(T - t_0) + z^2(S - s_0) + z(T - t_0 - t_1z)].
\end{aligned}$$

Then, we discuss the implicit functional system with the independent variable z :

$$\begin{aligned}
F_1(z, R, S, T) &= R - r_0 - r_1z - M[(R - r_0) + (R^2 - r_0^2) + r_0(T - t_0) + T(R - r_0)]z, \\
F_2(z, R, S, T) &= S - s_0 - s_1z - s_2z^2 - N[z(S - s_0 - s_1z) + z^2(S - s_0) + z^2(S^2 - s_0^2) \\
&\quad + (S - s_0)(R - r_0 - r_1z) + (R - r_0)(S - s_0 - s_1z)], \\
F_3(z, R, S, T) &= T - t_0 - t_1z - t_2z^2 - L[z^2(T - t_0) + z^2(S - s_0) + z(T - t_0 - t_1z)].
\end{aligned}$$

Based on the implicit function theorem [21], because F_1, F_2 and F_3 are analytic in the neighborhood of $(0, r_0, s_0, t_0)$ and $F_1(0, r_0, s_0, t_0) = F_2(0, r_0, s_0, t_0) = F_3(0, r_0, s_0, t_0) = 0$, and given the Jacobian determinant

$$\frac{\partial(F_1, F_2, F_3)}{\partial(R, S, T)} \Big|_{(0, r_0, s_0, t_0)} = 1 \neq 0,$$

we reach that $R = R(z)$, $S = S(z)$ and $T = T(z)$ are analytic in a neighborhood of the point $(0, r_0, s_0, t_0)$ and have a positive radius. This shows that Eq (4.1) converges in a neighborhood of the point $(0, r_0, s_0, t_0)$. The proof is completed.

Thus the power-series solution given by Eq (4.1) for Eq (3.1) is analytic and can be described as

$$\begin{aligned}
f(z) &= p_0 + p_1z + \sum_{n=1}^{\infty} p_{n+1}z^{n+1} \\
&= p_0 + p_0(-1 + p_0 + al_0)z - \sum_{n=1}^{\infty} \frac{1}{n+1} \left[p_n - \sum_{k=0}^n p_k(p_{n-k} + al_{n-k}) \right] z^{n+1}, \\
g(z) &= q_0 + q_1z + q_2z^2 + \sum_{n=1}^{\infty} q_{n+2}z^{n+2} \\
&= q_0 + q_1z + \frac{bq_0(q_0 - 1) + q_1(dp_1 - 1)}{2d(1 - p_0)} z^2
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)d(1-p_0)} \{ (dp_1 - 1)(n+1)q_{n+1} - bq_n + b \sum_{k=0}^n q_k q_{n-k} \\
& + \sum_{k=1}^n d(n-k+1)[(k+1)p_{k+1}q_{n-k+1} + (n-k+2)p_k q_{n-k+2}] \} z^{n+2}, \\
h(z) & = l_0 + l_1 z + l_2 z^2 + \sum_{n=1}^{\infty} l_{n+2} z^{n+2} \\
& = l_0 + l_1 z + \frac{1}{2} [c(l_0 - q_0) - l_1] z^2 + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} [c(l_n - q_n) - (n+1)l_{n+1}] z^{n+2}.
\end{aligned}$$

Moreover, the power-series solution of Eq (1.1) is

$$\begin{aligned}
u(x, t) & = p_0 + p_1(x-t) + \sum_{n=1}^{\infty} p_{n+1}(x-t)^{n+1} \\
& = p_0 + p_0(-1 + p_0 + al_0)(x-t) - \sum_{n=1}^{\infty} \frac{1}{n+1} [p_n - \sum_{k=0}^n p_k(p_{n-k} + al_{n-k})](x-t)^{n+1}, \\
v(x, t) & = q_0 + q_1(x-t) + q_2(x-t)^2 + \sum_{n=1}^{\infty} q_{n+2}(x-t)^{n+2} \\
& = q_0 + q_1(x-t) + \frac{bq_0(q_0 - 1) + q_1(dp_1 - 1)}{2d(1-p_0)}(x-t)^2 \\
& + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)d(1-p_0)} \{ (dp_1 - 1)(n+1)q_{n+1} - bq_n + b \sum_{k=0}^n q_k q_{n-k} \\
& + \sum_{k=1}^n d(n-k+1)[(k+1)p_{k+1}q_{n-k+1} + (n-k+2)p_k q_{n-k+2}] \} (x-t)^{n+2}, \quad (4.5) \\
w(x, t) & = l_0 + l_1(x-t) + l_2(x-t)^2 + \sum_{n=1}^{\infty} l_{n+2}(x-t)^{n+2} \\
& = l_0 + l_1(x-t) + \frac{1}{2} [c(l_0 - q_0) - l_1](x-t)^2 \\
& + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} [c(l_n - q_n) - (n+1)l_{n+1}](x-t)^{n+2},
\end{aligned}$$

