Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16907–16916.
DOI: 10.3934/math.2022928
Received: 18 March 2022
Revised: 02 June 2022
Accepted: 06 June 2022
Published: 18 July 2022

Research article

A generalization of identities in groupoids by functions

Hee Sik Kim[1], **J. Neggers**[2] **and Sun Shin Ahn**[3,*]

[1] Department of Mathematics, Hanyang University, Seoul, 04763, Korea
[2] Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, USA
[3] Department of Mathematics Education, Dongguk University, Seoul, 04620, Korea

* **Correspondence:** Email: sunshine@dongguk.edu.

Abstract: In this paper, we introduce the notions of a left and a right idenfunction in a groupoid by using suitable functions, and we apply this concept to several algebraic structures. Especially, we discuss its role in linear groupoids over a field. We show that, given an invertible function φ, there exists a groupoid such that φ is a right idenfunction. The notion of a right pseudo semigroup will be discussed in linear groupoids. The notion of an inversal is a generalization of an inverse element, and it will be discussed with idenfunctions in linear groupoids over a field.

Keywords: groupoid; (right, left) idenfunction; BCK-algebra; leftoid; right pseudo semigroup; (right, left) inversal
Mathematics Subject Classification: 06F35, 20N02

1. Introduction

R. H. Bruck [1] published a book, *A survey of binary systems*, and he discussed the theory of groupoids, loops and quasigroups, and several algebraic structures. O. Borůvka [2] stated the theory of decompositions of sets and its application to binary systems. P. Flondor [3] discussed a groupoid truncations which was induced at the procedure of MV-algebra operation from the group operation. M. H. Hooshmand [5] introduced the notion of grouplikes which is somthing between semigroup and groups. It is a generalization of a group. He showed that every grouplike is a semigroup containing the minimal ideal that is also a maximal subgroup. A. B. Saeid et al. [15] introduced the notion of e-group as a new generalization of a group, and they found conditions for a group to be an e-group. Y. L. Liu et al. [12] discussed special elements of a groupoid that are associated with pseudo-inverse functions, which are generalizations of the inverses associated with units of groupoids with identity elements.

The concept of a generalization in mathematics is very important to mathematicians for their research. Semirings, near-rings, quasi-rings are defined simply by deleting some axioms from rings, and they provided their own domains. This kind of activities happened in several mathematical research areas.

The notion of BCK-algebras was formulated in 1966 by Y. Imai and K. Iséki. We refer to [6, 8, 13] for more information. After that many generalized algebras of BCK/BCI-algebras, e.g., BCH-algebra, BH-algebra, BZ-algebra, BCC-algebra, have been investigated. The notion of a d-algebra was introduced by deleting two complicated axioms from the BCK-algebra. In sequel many algebraic structures, e.g., $B-, BE-, BF-, BM-, BO-, Q-$ algebras, were introduced by many researchers. Y. B. Jun et al. [9] edited a special issue "BCK-algebras and related algebraic systems". Hwang et al. [7] generalized the notion of the implicativity discussed in BCK-algebras, and applied it to several groupoids and BCK-algebras.

H. S. Kim and J. Neggers [11] introduced the notion of $Bin(X)$ of all binary systems(groupoids, algebras) defined on a set X, and showed that it becomes a semigroup under suitable operation.

On the other hand, there is a different way of a generalization in mathematics. J. S. Han et al. [4] developed Fibonacci numbers by introducing functions, and obtained a condition for constructing Fibonacci functions by using f-even and f-odd functions. B. Sroysang [17] developed Fibonacci functions with period k by using even and odd functions. K. K. Sharma [16] applied this concept, and obtained several properties of generalized tribonacci function and tribonacci numbers. If we use suitable functions to several axioms in algebras, then we may open the door of more generalized algebraic world.

The identity axiom in groups, semigroups, BCK-algebras and other general algebraic structures plays an important role for developing the theory. The inverse axiom plays also strong activity in the study of groups and fields. Using Fibonacci functions as we have discussed in Fibonacci numbers, it is our aim to generalize these two axioms by suitable functions, and we apply these to several algebraic structures. This method may apply to several axioms in various algebraic structures, especially many general algebraic structures related to BCK-algebras.

