



Research article

Dynamic analysis of fractional-order neural networks with inertia

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Abstract: The existence and the S-asymptotic ω -periodic of the solution in fractional-order Cohen-Grossberg neural networks with inertia are studied in this paper. Based on the properties of the Riemann-Liouville (R-L) fractional-order derivative and integral, the contraction mapping principle, and the Arzela-Ascoli theorem, sufficient conditions for the existence and the S-asymptotic ω -period of the system are achieved. In addition, an example is simulated to testify the theorem.

Keywords: fractional-order neural networks; contraction mapping principle; inertia; existence; S-asymptotic ω -periodic

Mathematics Subject Classification: 45M05

1. Introduction

As a generalization of integer-order calculus, one of the main advantages of fractional-order calculus is that it can be considered as a superset of the integer one. It has the ability to accomplish what the integer-order calculus can not. In the last few decades, it has received a lot of attention because of its increasing applications in various fields, including bioengineering, economics, signal and image processing, rheology, biology and electrochemistry [1–4]. Compared with the traditional integer-order model, the fractional-order differential model provides an effective tool to describe the inherent memory and heredity effects of real materials and processes, it provides a powerful tool for describing the memory and heredity of different things. Nowadays, fractional-order dynamical systems have been widely used in electromagnetic wave, electrolyte polarization, viscoelastic, economic, biology systems [5–9]. Moreover, there are strong links between fractional-order derivative operators and fractional-order Brownian motion, continuous time random walk method and generalized central limit theorem. In addition, fractional-order derivative operators allow the long-term memory and nonlocal dependence of many anomalous processes. Some significant results regarding fractional-order neural networks (FNNs) have been published, especially on the dynamics

of such systems, such as finite time stability, asymptotic stability, Mittag-Leffler stability, application and control of neural networks [10–15].

However, the FNNs they studied only contain one fractional-order derivative of the state in the system. Mathematics and physics consider them as super damping. But the neuron's state will change its dynamic properties when the inertia exceeds a critical point. The inertia in neural networks will increase the instability of the system. Furthermore, it has a profound meaning in biology, such as the cuttlefish's axon [16–18]. According to the biological background of the inertial neural system, the fractional-order inertial neural network simulates the dynamic behavior of the neurons more closely. Therefore, it is meaningful to consider damping in the system.

Integer-order neural networks with inertia have been studied a lot, and some exciting results have been obtained. Such as exponential synchronization, the fixed-time synchronization are studied in [19–22] and synchronization of coupled memristive neural networks (NNs) [23]. These inertial NNs have two fractional orders in the differential equations. Compared with the traditional NNs with a single order, the NNs with two different orders are more suitable to simulate the complex biological neural networks, and they can be used in many other fields, such as biology and cybernetics. Therefore, it is valuable to discuss the dynamics of FNNs with inertia. However, according to the enquires, most of the researches studied the stability of the FNNs without inertia, like Ke and Henriquez et al. who studied the asymptotic ω -periodicity of FNNs [24, 25], etc. Researchers have studied the FNNs with inertia, except that Gu et al. and Zhang et al. who considered the stability and synchronicity of Riemann-Liouville FNNs with inertia [26, 27].

In 1983, Cohen and Grossberg proposed a generalized neural network and ecological model, namely the Cohen-Grossberg neural network [28]. It contains many ecosystems and neural networks. At present, parallel processing, associative memory, and especially optimization computation have attracted wide interest. From the view of mathematics and physics, the Cohen-Grossberg neural network is a model of super-damping (damping tending to infinity), and the system with damping (weak damping) should also be considered in practical problems, that is, inertia should be considered in a Cohen-Grossberg neural network.

In this paper, we extend the inertial Cohen-Grossberg neural network model from integral-order to fractional-order. Inspired by the studies mentioned above, the existence and the S-asymptotic ω -periodic of the solution for fractional-order Cohen-Grossberg NNs with inertia (FCGNNIs) are discussed in this paper.

The main features of this paper are as follows:

1) A fractional-order Cohen-Grossberg neural network model with an inertia term is proposed for the first time. This model has practical application value in simulating complex biological neural networks.

2) By proper variable substitution, the model with two different fractional-order derivatives is simplified to a model with only one fractional-order derivative, which simplifies the study of the problem.

3) The sufficient conditions for the existence and S-asymptotic ω -periodic of the solution of the system are given.

The results are new. It is of theoretical significance and practical value to study the dynamic characteristics of the FCGNNI further.

Consider the following FCGNNI:

$$D_t^{2\alpha}(\sigma_i(t)) = -\gamma_i D_t^\alpha(\sigma_i(t)) - \alpha_i(\sigma_i(t))[h_i(\sigma_i(t)) - \sum_{j=1}^n a_{ij}f_j(\sigma_j(t)) - I_i(t)], \quad (1.1)$$

for all $i = 1, 2, \dots, n$, $t \geq 0$, where D_t^α is the R-L fractional-order derivative with order α ($0 < \alpha < 1$); $\sigma_i(t)$, $\alpha_i(\cdot) > 0$, $h_i(\cdot)$, $I_i(t)$ and $\gamma_i > 0$ are the state variable, the abstract amplification function, the behavior function, the external input and the damping coefficient of the i th neuron at time t ; a_{ij} represents the connection weight; $f_j(\cdot)$ is the activation function of the j th neuron.

The system's initial conditions are

$$\begin{cases} \sigma_i(s) = \varphi_{\sigma_i}(s), \\ D_t^\alpha(\sigma_i(s)) = \psi_{\sigma_i}(s), \end{cases} \quad s \leq 0, \quad i = 1, 2, \dots, n, \quad (1.2)$$

where $\varphi_{\sigma_i}(s)$ and $\psi_{\sigma_i}(s)$ are bounded and continuous in $(-\infty, 0]$.

Remark. If $\alpha = 1$, (1.1) is the integer-order Cohen-Grossberg NNs with inertia:

$$\frac{d^2\sigma_i(t)}{dt^2} = -\gamma_i \frac{d\sigma_i(t)}{dt} - \alpha_i(\sigma_i(t))[h_i(\sigma_i(t)) - \sum_{j=1}^n a_{ij}f_j(\sigma_j(t)) - I_i(t)],$$

$$i = 1, 2, \dots, n.$$

2. Preliminaries

The discussion in this paper has been established on the following assumptions ($i, j = 1, 2, \dots, n$):

H₁ : $\alpha_i(\cdot)$ and $\alpha'_i(\cdot)$ are bounded. That is, there exist $\underline{\alpha}_i \geq 0$, $\bar{\alpha}_i > 0$ and $A_i > 0$ such that

$$0 \leq \underline{\alpha}_i \leq \alpha_i(\sigma_i(t)) \leq \bar{\alpha}_i, \quad |\alpha'_i(\cdot)| \leq A_i.$$

H₂ : $f_j(\cdot)$ is bounded and Lipschitz-continuous. That is, there exist $l_j > 0$ and $\bar{f}_j > 0$ such that

$$|f_j(s) - f_j(v)| \leq l_j|s - v|, \quad |f_j(\cdot)| \leq \bar{f}_j.$$

H₃ : Let $p_i(\zeta_i) = \alpha_i(\zeta_i)h_i(\zeta_i)$, its derivative is bounded with $\underline{p}_i > 0$ and $\bar{p}_i > 0$. That is,

$$0 \leq \underline{p}_i \leq p'_i(\zeta_i) \leq \bar{p}_i.$$

H₄ : $I_i(t)$ is bounded with the boundary $I_i > 0$. That is, $|I_i(t)| \leq I_i$.

