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Research article

Solving a fractional differential equation via θ-contractions in $\Re$-complete metric spaces

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Abstract: In this manuscript, we introduce the notion of $\Re\alpha$-θ-contractions and prove some fixed-point theorems in the sense of $\Re$-complete metric spaces. These results generalize existing ones in the literature. Also, we provide some illustrative non-trivial examples and applications to a non-linear fractional differential equation.

Keywords: $\Re$-metric space; fixed point results; non-linear fractional differential equation
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The spaciousness of fixed-point theory can be glanced in different fields by looking at its applications. Fixed-point theorems say that functions must have at least one fixed point, under some circumstances. We can see that these results are usually beneficial in the region of mathematics and play a prissy character in detecting the existence and uniqueness of solutions of different mathematical models. Some scientists gave circumstances to find fixed points, in this manner, Banach and Caccioppoli gave Banach–Caccioppoli fixed-point theorem, which was started by Banach [6] in 1922 and was proved by Caccioppoli [7] in 1931. Banach–Caccioppoli fixed-point theorem guaranteed that if it seized, the function must have a fixed-point, under some circumstances. After this meritorious result of Banach and Caccioppoli, the fixed-point theory has taken on new elevations.

Branciari [2] proved Banach–Caccioppoli fixed-point theorem on a class of generalized metric spaces. In 2014, Jleli and Samet [1] coined a new concept of Θ-contraction mappings and established numerous fixed-point theorems for such mappings in complete metric spaces (CMS). Samet et al. [5] proved fixed-point theorems for α-ψ-contractive mappings. Ahmad et al. [4] proved fixed-point results for generalized Θ-contractions. Arshad et al. [8] proved some fixed-point results by using generalized contractions via triangular α-orbital admissibility in the sense of Branciari metric spaces.

Baghani et al. [3] (2017) presented a new generalization of the Banach fixed point theorem (BFPT) by defining the notion of $\mathfrak{R}$-sets. The $\mathfrak{R}$-set is a non-empty set equipped with a binary relation (called $\mathfrak{R}$-relation) having a special structure (see [3]). The metric defined on the $\mathfrak{R}$-set is called an $\mathfrak{R}$-metric space. The $\mathfrak{R}$-metric space contains partially ordered metric spaces and graphical metric spaces. Khalehoghli et al. [19] extended the work in [3] to $\mathfrak{R}$-metric spaces, Ali et al. [20] extended the work [19] to partial b-metric space and Khalil et al. [15] extended the work in [3] to ordered theoretic fuzzy metric spaces. Further fixed-point results on $\mathfrak{R}$-(generalized) metric spaces have been provided by Javed et al. [15] who initiated the notion of an $\mathfrak{R}$-structure and established the Banach contraction principle.

We introduce the concept of $\mathfrak{R}$-α-Θ-contractions ($\alpha_{\mathfrak{R}}$-$\Theta_{\mathfrak{R}}$-contractions), establish some fixed-point theorems for these contractions in the sense of $\mathfrak{R}$-complete metric spaces and some constructive examples and an application are also imparted. After proving that these contractions have fixed points, we give some examples to validate our results. For some necessary definitions and results, please see [9–18].

This manuscript is organized as follows. In section 2, some rudimentary concepts as $\mathfrak{R}$-sequence, Cauchy $\mathfrak{R}$-sequence, $\mathfrak{R}$-preserving, $\mathfrak{R}$-complete, $\mathfrak{R}$-continuous, $\mathfrak{R}$-convergent, Θ-contraction, α-Θ-contraction, and α-admissible are given. In section 3, the concept of $\alpha_{\mathfrak{R}}$-$\Theta_{\mathfrak{R}}$-contractions is introduced and some fixed point results are proved in the sense of $\mathfrak{R}$-CMSs and some constructive examples are also provided. In section 4, an application to non-linear fractional differential equations is provided.

2. Preliminaries

In this section, we recall some definitions that are necessary for the main work.

Definition 2.1. [3] Let $(\mathfrak{B},\mathfrak{R})$ be an $\mathfrak{R}$-set. A sequence $\{\beta_\omega\}$ is said to be an $\mathfrak{R}$-sequence if

$$(\forall\omega,k\in\mathbb{N},\beta_\omega\mathfrak{R}\beta_{\omega+k}) \text{ or } (\forall\omega,k\in\mathbb{N},\beta_{\omega+k}\mathfrak{R}\beta_\omega).$$

Also, $\{\beta_\omega\}$ is called a Cauchy $\mathfrak{R}$-sequence if for every $\varepsilon>0$ there exists an integer N such that $\mathfrak{D}(\beta_\omega,\beta_k)<\varepsilon$ if $\omega\geq N$ and $k\geq N$. It is clear that $\beta_\omega\mathfrak{R}\beta_k$ *or* $\beta_k\mathfrak{R}\beta_\omega$.