where $p_0 \neq 1, q_0, l_0, q_1$ and l_1 are arbitrary constants; the other terms p_n, q_n and l_n ($n \geq 2$) can be provided according to Eqs (4.3) and (4.4).

4.2. Numerical simulation of power-series solutions of Eq (3.1)

We take the first six terms of Eq (4.5) as approximate to u, v and w for $a = 1.5, b = 1, c = 2, d = 4 \times 10^{-10}, p_0 = 0.5, q_0 = 3, l_0 = 5, q_1 = 4$ and $l_1 = 6$. Then the approximation is depicted in Figure 1.

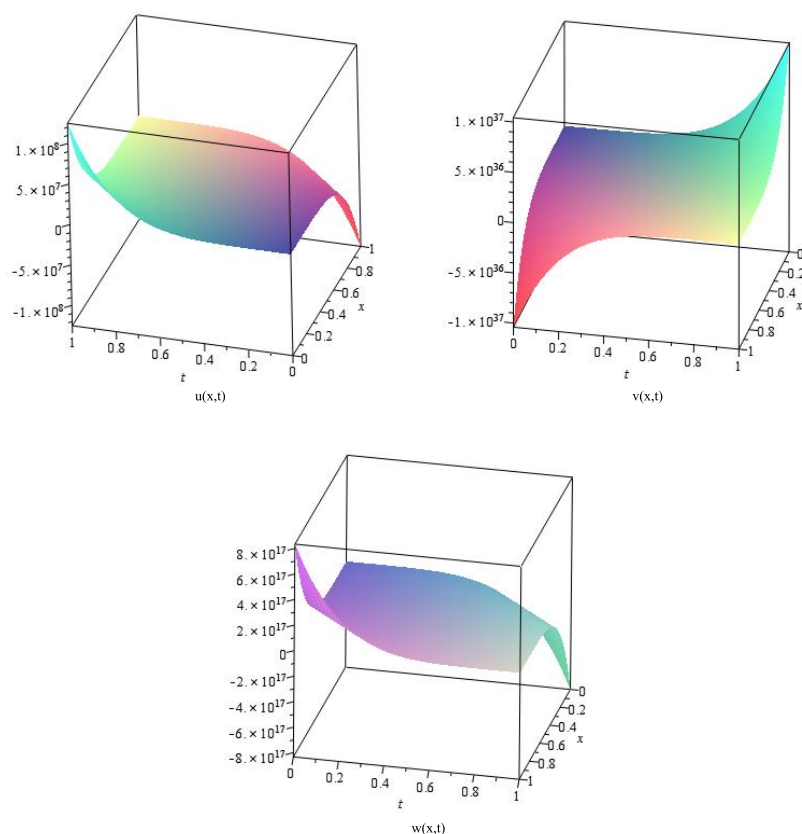


Figure 1. Power series-solution of Eq (4.5).

Figure 1 shows that the values of u , v and w tend to be stable when $x \in (0, 1)$ and $t \in (0, 1)$. However, when $x \rightarrow 0$ and $t \rightarrow 1$ or $x \rightarrow 1$ and $t \rightarrow 0$, u and w change suddenly in one direction and v changes sharply in the other direction. The rate of change of v is faster than that of u , and that of w is between them. This shows that healthy tissue may be destroyed before malignant cells arrive. Tumor progression is mediated by the acidification of surrounding tissues. Due to anaerobic glycolysis and metabolism, tumor cells produce excessive H^+ ions. This leads to local acidification, which then destroys the surrounding healthy tissue and promotes tumor invasion.