Let $\chi_i(t) = D_t^\alpha \sigma_i(t) + \eta_i \sigma_i(t)$, $\eta_i > 0$, from (1.1) one gets

$$\begin{cases} D_t^\alpha \sigma_i(t) = -\eta_i \sigma_i(t) + \chi_i(t), \\ D_t^\alpha \chi_i(t) = -\eta_i(\eta_i - \gamma_i)\sigma_i(t) - (\gamma_i - \eta_i)\chi_i(t) \\ \quad - \alpha_i(\sigma_i(t))[h_i(\sigma_i(t)) - \sum_{j=1}^n a_{ij}f_j(\sigma_j(t)) - I_i(t)]. \end{cases} \quad (2.1)$$

3. Results

In this section, three theorems about the existence and the S -asymptotic ω -periodic of the solution to the system given by (1.1) are proposed.

Theorem 3.1. Under the assumptions $H_1 - H_4$, if

$$\Gamma(\alpha + 1) > [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + \sum_{j=1}^n |a_{ij}|(\bar{\alpha}_j l_i + A_j \bar{f}_j) + A_i I_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\}] T^\alpha, \quad (3.1)$$

then, the system given by (1.1) has a unique solution on $[0, T]$ ($T > 0$).

Proof. Derive from Lemma 2 and (2.1):

$$\begin{cases} \sigma_i(t) = -\eta_i D_t^{-\alpha} \sigma_i(t) + D_t^{-\alpha} \chi_i(t), \\ \chi_i(t) = -\eta_i(\eta_i - \gamma_i) D_t^{-\alpha} \sigma_i(t) - (\gamma_i - \eta_i) D_t^{-\alpha} \chi_i(t) \\ \quad - D_t^{-\alpha} \alpha_i(\sigma_i(t)) [h_i(\sigma_i(t)) - \sum_{j=1}^n a_{ij} f_j(\sigma_j(t)) - I_i(t)]. \end{cases} \quad (3.2)$$

Let

$U(t) = \{u(t) | u(t) = (\zeta_1(t), \zeta_2(t), \dots, \zeta_n(t), \chi_1(t), \chi_2(t), \dots, \chi_n(t))^T, \sigma_i(t), \chi_i(t) \in C[0, t], i = 1, 2, \dots, n\}$, it is clear that it is a Banach space, where the norm is $\|U\| = \sup_{0 \leq t \leq T} \sum_{i=1}^n (|\sigma_i(t)| + |\chi_i(t)|)$.

Define $P : U(t) \rightarrow U(t)$ as

$$(Pu)(t) = \{(P\sigma_1)(t), (P\sigma_2)(t), \dots, (P\sigma_n)(t), (P\chi_1)(t), (P\chi_2)(t), \dots, (P\chi_n)(t)\}^T,$$

where

$$\begin{cases} (P\zeta_i)(t) = -\eta_i D_t^{-\alpha} \sigma_i(t) + D_t^{-\alpha} \chi_i(t), \\ (P\chi_i)(t) = -\eta_i(\eta_i - \gamma_i) D_t^{-\alpha} \sigma_i(t) - (\gamma_i - \eta_i) D_t^{-\alpha} \chi_i(t) \\ \quad - D_t^{-\alpha} \alpha_i(\sigma_i(t)) [h_i(\sigma_i(t)) - \sum_{j=1}^n a_{ij} f_j(\sigma_j(t)) - I_i(t)]. \end{cases} \quad (3.3)$$

The first thing is to prove that $PU_\delta \subset U_\delta$, where $U_\delta = \{u(t) \in U(t) : \|u\| \leq \delta\}$,

$$\delta \geq \frac{\sum_{i=1}^n \bar{\alpha}_i (|h_i(0)| + \sum_{j=1}^n |a_{ij}| \bar{f}_j + I_i) T^\alpha}{\Gamma(\alpha + 1) - [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\}] T^\alpha}.$$

Derive from H_1 and H_3 that

$$\begin{cases} \alpha_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t)) = \alpha'_i(\xi_i)(\sigma_i(t) - \bar{\sigma}_i(t)), \\ \alpha_i(\sigma_i(t)) h_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t)) h_i(\bar{\sigma}_i(t)) = p'_i(\bar{\xi}_i)(\sigma_i(t) - \bar{\sigma}_i(t)), \end{cases} \quad (3.4)$$

where ξ_i and $\bar{\xi}_i$ are between σ_i and $\bar{\sigma}_i$.

From (3.3), one has

$$\begin{aligned}
\|Pu\| &= \sup_{0 \leq t \leq T} \sum_{i=1}^n (|(P\sigma_i)(t)| + |(P\chi_i)(t)|) \\
&\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \left\{ \frac{\eta_i}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\sigma_i(z)| dz \right. \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\chi_i(z)| dz \\
&\quad + \frac{\eta_i |\eta_i - \gamma_i|}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\sigma_i(z)| dz \\
&\quad + \frac{|\gamma_i - \eta_i|}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\chi_i(z)| dz \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\alpha_i(\sigma_i(z)) h_i(\sigma_i(z)) - \sigma_i(0) h_i(0)| dz \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} [\bar{\alpha}_i |h_i(0)| + \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j + \bar{\alpha}_i I_i] dz \right\} \\
&\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \left\{ \frac{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\sigma_i(z)| dz \right. \\
&\quad + \frac{1 + |\gamma_i - \eta_i|}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\chi_i(z)| dz \\
&\quad + \frac{\bar{\alpha}_i |h_i(0)| + \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j + \bar{\alpha}_i I_i}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \Big\} \\
&\leq \max_{1 \leq i \leq n} \left\{ \frac{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i}{\Gamma(\alpha)} \right\} \int_0^t (t-z)^{\alpha-1} \|U\| dz \\
&\quad + \max_{1 \leq i \leq n} \left\{ \frac{1 + |\gamma_i - \eta_i|}{\Gamma(\alpha)} \right\} \int_0^t (t-z)^{\alpha-1} \|U\| dz \\
&\quad + \frac{\bar{\alpha}_i |h_i(0)| + \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j + \bar{\alpha}_i I_i}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \\
&\leq \left[\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\} \right] \frac{\|u\| T^\alpha}{\Gamma(\alpha + 1)} \\
&\quad + \frac{\sum_{i=1}^n \bar{\alpha}_i (|h_i(0)| + \sum_{j=1}^n |a_{ij}| \bar{f}_j + I_i)}{\Gamma(\alpha + 1)} T^\alpha \\
&\leq \left[\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\} \right] \frac{\delta T^\alpha}{\Gamma(\alpha + 1)} \\
&\quad + \frac{\Gamma(\alpha + 1) - [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\}] T^\alpha}{\Gamma(\alpha + 1)} \delta
\end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \delta \\
&= \delta.
\end{aligned} \tag{3.5}$$

Therefore, $PU_\delta \subset U_\delta$.