Definition 2.2. [3] Let $(\mathfrak{B},\mathfrak{R})$ be an $\mathfrak{R}$-set. A mapping $\xi_{\mathfrak{R}}:\mathfrak{B}\to\mathfrak{B}$ is called $\mathfrak{R}$-preserving if $\xi_{\mathfrak{R}}\beta\mathfrak{R}\xi_{\mathfrak{R}}\delta$, whenever $\beta\mathfrak{R}\delta$.

Definition 2.3. [3] Let $(\mathfrak{B},\mathfrak{R},\mathfrak{D})$ be an $\mathfrak{R}$-MS and $\mathfrak{R}$ be a binary relation over $\mathfrak{B}$. Then $\mathfrak{B}$ is said to be $\mathfrak{R}$-regular if for each sequence $\{\beta_\omega\}$ such that $\beta_\omega\mathfrak{R}\beta_{\omega+1}$, for all $\omega\in\mathbb{N}$, and $\beta_\omega\to e$, for some $e\in\mathfrak{B}$, then $\beta_\omega\mathfrak{R}$e, for all $\omega\in\mathbb{N}$ (briefly, $(\mathfrak{B},\mathfrak{R},\mathfrak{D})$ is called $\mathfrak{R}$-regular metric space).

Definition 2.4 [3] Let $(\mathfrak{B},\mathfrak{R},\mathfrak{D})$ be an $\mathfrak{R}$-MS. Then $\xi_{\mathfrak{R}}:\mathfrak{B}\to\mathfrak{B}$ is called $\mathfrak{R}$-continuous at $\beta\in\mathfrak{B}$

if for each $\mathfrak{R}$-sequence $\{\beta_\omega\}$ in $\mathfrak{B}$ with $\beta_\omega \to \beta$, we have $\xi_\mathfrak{R}\beta_\omega \to \xi_\mathfrak{R}\beta$. Also, $\xi_\mathfrak{R}$ is said to be $\mathfrak{R}$-continuous on $\mathfrak{B}$ if $\xi_\mathfrak{R}$ is $\mathfrak{R}$-continuous at each $\beta \in \mathfrak{B}$.

Definition 2.5. [3] Let $(\mathfrak{B}, \mathfrak{R}, \mathfrak{O})$ be an $\mathfrak{R}$-MS. Then $\mathfrak{B}$ is said to be an $\mathfrak{R}$-CMS if every Cauchy $\mathfrak{R}$-sequence is convergent in $\mathfrak{B}$.

Definition 2.6. [4] Let $\Theta: (0,\infty) \to (1,\infty)$ be a function satisfying the below circumstances:

(Θ_1) Θ is non-decreasing.

(Θ_2) For a sequence $\{\beta_\omega\} \subseteq \mathbb{R}^+$.

$$lim_{\omega\to\infty}\Theta(\beta_\omega) = 1 \Leftrightarrow lim_{\omega\to\infty}\beta_\omega = 0.$$

(Θ_3) There exist $k \in (0,1)$ and $l \in (0,\infty]$ such that

$$lim_{t\to\infty}\frac{\Theta(t)-1}{(t)^k} = l.$$

Let $(\mathfrak{B},\mathfrak{O})$ be a MS. A mapping $\xi: \mathfrak{B} \to \mathfrak{B}$ is said to be an Θ-contraction [4] if there exist $k \in (0,1)$ and a function Θ fulfiling (Θ_1)–(Θ_3) such that

$$\mathfrak{O}(\xi\beta,\xi\delta) \neq 0 \Rightarrow \Theta\big(\mathfrak{O}(\xi\beta,\xi\delta)\big) \leq [\Theta(\mathfrak{O}(\beta,\delta)]^k \qquad \forall\beta,\delta \in \mathfrak{B}.$$

Let Ω denote the set of all functions satisfying $(\Theta_1) - (\Theta_3)$.

Definition 2.7. [16] Let $(\mathfrak{B},\mathfrak{O})$ be a MS and $\xi: \mathfrak{B} \to \mathfrak{B}$ be a self-mapping. We say that ξ is an α-Θ-contraction if there exist $k \in (0,1)$ and two functions $\alpha: \mathfrak{B}\times\mathfrak{B} \to [0,\infty)$ and $\Theta \in \Omega$ such that

$$\mathfrak{O}(\xi\beta,\xi\delta) \neq 0 \Rightarrow \alpha(\beta,\delta)\Theta\big(\mathfrak{O}(\xi\beta,\xi\delta)\big) \leq [\Theta(d(\beta,\delta)]^k \qquad \forall\beta,\delta \in \mathfrak{B}.$$

3. Main results

In this section, we introduce the concept of $\alpha_\mathfrak{R}$-$\Theta_\mathfrak{R}$-contractions and some fixed-point results are also imparted in the sense of $\mathfrak{R}$-CMSs.