In this paper, by using suitable functions, we introduce the notions of a left and a right idenfunction in a groupoid, and we apply this concept to leftoids, abelian groups and left zero semigroups. Especially, we investigate its role in linear groupoids over a field, and we obtain two different idenfunctions related to linear groupoids. We show that, given an invertible function φ, there exists a groupoid such that φ is a right idenfunction. Moreover, we apply this concept to a groupoid which was derived from a group, and we classify the left and right idenfunctions. The notion of a right pseudo semigroup will be discussed in linear groupoids. The notion of an inversal is a generalization of an inverse element, and it will be discussed with idenfunctions in linear groupoids over a field.

2. Preliminaries

A d-algebra [14] is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra [6, 8, 13] is a d-algebra X satisfying the following additional axioms:

(IV) $(x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

A groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a *right zero semigroup* if $x * y = y$ for any $x, y \in X$. A groupoid $(X, *)$ is said to be a *leftoid* for $f : X \to X$ if $x * y = f(x)$ for any $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a *rightoid* for $f : X \to X$ if $x * y = f(y)$ for any $x, y \in X$. Note that a left (right, resp.) zero semigroup is a special case of a leftoid (right, resp.) (see [11]).

In a BCK-algebra X, we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$. We then observe that a BCK-algebra determines a poset structure on it. Let $(X, \leq)$ be a poset with the least element 0. If we define a binary operation $*$ on X as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise}, \end{cases}$$

then $(X, *, 0)$ is a BCK-algebra (see [10]). Such an algebra is said to be a *standard* BCK-algebra inherited from the poset $(X, \leq)$. Given a poset $(X, \leq)$, an element x is said to be *less than or equal* to y if $x \leq y$. Two distinct elements x and y in $(X, \leq)$ are said to be *incomparable* if neither $x \leq y$ nor $y \leq x$, and we denote it by $x \| y$.

Given two groupoids $(X, *)$ and $(X, \bullet)$, we define a new binary operation $\Box$ by $x \Box y := (x * y) \bullet (y * x)$ for all $x, y \in X$. Then we obtain a new groupoid $(X, \Box)$, i.e., $(X, \Box) = (X, *)\Box(X, \bullet)$. We denote the collection of all binary systems(groupoid, algebras) defined on X by $Bin(X)$ [11].

Theorem 2.1. [11] $(Bin(X), \Box)$ *is a semigroup and the left zero semigroup is an identity.*

3. Idenfunctions in groupoids

Given a groupoid $(X, *)$, i.e., $(X, *) \in Bin(X)$, a map $\varphi : X \to X$ is said to be a *right idenfunction* for $(X, *)$ if $x * \varphi(x) = x$ for any $x \in X$. A map $\varphi : X \to X$ is said to be a *left idenfunction* for $(X, *)$ if $\varphi(x) * x = x$ for any $x \in X$. A map $\varphi : X \to X$ is said to be an *idenfunction* for $(X, *)$ if $x * \varphi(x) = \varphi(x) * x = x$ for any $x \in X$.

Example 3.1. (a). Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := x(x - y)$ for any $x, y \in \mathbf{R}$. If we define $\varphi(x) := x - 1$ for any $x \in \mathbf{R}$, then $x * \varphi(x) = x * (x - 1) = x(x - (x - 1)) = x$ for any $x \in \mathbf{R}$. This shows that $\varphi(x) = x - 1$ is a right idenfunction for $(\mathbf{R}, *)$.
(b). Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := g(x)$ for any $x, y \in \mathbf{R}$ where $g(x) := x^3$ for all $x \in \mathbf{R}$. Then $h(x) := x^{\frac{1}{3}}$ is a left idenfunction for $(\mathbf{R}, *)$, since $h(x) * x = g(h(x)) = g(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$ for all $x \in \mathbf{R}$.
(c). Let $(X, *, 0)$ be a BCK-algebra. Then the zero map $\varphi(x) := 0$ for all $x \in X$ is a right idenfunction for $(X, *, 0)$, since $x = x * 0 = x * \varphi(x)$ for all $x \in X$.
(d). Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := 1 + 2(x + y)$ for any $x, y \in \mathbf{R}$. If we define $\varphi(x) := -\frac{1}{2}(x + 1)$ for any $x \in \mathbf{R}$, then $x * \varphi(x) = \varphi(x) * x = x$ for any $x \in \mathbf{R}$, i.e., $\varphi(x) = -\frac{1}{2}(x + 1)$ is an idenfunction for $(\mathbf{R}, *)$.