Let

$$u(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_n(t), \chi_1(t), \chi_2(t), \dots, \chi_n(t))^T \in U(t),$$

$$\bar{u}(t) = (\bar{\sigma}_1(t), \bar{\sigma}_2(t), \dots, \bar{\sigma}_n(t), \bar{\chi}_1(t), \bar{\chi}_2(t), \dots, \bar{\chi}_n(t))^T \in U(t).$$

Then,

$$\begin{aligned}
\|Pu - P\bar{u}\| &= \sup_{0 \leq t \leq t} \sum_{i=1}^n \{|P\sigma_i(t) - (P\bar{\sigma}_i)(t)| + |P\chi_i(t) - (P\bar{\chi}_i)(t)|\} \\
&\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{\eta_i D_t^{-\alpha} |\sigma_i(t) - \bar{\sigma}_i(t)| + D_t^{-\alpha} |\chi_i(t) - \bar{\chi}_i(t)| \\
&\quad + \eta_i |\eta_i - \gamma_i| D_t^{-\alpha} |\sigma_i(t) - \bar{\sigma}_i(t)| + |\gamma_i - \eta_i| D_t^{-\alpha} |\chi_i(t) - \bar{\chi}_i(t)| \\
&\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) h_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t)) h_i(\bar{\sigma}_i(t))| \\
&\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t))| \sum_{j=1}^n |a_{ij}| |f_j(\sigma_j(t)) - f_j(\bar{\sigma}_j(t))| \\
&\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t))| \sum_{j=1}^n |a_{ij}| |f_j(\bar{\sigma}_j(t))| \\
&\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t))| |I_i(t)|\} \\
&\leq \sup_{0 \leq t \leq t} \sum_{i=1}^n \{[\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + \sum_{j=1}^n \bar{\alpha}_j |a_{ji}| l_i \\
&\quad + \sum_{j=1}^n A_j |a_{ji}| \bar{f}_i + A_i I_i] \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\sigma_i(z) - \bar{\sigma}_i(z)| dz \\
&\quad + (1 + |\gamma_i - \eta_i|) \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} |\chi_i(z) - \bar{\chi}_i(z)| dz\} \\
&\leq \max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + \sum_{j=1}^n \bar{\alpha}_j |a_{ji}| l_i \\
&\quad + \sum_{j=1}^n A_j |a_{ji}| \bar{f}_i + A_i I_i\} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u - \bar{u}\| dz \\
&\quad + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u - \bar{u}\| dz \\
&\leq [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + \sum_{j=1}^n \bar{\alpha}_j |a_{ji}| l_i
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n A_j |a_{ji}| \bar{f}_i + A_i I_i + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\} \frac{T^\alpha}{\Gamma(\alpha + 1)} \|u - \bar{u}\| \\
& \leq \|u - \bar{u}\|.
\end{aligned} \tag{3.6}$$

One can see from (3.6) that P is a contraction mapping. Hence, there exists a unique fixed point $u^*(t)$, which satisfies $P(u^*(t)) = u^*(t)$. That is,

$$\begin{cases} \sigma_i^*(t) = -\eta_i D_t^{-\alpha} \sigma_i^*(t) + D_t^{-\alpha} \chi_i^*(t), \\ \chi_i^*(t) = -\eta_i(\eta_i - \gamma_i) D_t^{-\alpha} \sigma_i^*(t) - (\gamma_i - \eta_i) D_t^{-\alpha} \chi_i^*(t) \\ \quad - D_t^{-\alpha} \alpha_i(\sigma_i^*(t)) [h_i(\sigma_i^*(t)) - \sum_{j=1}^n a_{ij} f_j(\sigma_j^*(t)) - I_i(t)]. \end{cases}$$

Therefore, the system described by (1.1) has a unique solution. \square

Theorem 3.2. If $H_1 - H_4$ are satisfied and

$$\Gamma(\alpha + 1) > [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + A_i I_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\}] T^\alpha,$$

then system (1.1) has at least one solution in $[0, T]$.

Proof. Let

$$U(t) = \{u(t) | u(t) = (\sigma_1(t), \dots, \sigma_n(t), \chi_1(t), \dots, \chi_n(t))^T, \sigma_i(t), \chi_i(t) \in C[0, t], i = 1, 2, \dots, n\},$$

$$V(t) = \{v(t) | v(t) = (0, 0, \dots, 0, v_1(t), v_2(t), \dots, v_n(t))^T\},$$

one can see that $U(t)$ and $V(t)$ are Banach spaces with the norms

$$\|u\| = \sup_{0 \leq t \leq T} \sum_{i=1}^n (|\sigma_i(t)| + |\chi_i(t)|), \quad \|v\| = \sup_{0 \leq t \leq T} \sum_{i=1}^n |v_i(t)|$$

separately.

Respectively define operators $Q : U(t) \rightarrow U(t)$ and $R : V(t) \rightarrow V(t)$ as follows:

$$(Qu)(t) = ((Q\sigma_1)(t), (Q\sigma_2)(t), \dots, (Q\sigma_n)(t), (Q\chi_1)(t), (Q\chi_2)(t), \dots, (Q\chi_n)(t))^T,$$

$(Rv)(t) = (0, 0, \dots, 0, (Rv_1)(t), (Rv_2)(t), \dots, (Rv_n)(t))^T$, where

$$\begin{cases} (Q\sigma_i)(t) = -\eta_i D_t^{-\alpha} \sigma_i(t) + D_t^{-\alpha} \chi_i(t), \\ (Q\chi_i)(t) = -\eta_i(\eta_i - \gamma_i) D_t^{-\alpha} \sigma_i(t) - (\gamma_i - \eta_i) D_t^{-\alpha} \chi_i(t) \\ \quad - D_t^{-\alpha} \alpha_i(\sigma_i(t)) (h_i(\sigma_i(t)) - I_i(t)), \\ (Rv_i)(t) = D_t^{-\alpha} \alpha_i(\sigma_i(t)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(t)), i = 1, 2, \dots, n. \end{cases} \tag{3.7}$$

Let

$$\delta \geq \frac{\sum_{i=1}^n \bar{\alpha}_i [|h_i(0)| + I_i + \sum_{j=1}^n a_{ij} \bar{f}_j] T^\alpha}{\Gamma(\alpha + 1) - [\max_{1 \leq i \leq n} \{\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i\} + \max_{1 \leq i \leq n} \{1 + |\gamma_i - \eta_i|\}] T^\alpha},$$

$$B_\delta = \{u(t) \in U(t), v(t) \in V(t) : \|u\| \leq \delta, \|v\| \leq \delta\}.$$

It will begin with the proof that $Qu + Rv \in B_\delta$, where $u, v \in B_\delta$.