4.3. Power-series solutions of Eq (3.2)

For Case 3.2, similarly, we can also obtain the following power-series solution to Eq (1.1):

$$\begin{aligned}
 u(x, t) &= p_0 + \sum_{n=0}^{\infty} p_{n+1} t^{n+1} = p_0 + \sum_{n=0}^{\infty} \frac{1}{n+1} \left[p_n - \sum_{k=0}^n p_k (p_{n-k} + a l_{n-k}) \right] t^{n+1}, \\
 v(x, t) &= q_0 + \sum_{n=0}^{\infty} q_{n+1} t^{n+1} = q_0 + \sum_{n=0}^{\infty} \frac{b}{n+1} \left(q_n - \sum_{k=0}^n q_k q_{n-k} \right) t^{n+1}, \\
 w(x, t) &= l_0 + \sum_{n=0}^{\infty} l_{n+1} t^{n+1} = l_0 + \sum_{n=0}^{\infty} \frac{c}{n+1} (q_n - l_n) t^{n+1},
 \end{aligned} \tag{4.6}$$

where p_0 , q_0 and l_0 are arbitrary constants.

4.4. Numerical simulation of Power-series solutions of Eq (3.2)

We get the first six terms of the power-series solutions of (4.6) as approximate to u , v and w for $a = 1.5$, $b = 1$, $c = 2$, $d = 4 \times 10^{-10}$, $p_0 = 0.5$, $q_0 = 3$ and $l_0 = 5$ respectively, then, the approximations of u , v and w are illustrated in Figure 2.

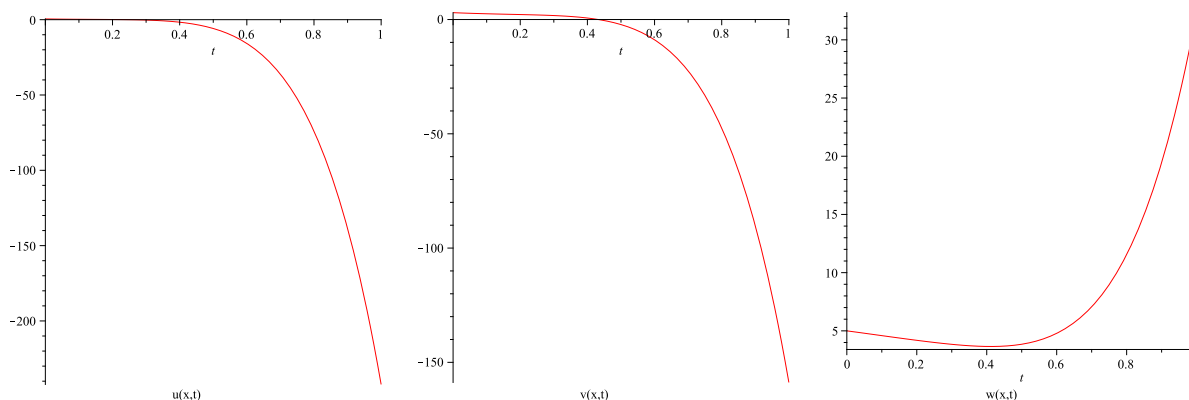


Figure 2. Power-series solution of Eq (4.6).

Figure 2 illustrates that, when $t \in (0, 1)$, the values of u , v and w are stable first and mutate over time. When u and v decrease at the same time, w increases. The rate of change of u is the same as that of v , and the rate of change of w is slightly slower than them. This indicates that when healthy cells and cancer cells decrease at the same time, H^+ ions will increase. We find that H^+ ions are sensitive to changes in healthy cells and cancer cells.

4.5. Power-series solutions of Eq (3.3)

For Case 3.3, the power-series solution to Eq (1.1) is described as follows:

$$\begin{aligned}
 u(x, t) &= p_0 + p_1x + \sum_{n=2}^{\infty} p_n x^n \\
 &= p_0 + \frac{ap_0l_1}{1 - 2p_0 - al_0}x + \sum_{n=2}^{\infty} \frac{ap_0l_n + \sum_{k=1}^{n-1} p_k(p_{n-k} + al_{n-k})}{1 - 2p_0 - al_0}x^n, \\
 v(x, t) &= q_0 + q_1x + q_2x^2 + \sum_{n=1}^{\infty} q_{n+2}x^{n+2} = q_0 + q_1x + \frac{dp_1q_1 - bq_0(1 - q_0)}{2d(1 - p_0)}x^2 \\
 &\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)d(1 - p_0)} \{d[(n+1)p_1q_{n+1} + \sum_{k=1}^n (n-k+1)((k+1)p_{k+1}q_{n-k+1} \\
 &\quad + (n-k+2)p_kq_{n-k+2})] - b(q_n - \sum_{k=0}^n q_kq_{n-k})\}x^{n+2}, \\
 w(x, t) &= l_0 + l_1x + \sum_{n=0}^{\infty} l_{n+2}x^{n+2} = l_0 + l_1x + \sum_{n=0}^{\infty} \frac{c(l_n - q_n)}{(n+1)(n+2)}x^{n+2},
 \end{aligned} \tag{4.7}$$

where p_0, l_0, q_0, q_1 and l_1 are arbitrary constants and that satisfy $1 - 2p_0 - al_0 \neq 0$ and $1 - p_0 \neq 0$.