Example 3.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Define a map $\varphi : X \to X$ by

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 3 & 1 \end{pmatrix}.$$

Then $x * \varphi(x) = x$ for any $x \in X$, i.e., φ is a right idenfunction for $(X, *)$.

Example 3.3. (a). Let X be the set of all natural numbers and let "+" be the usual addition on X. Then $(X, +)$ has no idenfunction. In fact, if we assume $\varphi : X \to X$ is an idenfunction on $(X, +)$. Then $x + \varphi(x) = x$ for any $x \in X$. It follows that $\varphi(x) = 0 \notin X$ for any $x \in X$, a contradiction.
(b). Let K be an algebraically closed field with $charK \neq 2$. Define a binary operation "$*$" on K by $x * y := x(x - y)$ for any $x, y \in K$. If we let $g(x) := \frac{x \pm \sqrt{x^2+4x}}{2}$, then $g^2(x) - xg(x) - x = 0$. Hence $g(x) * x = g(x)(g(x) - x) = (g^2(x) - xg(x) - x) + x = x$ for all $x \in K$. Hence $g(x)$ is a left idenfunction for $(K, *)$.

Proposition 3.4. *Let $(X, *)$ be a leftoid for $f : X \to X$. If $(X, *)$ has a left idenfunction φ, then φ is a right inverse function of f.*

Proof. Let φ be a left idenfunction for $(X, *)$. Then $\varphi(x) * x = x$ for all $x \in X$. Since $(X, *)$ is a leftoid for f, we have $f(\varphi(x)) = \varphi(x) * x = x$ for all $x \in X$, i.e., $f \circ \varphi = 1_X$. □

Proposition 3.5. *Let $(X, *)$ be a leftoid for $f : X \to X$. If $(X, *)$ has a right idenfunction φ, then $(X, *)$ is a left zero semigroup.*

Proof. Given $x \in X$, we have $x = x * \varphi(x) = f(x)$ for any $x \in X$. It follows that $f(x) = x$ for all $x \in X$, and hence $x * y = f(x) = x$ for any $x, y \in X$. This proves the proposition. □

Proposition 3.6. *Let $(X, +)$ be an abelian group and let $a \in X$. Define a binary operation "$*$" on X by $x * y := a + (x + y)$ for all $x, y \in X$. If φ is a right idenfunction for $(X, *)$, then φ is a constant function.*

Proof. Assume φ is a right idenfunction for $(X, *)$. Then $x * \varphi(x) = x$ for all $x \in X$. It follows that $a + (x + \varphi(x)) = x$ for all $x \in X$. Since $(X, +)$ is an abelian group, we obtain $\varphi(x) = -a$ for all $x \in X$, proving the proposition. □

Proposition 3.7. *Let $(X, *)$ be a left zero semigroup. Then the identity function $i_X : X \to X$ is the only left idenfunction for $(X, *)$.*

Proof. Assume $\varphi : X \to X$ is a left idenfunction for $(X, *)$. Then $\varphi(x) * x = x$ for all $x \in X$. Since $(X, *)$ is a left zero semigroup, we obtain $x = \varphi(x) * x = \varphi(x)$ for all $x \in X$. □

Proposition 3.8. *Let $(X, *)$ be a left zero semigroup. Then every function $\varphi : X \to X$ is a right idenfunction for $(X, *)$.*

Proof. Given a function $\varphi : X \to X$, we have $x * \varphi(x) = x$ for all $x \in X$, since $(X, *)$ is a left zero semigroup. □

Theorem 3.9. *Let K be a field and let $a, b, c \in K$ with $bc \neq 0$. Define a binary operation "$*$" on K by $x * y := a + bx + cy$ for all $x, y \in K$. If $\varphi : X \to X$ is an idenfunction for $(K, *)$, then*

(i) $\varphi(x) = x$ when $x * y = bx + (1 - b)y$,
(ii) $\varphi(x) = \frac{1-b}{b}x - \frac{a}{b}$ when $x * y = a + b(x + y)$, for all $x, y \in K$.