Based on (3.2),

$$\begin{aligned} \|Qu + Rv\| &\leq \|Qu\| + \|Rv\| \\ &\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ \eta_i D_t^{-\alpha} |\sigma_i(t)| + D_t^{-\alpha} |\chi_i(t)| \\ &\quad + \eta_i |\eta_i - \gamma_i| D_t^{-\alpha} |\sigma_i(t)| + |\gamma_i - \eta_i| D_t^{-\alpha} |\chi_i(t)| \\ &\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) (h_i(\sigma_i(t)) - I_i(t))| \} \\ &\quad + \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ D_t^{-\alpha} |\alpha_i(\sigma_i(t))| \sum_{j=1}^n a_{ij} |f_j(\sigma_j(t))| \} \\ &\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ (\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i) D_t^{-\alpha} |\sigma_i(t)| \\ &\quad + (1 + |\gamma_i - \eta_i|) D_t^{-\alpha} |\chi_i(t)| + D_t^{-\alpha} \bar{\alpha}_i (|h_i(0)| + I_i) \} \\ &\quad + \sup_{0 \leq t \leq T} \sum_{i=1}^n \left\{ \sum_{j=1}^n D_t^{-\alpha} \bar{\alpha}_i |a_{ij}| \bar{f}_j \right\} \\ &\leq \max_{1 \leq i \leq n} \{ \eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i \} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u\| dz \\ &\quad + \max_{1 \leq i \leq n} \{ 1 + |\gamma_i - \eta_i| \} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u\| dz \\ &\quad + \sum_{i=1}^n \bar{\alpha}_i (|h_i(0)| + I_i) \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \\ &\quad + \sum_{i=1}^n \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \\ &\leq \max_{1 \leq i \leq n} \{ \eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i \} \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\| \\ &\quad + \max_{1 \leq i \leq n} \{ 1 + |\gamma_i - \eta_i| \} \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\| \\ &\quad + \sum_{i=1}^n \bar{\alpha}_i (|h_i(0)| + I_i) + \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &= [\max_{1 \leq i \leq n} \{ \eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i \} \\ &\quad + \max_{1 \leq i \leq n} \{ 1 + |\gamma_i - \eta_i| \}] \frac{T^\alpha \delta}{\Gamma(\alpha+1)} \\ &\quad + \sum_{i=1}^n \bar{\alpha}_i (|h_i(0)| + I_i) + \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{T^\alpha}{\Gamma(\alpha+1)} \\ &\leq \delta, \end{aligned}$$

then $Qu + Rv \in B_\delta$.

Next, let $u(t) \in B_\delta$,

$$\bar{U}(t) = \{\bar{u}(t)|\bar{u}(t) = (\bar{\sigma}_1(t), \bar{\sigma}_2(t), \dots, \bar{\sigma}_n(t), \bar{\chi}_1(t), \bar{\chi}_2(t), \dots, \bar{\chi}_n(t))^T \in B_\delta\},$$

$$\begin{aligned} \|Qu - Q\bar{u}\| &\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ \eta_i D_t^{-\alpha} |\sigma_i(t) - \bar{\sigma}_i(t)| + D_t^{-\alpha} |\chi_i(t) - \bar{\chi}_i(t)| \\ &\quad + \eta_i |\eta_i - \gamma_i| D_t^{-\alpha} |\sigma_i(t) - \bar{\sigma}_i(t)| + |\gamma_i - \eta_i| D_t^{-\alpha} |\chi_i(t) - \bar{\chi}_i(t)| \\ &\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) h_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t)) h_i(\bar{\sigma}_i(t))| \\ &\quad + D_t^{-\alpha} |\alpha_i(\sigma_i(t)) - \alpha_i(\bar{\sigma}_i(t))| I_i(t) \} \\ &\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ [\eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + A_i I_i] D_t^{-\alpha} |\sigma_i(t) - \bar{\sigma}_i(t)| \\ &\quad + (1 + |\gamma_i - \eta_i|) D_t^{-\alpha} |\chi_i(t) - \bar{\chi}_i(t)| \} \\ &\leq \max_{1 \leq i \leq n} \{ \eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + A_i I_i \} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u - \bar{u}\| dz \\ &\quad + \max_{1 \leq i \leq n} \{ 1 + |\gamma_i - \eta_i| \} \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} \|u - \bar{u}\| dz \\ &= [\max_{1 \leq i \leq n} \{ \eta_i + \eta_i |\eta_i - \gamma_i| + \bar{p}_i + A_i I_i \} \\ &\quad + \max_{1 \leq i \leq n} \{ 1 + |\gamma_i - \eta_i| \}] \frac{T^\alpha}{\Gamma(\alpha + 1)} \|u - \bar{u}\| \\ &\leq \|u - \bar{u}\|. \end{aligned} \tag{3.8}$$

Therefore, Q is a contraction mapping.

Then, it comes to prove that R is continuous and compact.

Since $\alpha_i(\cdot)$ and $f_j(\cdot)$ are continuous, one can see that R is continuous.

Let $v(t) \in B_\delta$, one has

$$\begin{aligned} \|(Rv)(t)\| &\leq \sup_{0 \leq t \leq T} \sum_{i=1}^n \{ D_t^{-\alpha} |\alpha_i(\sigma_i(t))| \sum_{j=1}^n |a_{ij}| f_j(\sigma_j(t)) \} \\ &\leq \sum_{i=1}^n D_t^{-\alpha} \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \\ &= \sum_{i=1}^n \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} dz \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \bar{\alpha}_i |a_{ij}| \bar{f}_j \frac{T^\alpha}{\Gamma(\alpha + 1)}, \end{aligned}$$

which means that R is uniformly bounded in B_δ .

Next, it will be proved that $(Qv)(t)$ is uniformly continuous.

In fact, for $v \in B_\delta$, when $0 < t_1 < t_2$, one has

$$|(Rv_i)(t_2) - (Rv_i)(t_1)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \right.$$

$$\begin{aligned}
& -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \\
= & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \right. \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \right| \\
= & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - z)^{\alpha-1} - (t_1 - z)^{\alpha-1}] \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - z)^{\alpha-1} \alpha_i(\sigma_i(z)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(z)) dz \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - z)^{\alpha-1} - (t_1 - z)^{\alpha-1}] |\alpha_i(\sigma_i(z))| \sum_{j=1}^n |a_{ij}| |f_j(\sigma_j(z))| dz \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - z)^{\alpha-1} |\alpha_i(\sigma_i(z))| \sum_{j=1}^n |a_{ij}| |f_j(\sigma_j(z))| dz \\
= & \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - z)^{\alpha-1} - (t_2 - z)^{\alpha-1}] dz \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - z)^{\alpha-1} dz \right\} \\
= & \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{1}{\Gamma(\alpha+1)} [t_1^\alpha + (t_2 - t_1)^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha] \\
= & \bar{\alpha}_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \frac{1}{\Gamma(\alpha+1)} [t_1^\alpha + 2(t_2 - t_1)^\alpha - t_2^\alpha]. \tag{3.9}
\end{aligned}$$