4.6. Numerical simulation of power-series solutions of Eq (3.3)

We acquire the first six terms of the power-series solutions of Eq (4.7) as approximate to u , v and w for $a = 0.1$, $b = 1$, $c = 2$, $d = 4 \times 10^{-10}$, $p_0 = 0.5$, $q_0 = 3$, $l_0 = 5$, $q_1 = 4$ and $l_1 = 6$ respectively; then, the approximations of u , v and w are portrayed in Figure 3.

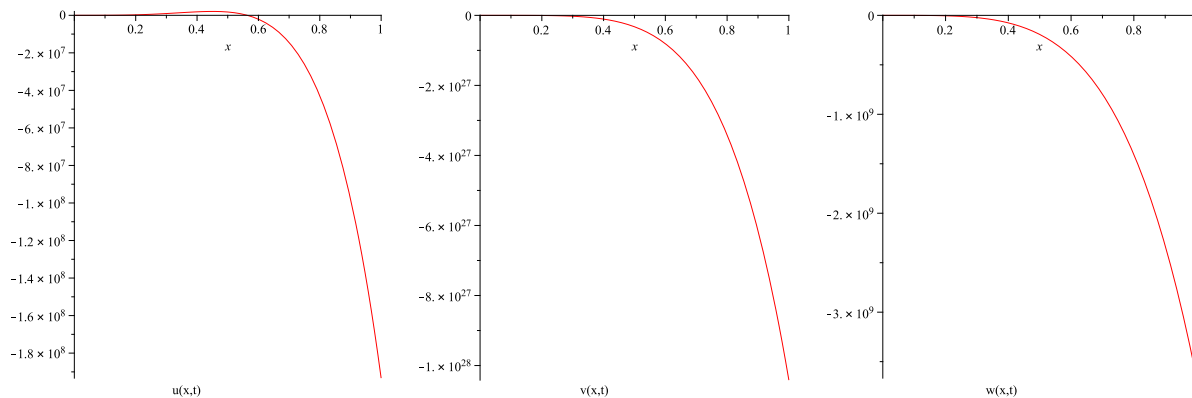


Figure 3. Power-series solution of Eq (4.7).

Figure 3 shows that, when $x \in (0, 1)$, the values of u , v and w are initially stable and mutate in the same direction. The rate of change of v is the fastest, and the rates of change of u and w are basically the same. It describes that cancer cells decline faster than healthy cells. This shows a process of complete destruction of healthy tissue after tumor tissue invasion.

Remark 4.1. For Case 3.2 and Case 3.3, the proofs of convergence of the power series solutions are similar to that for Case 3.1. The details have been omitted here.

5. Conclusions

In this study, we applied the Lie group analysis method to an acid-mediated cancer invasion model. An important feature of this model is that tumor progression is mediated by acidification of the surrounding tissue. Especially, the model presumes that an excess of H^+ ions is produced by tumor cells as a consequence of their anaerobic, glycolytic metabolism. Based on this method, the symmetries and reduced equations of Eq (1.1) were derived. Furthermore, explicit solutions of the reduced equations were obtained using the power-series method. Finally, this paper also demonstrates the stability behavior of the model for different parameters as achieved through the use of graphical analysis. In this way, we have found that H^+ is decreased ahead of the advancing tumor front. Moreover, for certain parameter values, healthy tissue could be destroyed prior to the arrival of malignant cells. In the future, we can use this method to solve more tumor-related mathematical problems.

Acknowledgments

This research was supported by the Natural Science Foundation of Shanxi (No. 202103021224068).

Conflict of interest

The authors declare that they have no competing interests.