Proof. Assume φ is a right idenfunction for $(X, *)$. Then, for any $x \in X$, we have $x = x * \varphi(x) = a + bx + c\varphi(x)$. Since $c \neq 0$, we obtain

$$\varphi(x) = \frac{1}{c}[(1 - b)x - a]. \tag{1}$$

Assume g is a left idenfunction for $(X, *)$. Then, for any $x \in X$, we have $x = g(x) * x = a + bg(x) + cx$. Since $b \neq 0$, we obtain

$$g(x) = \frac{1}{b}[(1 - c)x - a]. \tag{2}$$

If we assume that $\varphi(x) = g(x)$, then we obtain

$$(b - c)[1 - b - c]x = a(b - c). \tag{3}$$

Case (i): $b - c \neq 0$. If $b + c \neq 1$, then $x = \frac{a}{1-b-c}$ for all $x \in K$, a contradiction. Hence there is no such an idenfunction φ. If $b+c = 1$, then $c = 1-b$. By (3), we obtain $a = 0$. Hence $x*y = bx+cy = bx+(1-b)y$ for all $x, y \in K$. In this case, by (1) and (2), we obtain $\varphi(x) = \frac{1}{c}[(1 - b)x - a] = \frac{1}{1-b}[(1 - b)x - 0] = x$ and $g(x) = \frac{1}{b}[(1 - c)x - a] = x$. Case (ii): $b - c = 0$. Then $b = c$ and hence $x * y = a + b(x + y)$. Since φ is a right idenfunction, we have $x = x * \varphi(x) = a + b(x + \varphi(x))$. It follows that $\varphi(x) = \frac{1-b}{b}x - \frac{a}{b}$. □

Given a groupoid $(X, *)$ and maps $\varphi_i : X \to X$, $(i = 1, 2)$, we define a map $\varphi_1 \odot \varphi_2 : X \to X$ by $(\varphi_1 \odot \varphi_2)(x) := \varphi_1(x) * \varphi_2(x)$ for all $x \in X$.

Proposition 3.10. *Let $(X, *)$ be a semigroup. If φ_1 and φ_2 are right idenfunction for $(X, *)$, then $\varphi_1 \odot \varphi_2$ is also a right idenfunction for $(X, *)$.*

Proof. Given $x \in X$, we obtain

$$\begin{aligned} x * (\varphi_1 \odot \varphi_2)(x) &= x * (\varphi_1(x) * \varphi_2(x)) \\ &= (x * \varphi_1(x)) * \varphi_2(x) \\ &= x * \varphi_2(x) \\ &= x. \end{aligned}$$

This shows that $\varphi_1 \odot \varphi_2$ is a right idenfunction for $(X, *)$. □

Proposition 3.11. *Let $(X, *, 0)$ be a standard BCK-algebra, i.e.,*

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Define a map $\varphi : X \to X$ satisfying $x * \varphi(x) \neq 0$ for any $x \in X$. Then φ is a right idenfunction for $(X, *, 0)$.

Proof. Given $x \in X$, we assign $\varphi(x)$ in X satisfying either $\varphi(x) \leq x$ or $x \| \varphi(x)$, i.e., x and $\varphi(x)$ are incomparable in the poset $(X, \leq)$. It follows that $x * \varphi(x) = x$ for any $x \in X \setminus \{0\}$. Since $(X, *, 0)$ is a BCK-algebra, we have $0 * \varphi(0) = 0$. This shows that φ is a right idenfunction for $(X, *)$. □

Theorem 3.12. *Let* $X := [0, 1]$ *and let* $x * y := \max\{x - y, 0\}$ *for all* $x, y \in X$*. If* $\varphi : X \to X$ *is a right idenfunction for* $(X, *)$*, then* φ *is of the form:*