The right side of (3.9) tends to 0 when $t_1 \rightarrow t_2$. Therefore, $R(B_\delta)$ is a compact set. It comes out from the Arzela-Ascoli theorem that R is compact. Therefore, Q is a contraction mapping and R is continuous and compact. According to Lemma 3, there exists at least one $u^*(t) \in B_\delta$ such that $Qu^*(t) + Ru^*(t) = u^*(t)$. That is,

$$\begin{cases}
\sigma_i^*(t) = -\eta_i D_t^{-\alpha} \sigma_i^*(t) + D_t^{-\alpha} \chi_i^*(t), \\
\chi_i^*(t) = -\eta_i (\eta_i - \gamma_i) D_t^{-\alpha} \sigma_i^*(t) - (\beta_i - \eta_i) D_t^{-\alpha} \chi_i^*(t) \\
\quad - D_t^{-\alpha} \alpha_i(\sigma_i^*(t)) [h_i(\sigma_i^*(t)) - \sum_{j=1}^n a_{ij} f_j(\sigma_j^*(t)) - I_i(t)],
\end{cases} \tag{3.10}$$

which means that the system described by (1.1) has at least one solution. \square

Theorem 3.3. If the assumptions $H_1 - H_4$ are satisfied, $I_i(t + \omega) = I_i(t)$ for a certain constant $\omega > 0$,

and

$$1 + \eta_i(\gamma_i - \eta_i) - \bar{p}_i > 0,$$

$$\delta = \min_{1 \leq i \leq n} \{2\eta_i + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j - \sum_{j=1}^n |a_{ji}| l_i \bar{\alpha}_j,$$

$$2(\gamma_i - \eta_i) + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j - \sum_{j=1}^n |a_{ij}| l_j \bar{\alpha}_i - 2\bar{\alpha}_i I_i\} > 0,$$

$i = 1, 2, \dots, n$, then the solution of the system described by (1.1) is S-asymptotic ω -periodic, where $0 < T < \infty$.

Proof. Let $\sigma^T(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_n(t))$ be a solution of the system given by (1.1) with the initial values $\sigma_i(s) = \varphi_i(s)$, ($i = 1, 2, \dots, n$, $s \leq 0$).

From the assumptions H_1 and H_3 ,

$$\alpha_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t)) = \alpha'_i(\xi_i^*)[\sigma_i(t + \omega) - \sigma_i(t)],$$

$$\alpha_i(\sigma_i(t + \omega))h_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t))h_i(\sigma_i(t)) = p'_i(\xi_i)[\sigma_i(t + \omega) - \sigma_i(t)],$$

where ξ_i is between 0 and ζ_i .

From the conditions in Theorem 3.3,

$$1 + \eta_i(\gamma_i - \eta_i) - \underline{p}_i \geq 1 + \eta_i(\gamma_i - \eta_i) - p'_i(\xi_i) \geq 1 + \eta_i(\gamma_i - \eta_i) - \bar{p}_i > 0,$$

then it follows from (3.2) and Lemma 5 that

$$\begin{aligned} D_t^\alpha [\sigma_i(t + \omega) - \sigma_i(t)]^2 &\leq 2[\sigma_i(t + \omega) - \sigma_i(t)] D_t^\alpha [\sigma_i(t + \omega) - \sigma_i(t)] \\ &= 2[\sigma_i(t + \omega) - \sigma_i(t)] [-\eta_i \sigma_i(t + \omega) + \chi_i(t + \omega) + \eta_i \sigma_i(t) - \chi_i(t)] \\ &= -2\eta_i [\sigma_i(t + \omega) - \sigma_i(t)]^2 + 2[\sigma_i(t + \omega) - \sigma_i(t)] [\chi_i(t + \omega) - \chi_i(t)] \\ D_t^\alpha [\chi_i(t + \omega) - \chi_i(t)]^2 &\leq 2[\chi_i(t + \omega) - \chi_i(t)] D_t^\alpha [\chi_i(t + \omega) - \chi_i(t)] \\ &= 2[\chi_i(t + \omega) - \chi_i(t)] \{-\eta_i(\eta_i - \gamma_i)[\sigma_i(t + \omega) - \sigma_i(t)] \\ &\quad - (\gamma_i - \eta_i)[\chi_i(t + \omega) - \chi_i(t)] \\ &\quad - [\alpha_i(\sigma_i(t + \omega))h_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t))h_i(\sigma_i(t))] \\ &\quad + [\alpha_i(\sigma_i(t + \omega)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(t + \omega)) - \alpha_i(\sigma_i(t)) \sum_{j=1}^n a_{ij} f_j(\sigma_j(t))] \\ &\quad + [\alpha_i(\sigma_i(t + \omega))I_i(t + \omega) - \alpha_i(\sigma_i(t))I_i(t)]\} \\ &= 2[\chi_i(t + \omega) - \chi_i(t)] \{-\eta_i(\eta_i - \gamma_i)[\sigma_i(t + \omega) - \sigma_i(t)] \\ &\quad - (\gamma_i - \eta_i)[\chi_i(t + \omega) - \chi_i(t)] - p'_i(\xi_i)[\sigma_i(t + \omega) - \sigma_i(t)] \\ &\quad + [\alpha_i(\sigma_i(t + \omega)) \sum_{j=1}^n a_{ij} [f_j(\sigma_j(t + \omega)) - f_j(\sigma_j(t))] \\ &\quad + [\alpha_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t))] \sum_{j=1}^n a_{ij} f_j(\sigma_j(t))\} \end{aligned}$$