References

1. D. S. Jones, B. D. Sleeman, *Differential equations and mathematical biology*, Chapman and Hall/CRC, 2003.
2. P. Veerasha, E. Ilhan, D. G. Prakasha, H. M. Baskonus, W. Gao, Regarding on the fractional mathematical model of tumour invasion and metastasis, *CMES-Comp. Model. Eng.*, **127** (2021), 1013–1036. <https://doi.org/10.32604/cmcs.2021.014988>
3. A. Bertuzzi, A. Fasano, A. Gandolfi, C. Sinisgalli, ATP production and necrosis formation in a tumour spheroid model, *Math. Model. Nat. Phenom.*, **2** (2007), 30–46. <https://doi.org/10.1051/mmnp:2007002>
4. R. Venkatasubramanian, M. A. Henson, N. S. Forbes, Incorporating energy metabolism into a growth model of multicellular tumor spheroids, *J. Theor. Biol.*, **242** (2006), 440–453. <https://doi.org/10.1016/j.jtbi.2006.03.011>
5. T. Telksnys, I. Timofejeva, Z. Navickas, R. Marcinkevicius, R. Mickevicius, M. Ragulskis, Solitary solutions to an androgen-deprivation prostate cancer treatment model, *Math. Method. Appl. Sci.*, **43** (2020), 3995–4006. <https://doi.org/10.1002/mma.6168>
6. N. Bellomo, N. K. Li, P. K. Maini, On the foundations of cancer modelling: selected topics, speculations, and perspectives, *Math. Mod. Meth. Appl. S.*, **18** (2008), 593–646. <https://doi.org/10.1142/S0218202508002796>
7. T. Telksnys, Z. Navickas, M. A. F. Sanjuan, R. Marcinkevicius, M. Ragulskis, Kink solitary solutions to a hepatitis C evolution model, *Discrete Cont. Dyn. Syst. B*, **25** (2020), 4427–4447. <https://doi.org/10.3934/dcdsb.2020106>
8. T. Telksnys, Z. Navickas, R. Marcinkevicius, M. S. Cao, M. Ragulskis, Homoclinic and heteroclinic solutions to a hepatitis C evolution model, *Open Math.*, **16** (2018), 1537–1555. <https://doi.org/10.1515/math-2018-0130>
9. R. A. Gatenby, E. T. Gawlinski, A reaction-diffusion model for cancer invasion, *Cancer Res.*, **56** (1996), 5745–5753.
10. R. A. Gatenby, E. T. Gawlinski, The glycolytic phenotype in carcinogenesis and tumour invasion: insights through mathematical modelling, *Cancer Res.*, **63** (2003), 3847–3854.
11. A. Fasano, M. A. Herrero, M. R. Rodrigo, Slow and fast invasion waves in a model of acid-mediated tumour growth, *Math. Biosci.*, **220** (2009), 45–56. <https://doi.org/10.1016/j.mbs.2009.04.001>
12. B. Gao, Y. Zhang, Symmetry analysis of the time fractional Gaudrey-Dodd-Gibbon equation, *Physica A*, **525** (2019), 1058–1062. <https://doi.org/10.1016/j.physa.2019.04.023>
13. Z. G. Wang, Symmetries and solutions of hyperbolic mean curvature flow with a constant forcing term, *Appl. Math. Comput.*, **235** (2014), 560–566. <https://doi.org/10.1016/j.amc.2013.12.134>

14. B. Gao, C. F. He, Analysis of a coupled short pulse system via symmetry method, *Nonlinear Dyn.*, **90** (2017), 2627–2636. <https://doi.org/10.1007/s11071-017-3827-0>
15. J. H. Wang, Symmetries and solutions to geometrical flows, *Sci. China Math.*, **56** (2013), 1689–1704. <https://doi.org/10.1007/s11425-013-4635-8>
16. P. J. Olver, *Applications of Lie groups to differential equations*, New York: Springer, 1993. <https://doi.org/10.1007/978-1-4612-4350-2>
17. G. W. Bluman, S. Kumei, *Symmetries and differential equations*, New York, NY: Springer, 1989. <https://doi.org/10.1007/978-1-4757-4307-4>
18. B. Gao, Y. X. Wang, Invariant solutions and nonlinear self-adjointness of the two-component Chaplygin gas equation, *Discrete. Dyn. Nat. Soc.*, **2019** (2019), 9609357. <https://doi.org/10.1155/2019/9609357>
19. G. W. Bluman, S. C. Anco, *Symmetry and integration methods for differential equations*, New York, NY: Springer, 2002. <https://doi.org/10.1007/b97380>
20. N. H. Asmar, *Partial differential equations with Fourier series and boundary value problems*, Beijing: China Machine Press, 2005.
21. W. Rudin, *Principles of mathematical analysis*, Beijing: China Machine Press, 2004.



AIMS Press

©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)