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in (0, 1], \\ \alpha & \text{if } x = 0, \end{cases}$$

for some $\alpha \geq 0$*.*

Proof. If $\varphi : X \to X$ is a right idenfunction for $(X, *)$, then $x = x * \varphi(x) = \max\{x - \varphi(x), 0\}$ for all $x \in X$. We claim that if $x \in (0, 1]$, then $x - \varphi(x) \geq 0$. Assume that there exists $x_0 \in (0, 1]$ such that $x_0 < \varphi(x_0)$. It follows that $x_0 \neq \max\{x_0 - \varphi(x_0), 0\} = x_0 * \varphi(x_0)$, which shows that φ is not a right idenfunction, a contradiction. Hence $x \geq \varphi(x)$ for all $x \in (0, 1]$, which implies that $x = x * \varphi(x) = \max\{x - \varphi(x), 0\} = x - \varphi(x)$ for all $x \in (0, 1]$. Now, $0 = 0 * \varphi(0) = \max\{0 - \varphi(0), 0\}$, which shows that $\varphi(0) = \alpha$ for some $\alpha \geq 0$. □

Proposition 3.13. *If* $\varphi : X \to X$ *is an invertible function, then there exists a groupoid* $(X, *)$ *such that* φ *is a right idenfunction for* $(X, *)$*.*

Proof. Let $\varphi : X \to X$ be an invertible function. Define a binary operation "$*$" on X by $x * y := \varphi^{-1}(y)$ for any $x, y \in X$. Then $x * \varphi(x) = \varphi^{-1}(\varphi(x)) = x$ for all $x \in X$. □

Proposition 3.13 shows that every invertible function $\varphi : X \to X$ is a right idenfunction for at least one groupoid $(X, *)$. Let $(X, *)$ be a left zero semigroup and let $\varphi : X \to X$ be an arbitrary function. Then $x * \varphi(x) = x$ for all $x \in X$. Thus any map $\varphi : X \to X$ is a right idenfunction for at least one groupoid $(X, *)$.

Proposition 3.14. *Let* $(X, *)$ *be a groupoid and let* $(X, \bullet)$ *be an idempotent groupoid, i.e.,* $x \bullet x = x$ *for all* $x \in X$*. If* $\varphi : X \to X$ *is an idenfunction for* $(X, *)$*, then* φ *is also an idenfunction for* $(X, *)\Box(X, \bullet)$*.*

Proof. If φ is an idenfunction for $(X, *)$, then $x * \varphi(x) = x = \varphi(x) * x$ for all $x \in X$. It follows that $x\Box\varphi(x) = (x * \varphi(x)) \bullet (\varphi(x) * x) = x \bullet x = x$. Similarly, we have $\varphi(x)\Box x = x$ for all $x \in X$. This proves the proposition. □

Proposition 3.15. *Let* $(X, *, e)$ *be a group. If* $\varphi : X \to X$ *is a right idenfunction for* $(X, *)$*, then* φ *is the zero map.*

Proof. If $\varphi : X \to X$ is an idenfunction, then $x * \varphi(x) = \varphi(x) * x = x$ for all $x \in X$. Since $(X, *, e)$ is a group, we obtain $\varphi(x) = e$ for all $x \in X$, proving the proposition. □

Proposition 3.16. *Let* $(X, \bullet, e)$ *be a group and let* $a \in X$*. Define a binary operation "$*$" on* X *by* $x * y := a \bullet x \bullet y$ *for any* $x, y \in X$*. Then the following hold:*

(i) *If* φ *is a right idenfunction for* $(X, *)$*, then* $\varphi(x) = x^{-1} \bullet a^{-1} \bullet x$ *for all* $x \in X$*.*
(ii) *If* ϱ *is a left idenfunction for* $(X, *)$*. then* $\varrho(x) = a^{-1}$ *for all* $x \in X$*.*
(iii) *If* $\varphi(x) = \varrho(x)$ *for all* $x \in X$*, then* a *is in the center of* $(X, \bullet, e)$ *and* $\varphi(x) = \varrho(x) = a^{-1}$ *for all* $x \in X$*.*