$$+[\alpha_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t))]I_i(t)\}.$$

Therefore,

$$\begin{aligned} & D_i^\alpha \{[\sigma_i(t + \omega) - \sigma_i(t)]^2 + [\chi_i(t + \omega) - \chi_i(t)]^2\} \\ &= -2\eta_i[\sigma_i(t + \omega) - \sigma_i(t)]^2 - 2(\gamma_i - \eta_i)[\chi_i(t + \omega) - \chi_i(t)]^2 \\ &\quad + [2 + 2\eta_i(\gamma_i - \eta_i) - 2p_i'(\xi_i)][\sigma_i(t + \omega) - \sigma_i(t)][\chi_i(t + \omega) - \chi_i(t)] \\ &\quad + 2\alpha_i(\sigma_i(t + \omega))[\chi_i(t + \omega) - \chi_i(t)] \sum_{j=1}^n a_{ij}[f_j(\sigma_j(t + \omega)) - f_j(\sigma_j(t))] \\ &\quad + 2[\alpha_i(\sigma_i(t + \omega)) - \alpha_i(\sigma_i(t))][\chi_i(t + \omega) - \chi_i(t)] \left[\sum_{j=1}^n a_{ij}f_j(\sigma_j(t)) + I_i(t) \right] \\ &\leq -2\eta_i[\sigma_i(t + \omega) - \sigma_i(t)]^2 - 2(\gamma_i - \eta_i)[\chi_i(t + \omega) - \chi_i(t)]^2 \\ &\quad + [2 + 2\eta_i(\gamma_i - \eta_i) - 2\underline{p}_i][\sigma_i(t + \omega) - \sigma_i(t)][\chi_i(t + \omega) - \chi_i(t)] \\ &\quad + 2\bar{\alpha}_i|\chi_i(t + \omega) - \chi_i(t)| \sum_{j=1}^n |a_{ij}l_j|\sigma_j(t + \omega) - \sigma_j(t) + 4\bar{\alpha}_iI_i|\chi_i(t + \omega) - \chi_i(t)| \\ &\quad + 2A_i|\sigma_i(t + \omega) - \sigma_i(t)|[\chi_i(t + \omega) - \chi_i(t)] \sum_{j=1}^n |a_{ij}|\bar{f}_j \\ &\leq -2\eta_i[\sigma_i(t + \omega) - \sigma_i(t)]^2 \\ &\quad + [1 + \eta_i(\gamma_i - \eta_i) - \underline{p}_i]\{[\sigma_i(t + \omega) - \sigma_i(t)]^2 + [\chi_i(t + \omega) - \chi_i(t)]^2\} \\ &\quad - 2(\gamma_i - \eta_i)[\chi_i(t + \omega) - \chi_i(t)]^2 + \bar{\alpha}_i \sum_{j=1}^n |a_{ij}l_j|[\sigma_j(t + \omega) - \sigma_j(t)]^2 \\ &\quad + \bar{\alpha}_i \sum_{j=1}^n |a_{ij}l_j|[\chi_i(t + \omega) - \chi_i(t)]^2 + 2\bar{\alpha}_iI_i + 2\bar{\alpha}_iI_i[\chi_i(t + \omega) - \chi_i(t)]^2\} \\ &\quad + A_i \sum_{j=1}^n |a_{ij}|\bar{f}_j\{[\sigma_i(t + \omega) - \sigma_i(t)]^2 + [\chi_i(t + \omega) - \chi_i(t)]^2\} \\ &= \{-2\eta_i + 1 + \eta_i(\gamma_i - \eta_i) - \underline{p}_i + A_i \sum_{j=1}^n |a_{ij}|\bar{f}_j\}[\sigma_i(t + \omega) - \sigma_i(t)]^2 \\ &\quad + [1 + \eta_i(\gamma_i - \eta_i) - \underline{p}_i - 2(\gamma_i - \eta_i) + A_i \sum_{j=1}^n |a_{ij}|\bar{f}_j \\ &\quad + \sum_{j=1}^n |a_{ij}l_j\bar{\alpha}_i + 2\bar{\alpha}_iI_i][\chi_i(t + \omega) - \chi_i(t)]^2 \\ &\quad + \sum_{j=1}^n |a_{ij}l_j\bar{\alpha}_i|[\sigma_j(t + \omega) - \sigma_j(t)]^2 + 2\bar{\alpha}_iI_i. \end{aligned} \tag{3.11}$$

From (3.11),

$$\begin{aligned}
& D_t^\alpha \sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 + [\chi_i(t+\omega) - \chi_i(t)]^2\} \\
& \leq \sum_{i=1}^n [-2\eta_i + 1 + \eta_i(\gamma_i - \eta_i) - \underline{p}_i + A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \\
& \quad + \sum_{j=1}^n |a_{ji}| l_j \bar{\alpha}_j] [\sigma_i(t+\omega) - \sigma_i(t)]^2 \\
& \quad + \sum_{i=1}^n \{[1 + \eta_i(\gamma_i - \eta_i)] - 2(\gamma_i - \eta_i) - \underline{p}_i + A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j \\
& \quad + \sum_{j=1}^n |a_{ij}| l_j \bar{\alpha}_i + 2\bar{\alpha}_i I_i\} [\chi_i(t+\omega) - \chi_i(t)]^2 + \sum_{i=1}^n 2\bar{\alpha}_i I_i \\
& \leq -\delta \sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 + [\chi_i(t+\omega) - \chi_i(t)]^2\} + A, \tag{3.12}
\end{aligned}$$

where $\delta = \min_{1 \leq i \leq n} \{2\eta_i + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j - \sum_{j=1}^n |a_{ji}| l_j \bar{\alpha}_j, 2(\gamma_i - \eta_i) + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^n |a_{ij}| \bar{f}_j - \sum_{j=1}^n |a_{ij}| l_j \bar{\alpha}_i - 2\bar{\alpha}_i I_i\}$, and $A = \sum_{i=1}^n 2\bar{\alpha}_i I_i$.

Therefore,

$$\begin{aligned}
\sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 + [\chi_i(t+\omega) - \chi_i(t)]^2\} & \leq -\delta D_t^{-\alpha} \sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 \\
& \quad + [\chi_i(t+\omega) - \chi_i(t)]^2\} + D_t^{-\alpha}(A) \\
& \leq -\delta D_t^{-\alpha} \sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 \\
& \quad + [\chi_i(t+\omega) - \chi_i(t)]^2\} + \frac{AT^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

for $0 < t < T < +\infty$.

Let $P(t) = \sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2 + [\chi_i(t+\omega) - \chi_i(t)]^2\}$.

Obviously, $P(0) > 0$, and

$$P(t) \leq -\delta D_t^{-\alpha} P(t) + \frac{AT^\alpha}{\Gamma(\alpha+1)}.$$

From Lemma 4,

$$P(t) \leq \frac{AT^\alpha}{\Gamma(\alpha+1)} E_\alpha(-\delta t^\alpha),$$

then

$$\sum_{i=1}^n \{[\sigma_i(t+\omega) - \sigma_i(t)]^2\} \leq \frac{AT^\alpha}{\Gamma(\alpha+1)} E_\alpha(-\delta t^\alpha).$$

Therefore,

$$\lim_{t \rightarrow \infty} [\sigma_i(t+\omega) - \sigma_i(t)]^2 = 0, \quad i = 1, 2, \dots, n.$$

from Definition 4, one sees that the solution of the system given by (1.1) is S -asymptotic ω -periodic. \square

Remark. Among Theorems 3.1–3.3, under the condition of Theorem 3.1, we reveal that the system given by (1.1) has a unique solution; under the condition of Theorem 3.2, we reveal that the system given by (1.1) has at least one solution; if the solution of the system given by (1.1) exists and the conditions in Theorem 3.3 are satisfied, we reveal that the solution of the system given by (1.1) is S -asymptotic ω -periodic.

4. Numerical example

In this section, a numerical example is simulated to verify the results.

Example. Consider the following FCGNNI:

$$D_t^{2\alpha}(\sigma_i(t)) = -\gamma_i D_t^\alpha(\sigma_i(t)) - \alpha_i(\sigma_i(t))[h_i(\sigma_i(t)) - \sum_{j=1}^2 a_{ij} f_j(\sigma_j(t)) - I_i(t)], \quad (4.1)$$

where $i = 1, 2$, $0 < \alpha < 1$.