Proof. (i). Assume that φ is a right idenfunction for $(X, \bullet)$. Then $x * \varphi(x) = x$ for all $x \in X$. It follows that $a \bullet x \bullet \varphi(x) = x$ for all $x \in X$, which implies that $\varphi(x) = x^{-1} \bullet a^{-1} \bullet x$ for all $x \in X$.
(ii). If ϱ is a left idenfunction for φ, then $\varrho(x) * x = x$ for all $x \in X$. It follows that $a \bullet \varrho(x) \bullet x = x$ for all $x \in X$, which implies that $\varrho(x) = a^{-1}$ for all $x \in X$.
(iii). Assume $\varphi(x) = \varrho(x)$ for all $x \in X$. Then $x^{-1} \bullet a^{-1} \bullet x = a^{-1}$ for all $x \in X$. It follows that $x^{-1} \bullet a^{-1} = a^{-1} \bullet x^{-1}$ for all $x \in X$. This shows that a is in the center of X and $\varphi(x) = \varrho(x) = a^{-1}$ for all $x \in X$. □

4. Right pseudo semigroups

A groupoid $(X, *)$ is said to be a *right pseudo semigroup* if, for any $x, y, z \in X$, there exists $w \in X$ such that $(x * y) * z = x * w$. In this case, w is said to be a *pseudo product* of y and z with respect to x. Note that w may not be unique. If $(X, *)$ is a left zero semigroup, then every element of X becomes a pseudo product.

Example 4.1. Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x*y := x+2y$ for all $x, y \in \mathbf{R}$. Then $(x * y) * z = x + 2(y + z)$. If we take $w := y + z$, then $(x * y) * z = x * w$ and $(\mathbf{R}, *)$ is a right pseudo semigroup. Note that $(\mathbf{R}, *)$ is not a semigroup, since $(x * y) * z = x + 2(y + z)$, but $x * (y * z) = x + 2y + 4z$.

Theorem 4.2. *Let $X := K$ be a field and let $a, b, c \in K$. Define a binary operation "$*$" on K by $x * y := a + bx + cy$ for all $x, y \in K$. If $(X, *)$ is a right pseudo semigroup, then it is one of the following forms:*

(i) $x * y = x + cy$, $w = y + z$,
(ii) $x * y = a$, *w is an arbitrary element in K,*
(iii) $x * y = a + cy$ *and* $w = z$ *for all* $x, y, z \in K$.

Proof. Assume $(X, *)$ is a right pseudo semigroup. Then, for any $x, y, z \in K$,

$$\begin{aligned}(x * y) * z &= a + b(x * y) + cz \\ &= a + b(a + bx + cy) + cz \\ &= a(1 + b) + b^2 x + bcy + cz,\end{aligned}$$

and $x * w = a + bx + cw$ for some $w \in K$. It follows that

$$a(1 + b) = a, \; b^2 = b, \; cw = c(by + z). \tag{4}$$

If $a = 0$, then (4) becomes

$$b^2 = b, \;\; cw = c(by + z). \tag{5}$$

If $b = 0$, then $x * y = cy, (x * y) * z = cz$, and hence $x * w = cw = (x * y) * w$. If $b = 1$, then we take $w := y + z$. It follows that $(x * y) * z = x + cy + cz$ and $x * w = x * (y + z) = 0 + 1x + c(y + z)$, which proves $(x * y) * z = x * w$. Hence $x * y = x + cy$ and $w = y + z$. If $a \neq 0$, then (4) becomes

$$a(1 + b) = a, \; b^2 = b, \; cw = bcy + cz. \tag{6}$$

It follows that $b = 0$ and $cw = cz$. We consider two cases: $c = 0$ or $w = z$. If $c = 0$, then $(x * y) * z = a = x * y$. Hence $x * y = a$ and w can be any element of K. If $w = z$, then $x * y = a + cy$ and $(x * y) * z = a + cz = a + cw = x * w$. This proves the theorem. □

5. Inversals in groupoids

Let $(X, *)$ be a groupoid and let $\varphi : X \to X$ be a right idenfunction for $(X, *)$. A map $\psi : X \to X$ is said to be a *right inversal* for φ if $x * \psi(x) = \varphi(x)$ for all $x \in X$. Similarly, a map $\xi : X \to X$ is said to be a *left inversal* for φ if $\xi(x) * x = \varphi(x)$ for all $x \in X$.

Proposition 5.1. *Let $(X, *)$ be a right zero semigroup. If $\varphi : X \to X$ is a right idenfunction for $(X, *)$ and $\xi : X \to X$ is a right inversal for φ, then $\xi(x) = \varphi(x) = x$ for all $x \in X$.*

Proof. Let $(X, *)$ be a right zero semigroup and let $\varphi : X \to X$ be a right idenfunction for $(X, *)$. Then $x = x * \varphi(x) = \varphi(x)$ for all $x \in X$. If $\xi : X \to X$ is a right inversal for φ, then $\xi(x) = x * \xi(x) = \varphi(x) = x$ for all $x \in X$, proving the proposition. □

Theorem 5.2. *Let K be a field and let $a, b, c \in K$ with $c \neq 0$. Define a binary operation "$*$" on K by $x * y := a + bx + cy$ for all $x, y \in K$. Then the following hold:*

(i) *If φ is a right idenfunction for $(K, *)$, then $\varphi(x) = \frac{1}{c}[(1 - b)x - a]$.*
(ii) *If ξ is a right inversal for φ, then $\xi(x) = \frac{1}{c^2}[(1 - b - bc)x - a(1 + c)]$.*

Proof. (i). If φ is a right idenfunction for $(K, *)$, then $x = x * \varphi(x) = a + bx + c\varphi(x)$ for all $x \in K$. It follows that $\varphi(x) = \frac{1}{c}[(1 - b)x - a]$.
(ii). Assume ξ is a right inversal for φ. Then $x * \xi(x) = \varphi(x)$ for all $x \in K$. It follows that

$$\begin{aligned} \frac{1}{c}[(1-b)x - a] &= \varphi(x) \\ &= x * \xi(x) \\ &= a + bx + c\xi(x). \end{aligned}$$

This shows that $\xi(x) = \frac{1}{c^2}[(1 - b - bc)x - a(1 + c)]$. □

Example 5.3. Let $\mathbf{R}$ be the real field. We let $a = 0, b = 2$ and $c = -2$ in Theorem 5.2. Then we have $x * y = 2x - 2y$, $\xi(x) = \frac{3}{4}x$ and $\varphi(x) = \frac{1}{2}x$. If we define $a = 0, b = 1$ and $c \neq 0$, then $x * y = x + cy$, $\varphi(x) = 0$ and $\xi(x) = -\frac{1}{c}x$. In particular, if $c = -1$, then $\varphi(x) = 0, x * y = x - y$ and $\xi(x) = x$.

Proposition 5.4. *Let K be a field and let $a, b, c \in K$ with $c \neq 0$. Define a binary operation "$*$" on K by $x * y := a + bx + cy$ for all $x, y \in K$. Let φ be a right idenfunction for $(K, *)$ and let ξ be a right inversal for φ. If $\xi^{(1)}(x) := \xi(x)$ and $x * \xi^{(n+1)}(x) = \xi^{(n)}(x)$ for all $x \in X$, where n is a natural number, then*

$$\xi^{(n+1)}(x) = \frac{1}{c}[\xi^{(n)}(x) - bx - a].$$

Proof. For any $x \in X$ and a natural number n, we obtain

$$\begin{aligned} \xi^{(n)}(x) &= x * \xi^{(n+1)}(x) \\ &= a + bx + c\xi^{(n+1)}(x). \end{aligned}$$

Since $c \neq 0$, we have $\xi^{(n+1)}(x) = \frac{1}{c}[\xi^{(n)}(x) - bx - a]$. □

Example 5.5. Let $a := 0$, $b := 2$ and $c := -2$ in Proposition 5.4. By Example 5.3, we have $x * y = 2x - 2y$ for all $x, y \in X$, and $\xi(x) = \frac{3}{4}x, \varphi(x) = \frac{1}{2}x$. It follows that $\xi^{(n+1)}(x) = -\frac{1}{2}\xi^{(n)}(x) + x$. In fact, $\xi^{(2)}(x) = -\frac{1}{2}\xi^{(1)}(x) + x = \frac{5}{8}x$, $\xi^{(3)}(x) = \frac{11}{16}x, \xi^{(4)}(x) = \frac{21}{32}x$ and $\xi^{(5)}(x) = \frac{41}{64}(x)$. Assume $\lim_{n\to\infty} \xi^{(n)}(x) := \alpha x$. Then $\alpha x = \lim_{n\to\infty} \xi^{(n+1)}(x) = \lim_{n\to\infty}[-\frac{1}{2}\xi^{(n)}(x) + x] = -\frac{1}{2}\alpha x + x$. It follows that $\alpha = \frac{2}{3}$. Hence $\lim_{n\to\infty} \xi^{(n)}(x) = \frac{2}{3}x$.