Let

$$\begin{aligned} \alpha &= 0.75, \quad \gamma_1 = 1.5, \gamma_2 = 1.3, \quad \eta_1 = 0.4, \eta_2 = 0.2, \\ \alpha_1(\sigma_1) &= \frac{3}{20} \left(2 + \frac{1}{1 + \sigma_1^2}\right), \quad \alpha_2(\sigma_2) = \frac{53}{500} \left(2 - \frac{1}{1 + \sigma_2^2}\right), \\ h_1(\sigma_1) &= 2.6\sigma_1, \quad h_2(\sigma_2) = 6\sigma_2, \\ f_j(\sigma_j) &= 0.1 \sin(\sigma_j(t)), \quad (j = 1, 2), \\ I_i &= 0.1 \cos \frac{2t}{3}, \quad i = 1, 2. \quad \omega = 3\pi, \\ a_{11} &= 0.15, a_{12} = 0.1, a_{21} = 0.15, a_{22} = 0.15. \end{aligned}$$

Therefore, the parameters in Theorem 3.3 are

$$\underline{\alpha}_1 = 0.3, \quad \bar{\alpha}_1 = 0.45, \quad \underline{\alpha}_2 = 0.106, \quad \bar{\alpha}_2 = 0.212, \quad A_1 = 0.15, A_2 = 0.106,$$

$$\bar{f}_j = l_j = 1, \quad I_i = 0.1, \quad i, j = 1, 2, \quad \underline{p}_1 = 0.7, \bar{p}_1 = 1.17, \underline{p}_2 = 1.1667, \bar{p}_2 = 1.1925.$$

After calculating, one has

$$1 + \eta_1(\gamma_1 - \eta_1) - \bar{p}_1 = 0.27 > 0, \quad 1 + \eta_2(\gamma_2 - \eta_2) - \bar{p}_2 = 0.0275 > 0,$$

$$\delta = \min_{1 \leq i \leq 2} \{2\eta_i + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^2 |a_{ij}| \bar{f}_j - \sum_{j=1}^2 |a_{ji}| l_i \bar{\alpha}_j,$$

$$2(\gamma_i - \eta_i) + \underline{p}_i - [1 + \eta_i(\gamma_i - \eta_i)] - A_i \sum_{j=1}^2 |a_{ij}| \bar{f}_j - \sum_{j=1}^2 |a_{ji}| l_j \bar{\alpha}_i - 2\bar{\alpha}_i I_i\} = 1.3827 > 0.$$

Due to Theorem 3.3, the solution of the system given by (4.1) is S -asymptotic 3π -periodic.

The states of $\sigma_1(t)$ and $\sigma_2(t)$ are shown in Figure 1. It is clear that the numerical simulation is consistent with Theorem 3.3.

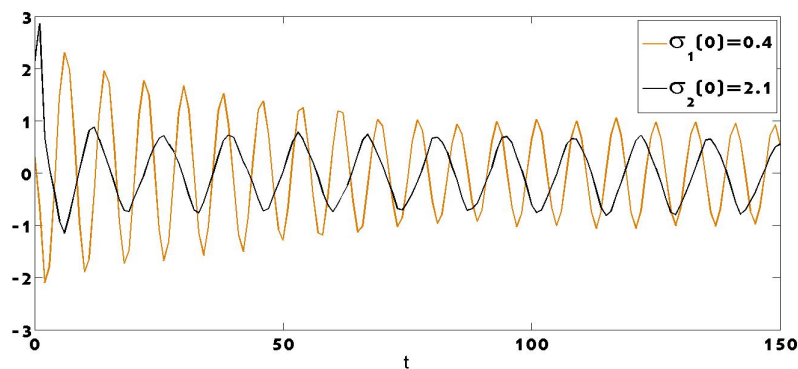


Figure 1. Time response curves for the state variables $\sigma_1(t)$ and $\sigma_2(t)$ in Example

5. Conclusions

To date, scholars have done a lot of research on the dynamic behavior of inertial neural networks, and most of them focused on the chaos, bifurcation, stability and synchronization dynamics of NNs with inertia. But the existence of the solution and the S-asymptotic ω -periodic have not been studied. In this article, the existence and the S-asymptotic ω -period of the solution in for FCGNNs are discussed. By employing the contraction mapping principle and the differential mean-value theorem, the existence and the S-asymptotic ω -periodic of the solution in the system, namely, Theorems 3.1–3.3. Finally, an example is simulated to demonstrate the correctness and validity of the results, which has a particular significance in both theory and application. The idea is innovative and meaningful. Similarly, the stability of other types of FNNs can be studied by using the methods employed in the theorems, including fractional-order inertial BAM neural networks, fractional-order inertial BAM Cohen-Grossberg neural networks, etc.

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Conflict of interest

We declare that we have no financial or personal relationships with other people or organizations that can inappropriately influence our work, and that there is no professional or other personal interest of any nature or kind in any product, service or company that could be construed as influencing the position presented in the manuscript.

References

1. V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers Ltd., 2009.

2. C. P. Li, W. H. Deng, *Remarks on fractional derivatives*, *Appl. Math. Comput.*, **187** (2007), 777–784. <https://doi.org/10.1016/j.amc.2006.08.163>
3. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
4. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1999.
5. O. Heaviside, *Electromagnetic theory*, New York: Chelsea, 1971.
6. H. Sun, A. Abdelwahab, B. Onaral, Linear approximation of transfer function with a pole of fractional power, *IEEE T. Automat. Contr.*, **29** (1984), 441–444. <https://doi.org/10.1109/TAC.1984.1103551>
7. R. C. Koeller, Application of fractional calculus to the theory of viscoelasticity, *J. Appl. Mech.*, **51** (1984), 299–307. <https://doi.org/10.1115/1.3167616>
8. W. C. Chen, Nonlinear dynamics and chaos in a fractional-order financial system, *Chaos Soliton. Fract.*, **36** (2008), 1305–1314. <https://doi.org/10.1016/j.chaos.2006.07.051>
9. T. J. Anastasio, The fractional-order dynamics of brainstem vestibulo-oculomotor neurons, *Biol. Cybern.*, **72** (1994), 69–79. <https://doi.org/10.1007/BF00206239>
10. C. J. Xu, P. L. Li, On finite-time Stability for fractional-order neural networks with proportional delays, *Neural Process. Lett.*, **50** (2019), 1241–1256. <https://doi.org/10.1007/s11063-018-9917-2>
11. J. D. Li, Z. B. Wu, N. J. Huang, Asymptotical stability of Riemann-Liouville fractional neutral-type delayed projective neural networks, *Neural Process. Lett.*, **50** (2019), 565–579. <https://doi.org/10.1007/s11063-019-10050-8>
12. F. Y. Zhou, C. Y. Ma, Mittag-Leffler stability and global asymptotically ω -periodicity of fractional BAM neural networks with time-varying delays, *Neural Process. Lett.*, **47** (2018), 71–98. <https://doi.org/10.1007/s11063-017-9634-2>
13. C. J. Xu, W. Zhang, C. Aouiti, Z. X. Liu, M. X. Liao, P. L. Li, Further investigation on bifurcation and their control of fractional-order BAM neural networks involving four neurons and multiple delays, *Math. Method. Appl. Sci.*, 2021. <https://doi.org/10.1002/mma.7581>
14. S. C. Xu, X. Y. Wang, X. L. Ye, A new fractional-order chaos system of Hopfield neural network and its application in image encryption, *Chaos Soliton. Fract.*, **157** (2022), 111889. <https://doi.org/10.1016/j.chaos.2022.111889>
15. Y. R. Liu, L. Wang, K. X. Gu, M. Li, Artificial Neural Network (ANN)-Bayesian probability framework (BPF) based method of dynamic force reconstruction under multi-source uncertainties, *Knowl.-Based Syst.*, **237** (2022), 107796. <https://doi.org/10.1016/j.knosys.2021.107796>
16. A. Mauro, F. Conti, F. Dodge, R. Schor, Subthreshold behavior and phenomenological impedance of the squid giant axon, *J. Gen. Physiol.*, **55** (1970), 497–523. <https://doi.org/10.1085/jgp.55.4.497>
17. D. E. Angelaki, M. J. Correia, Models of membrane resonance in pigeon semicircular canal type II hair cells, *Biol. Cybern.*, **65** (1991), 1–10. <https://doi.org/10.1007/BF00197284>
18. J. F. Ashmore, D. Attwell, Models for electrical tuning in hair cells, *Proc. Royal Soc. B: Biol. Sci.*, **226** (1985), 325–344.

19. L. Ke, W. L. Li, Exponential synchronization in inertial Cohen-Grossberg neural networks with time delays, *J. Franklin I.*, **356** (2019), 11285–11304. <https://doi.org/10.1016/j.jfranklin.2019.07.027>
20. L. Ke, W. L. Li, Exponential synchronization in inertial neural networks with time delays, *Electronics*, **8** (2019), 356. <https://doi.org/10.3390/electronics8030356>
21. S. Li, X. M. Wang, H. Y. Qin, X. M. Zhong, Synchronization criteria for neutral-type quaternion-valued neural networks with mixed delays, *AIMS Mathematics*, **6** (2021), 8044–8063. <https://doi.org/10.3934/math.2021467>
22. F. C. Kong, Q. X. Zhu, R. Sakthivel, A. Mohammadzadeh, Fixed-time synchronization analysis for discontinuous fuzzy inertial neural networks with parameter uncertainties, *Neurocomputing*, **422** (2021), 295–313. <https://doi.org/10.1016/j.neucom.2020.09.014>
23. W. Zhang, J. T. Qi, Synchronization of coupled memristive inertial delayed neural networks with impulse and intermittent control, *Neural Comput. Appl.*, **33** (2020), 7953–7964. <https://doi.org/10.1007/s00521-020-05540-z>
24. L. Ke, Mittag-Leffler stability and asymptotic ω -periodicity of fractional inertial neural networks with time-delays, *Neurocomputing*, **465** (2021), 53–62. <https://doi.org/10.1016/j.neucom.2021.08.121>
25. H. R. Henriquez, M. Pjerri, P. Tboas, On S-asymptotically ω -periodicity functions on Banach spaces and applications, *J. Math. Anal. Appl.*, **343** (2008), 1119–1130. <https://doi.org/10.1016/j.jmaa.2008.02.023>
26. Y. J. Gu, H. Wang, Y. G. Yu, Stability and synchronization for Riemann-Liouville fractional time-delayed inertial neural networks, *Neurocomputing*, **340** (2019), 270–280. <https://doi.org/10.1016/j.neucom.2019.03.005>
27. S. L. Zhang, M. L. Tang, X. G. Liu, Synchronization of a Riemann-Liouville fractional time-delayed neural network with two inertial terms, *Circuits Syst. Signal Process.*, **40** (2021), 5280–5308. <https://doi.org/10.1007/s00034-021-01717-6>
28. M. A. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE T. Syst. Man. Cy.*, **SMC-13** (1983), 815–826. <https://doi.org/10.1109/TSMC.1983.6313075>
29. M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, 1964.

Appendix

Definition 1. [4] The R-L fractional-order integral of the function $u(t)$ is defined as

$${}_{t_0}D_t^{-q}u(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-r)^{q-1}u(r)dr, \quad (5.1)$$

where $\Gamma(\tau) = \int_0^{+\infty} t^{\tau-1}e^{-t}dt$, where $0 < q < 1$.

Definition 2. [4] The R-L fractional-order derivative is defined as

$${}^{RL}D_x^\alpha v(x) = D^n({}_aD_x^{\alpha-n}v(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{v(t)}{(x-t)^{\alpha-n+1}} dt, \quad (5.2)$$

where the order α is a positive real number, and $n-1 \leq \alpha \leq n$.

Definition 3. [4] The Mittag-Leffler function with one parameter is

$$E_\rho(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(k\rho + 1)}, \quad (5.3)$$

where $\rho > 0$ and $s \in \mathbb{C}$.

Definition 4. [19] If $p(t) \in C_b([0, +\infty), \mathbf{R})$, there exists $\omega > 0$ such that $\lim_{t \rightarrow +\infty} [p(t+\omega) - p(t)] = 0$, then $p(t)$ is S -asymptotic ω -periodic and ω is an asymptotic period of $p(t)$.

Lemma 1. [2] If $s(t) \in C^r[0, +\infty)$, $n-1 \leq q < n$, $m-1 \leq p < m$, $n, m \in \mathbf{Z}^+$, $r = \max\{n, m\}$, then

$$(1) D_t^{-q} A = \frac{At^q}{\Gamma(q+1)}, \text{ where } A \text{ is a constant.}$$

$$(2) D_t^q (ks_1(t) + hs_2(t)) = kD_t^q s_1(t) + hD_t^q s_2(t), \text{ where } k \text{ and } h \text{ are constants.}$$

$$(3) D_t^p D_t^{-q} s(t) = D_t^{p-q} s(t), D_t^{-q} D_t^{-p} s(t) = D_t^{-p-q} s(t).$$

Lemma 2. [26] If $s(t) \in R$ is derivable in $[0, \delta]$ ($\delta > 0$) and $0 < q < 1$, $n-1 < p < n$, then

$$(1) D_t^p D_t^q s(t) = D_t^{p+q} s(t),$$

$$(2) D_t^{-p} D_t^q s(t) = D_t^{-p+q} s(t).$$

Lemma 3. [29] Z is a Banach space, C is a closed convex and nonempty subset of Z , and G_1, G_2 are operators which satisfy

$$(1) G_1 x + G_2 y \in C \text{ whenever } x, y \in C;$$

$$(2) G_1 \text{ is compact and continuous;}$$

$$(3) G_2 \text{ is a contraction mapping;}$$

then there exists $x \in C$ which satisfies $G_1 x + G_2 x = x$.

Lemma 4. [24] If $s(t) \in C[0, +\infty)$, and there exist $d_1 > 0$ and $d_2 > 0$ which satisfy $s(t) \leq -d_1 D_{0,t}^{-q} s(t) + d_2$, then

$$s(t) \leq d_2 E_q(-d_1 t^q). \quad (5.4)$$

Lemma 5. [26] If $v(t)$ has a continuous derivative, then

$$\frac{1}{2} D_t^q r^2(t) \leq v(t) D_t^q v(t), \quad 0 < q \leq 1. \quad (5.5)$$