6. Conclusions

By using a function, we introduced the notion of an idenfunction, which is a generalization of an identity axiom. We applied this notion to several algebraic structures, e.g., (linear) groupoid, leftoids, abelian groups and left zero semigroups. Especially, we investigated its role in linear groupoids over a field, and we obtained that there are two different idenfunctions according to different linear groupoids. We showed that, given an invertible function φ, there exists a groupoid such that φ is a right idenfunction. The notion of a right pseudo semigroup was discussed in linear groupoids. We introduced the notion of an inversalm which is a generalization of an inverse element. We discussed it with idenfunctions in linear groupoids over a field.

There are many algebraic structures related to BCK-algebras, and we will apply the notion of an idenfunction to several algebraic structures. As an example, the theory of a semigroup with identity will be developed as a (generalized) semigroup with idenfunctions. In sequel, we will apply this concept to several algebraic structures. A generalization of several axioms in algebraic structures by using suitable functions may contribute to extend research areas of mathematics in future.

Conflict of interest

The authors declare that they have no competing interests.

References

1. R. H. Bruck, *A survey of binary systems*, Berlin, Heidelberg: Springer, 1971.
2. O. Borůvka, *Foundations of the theory of groupoids and groups*, New York: John Wiley & Sons, 1976.
3. P. Flondor, Groupoid truncations, *U.P.B. Sci. Bull., Series A*, **A** (2010), 3–10.
4. J. S. Han, H. S. Kim, J. Neggers, On Fibonacci functions with Fibonacci numbers, *Adv. Differ. Equ.*, **2012** (2012), 126. https://doi.org/10.1186/1687-1847-2012-126
5. M. H. Hooshmand, Grouplikes, *Bull. Iran. Math. Soc.*, **39** (2013), 65–85.
6. Y. S. Huang, *BCI-algebras*, Beijing: Science Press, 2006.
7. I. Hwang, H. S. Kim, J. Neggers, Some implicativities for groupoids and BCK-algebras, *Mathematics*, **7** (2019), 973. http://doi.org/10.3390/math7100973
8. A. Iorgulescu, *Algebras of logic as BCK-algebras*, Bucharest: Editura ASE, 2008.
9. Y. B. Jun, I. Chajda, H. S. Kim, E. H. Roh, J. M. Zhan, A. Iorgulescu, BCK-algebras and related algebraic systems, *Int. J. Math. Math.*, **2011** (2011), 268683. https://doi.org/10.1155/2011/268683
10. J. Y. Kim, Y. B. Jun, H. S. Kim, BCK-algebras inherited from the posets, *Math. Jpn.*, **45** (1997), 119–123.
11. H. S. Kim, J. Neggers, The semigroups of binary systems and some perspectives, *B. Korean Math. Soc.*, **45** (2008), 651–661.
12. Y. L. Liu, H. S. Kim, J. Neggers, Some special elements and pseudo inverse functions in groupoids, *Mathematics*, **7** (2019), 173. http://doi.org/10.3390/math7020173

13. J. Meng, Y. B. Jun, *BCK-algebras*, Seoul: Kyungmoon Sa, 1994.

14. J. Neggers, H. S. Kim, On *d*-algebras, *Math. Slovaca*, **49** (1999), 19–26.

15. A. B. Saeid, A. Rezaei, A. Radfar, A generalization of groups, *Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis., Mat. Nat.*, **96** (2018), A4. http://doi.org/10.1478/AAPP.961A4

16. K. K. Sharma, Generalized tribonacci function and tribonacci numbers, *IJRTE*, **9** (2020), 1313–1316.

17. B. Sroysang, On Fibonacci functions with period *k*, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 418123. http://doi.org/10.1155/2013/418123