Definition 3.1. Let $(\mathfrak{B},\mathfrak{O})$ be an $\mathfrak{R}$-CMS and $\xi_\mathfrak{R}: \mathfrak{B} \to \mathfrak{B}$ be a mapping. We say that $\xi_\mathfrak{R}$ is an $\alpha_\mathfrak{R}$-$\Theta_\mathfrak{R}$-contraction if there exist $k \in (0,1)$ and two functions $\alpha_\mathfrak{R}: \mathfrak{B}\times\mathfrak{B} \to [0,\infty)$ and $\Theta \in \Omega$ such that

$$\mathfrak{O}(\xi\beta,\xi\delta) \neq 0 \Rightarrow \alpha_\mathfrak{R}(\beta,\delta)\Theta(\mathfrak{O}(\xi_\mathfrak{R}\beta,\xi_\mathfrak{R}\delta) \leq [\Theta(\mathfrak{O}(\beta,\delta)]^k, \qquad \forall\beta,\delta \in \mathfrak{B} \text{ with } \beta\mathfrak{R}\delta.$$

Definition 3.2. Let $\xi_\mathfrak{R}: \mathfrak{B} \to \mathfrak{B}$ and $\alpha_\mathfrak{R}: \mathfrak{B}\times\mathfrak{B} \to [0,\infty)$. We say that $\xi_\mathfrak{R}$ is $\alpha_\mathfrak{R}$-admissible if for all $\beta,\delta \in \mathfrak{B}$ with $\beta\mathfrak{R}\delta$,

$$\alpha_\mathfrak{R}(\beta,\delta) \geq 1 \Rightarrow \alpha_\mathfrak{R}(\xi_\mathfrak{R}\beta,\xi_\mathfrak{R}\delta) \geq 1.$$

Example 3.3. Let $\mathfrak{B} = (0,1] = A \cup B = (0,1]\backslash\{\frac{1}{4},\frac{1}{3},\frac{1}{2}\} \cup \{\frac{1}{4},\frac{1}{3},\frac{1}{2}\}$. Define $\xi_\mathfrak{R}: \mathfrak{B} \to \mathfrak{B}$ and $\alpha_\mathfrak{R}: \mathfrak{B}\times\mathfrak{B} \to [0,\infty)$ by

$$\xi_\mathfrak{R}(\beta) = \frac{5}{3}\beta$$

and

$$\alpha_{\Re}(\beta,\delta)=\frac{1}{max\,\{\beta,\delta\}},\forall\beta\in A,\delta\in B.$$

Define the $\Re$-relation: $\beta\Re\delta \Leftrightarrow \beta\le\delta$. Here, $\xi_{\Re}$ is $\alpha_{\Re}$-admissible. It is not α-admissible by taking $\beta=1$ and $\delta=\frac{1}{2}$.

Example 3.4. Let $\mathfrak{B}=(-2,2]$. Define the relation: $\beta\Re\delta \Leftrightarrow \beta+\delta\ge 0$.
Define the function $\alpha_{\Re}\colon \mathfrak{B}\times\mathfrak{B}\to[0,\infty)$ by

$$\alpha_{\Re}(\beta,\delta)=\begin{cases}\dfrac{min\{\beta,\delta\}}{1+max\{\beta,\delta\}}, & if\ \beta,\delta\in(0,2]\\ e^{-max\{\beta,\delta\}}, & if\ \beta,\delta\in[0,-2)\\ 0, & otherwise.\end{cases}$$

Define the mapping $\xi_{\Re}\colon\mathfrak{B}\to\mathfrak{B}$ by

$$\xi_{\Re}(\beta)=\begin{cases}1 & if\ \beta\in\left[\frac{-1}{2},\frac{1}{2}\right]\\ \dfrac{min\,\{1,\beta\}}{1+max\,\{1,\beta\}} & otherwise.\end{cases}$$

Clearly, $\xi_{\Re}$ is $\alpha_{\Re}$-admissible. It is not α-admissible. Indeed, for $\beta=0$ and $\delta=-1$, one has

$$\alpha(0,-1)=e^{0}=1.$$

But,

$$\alpha(\xi(0),\xi(-1))=\alpha\left(1,-\frac{1}{2}\right)=0.$$

Remark 3.5. The above example shows that an $\alpha_{\Re}$-admissible mapping need not to be an α-admissible mapping. But the converse holds.

Theorem 3.6. Let $(\mathfrak{B},\Re,\mathfrak{D})$ be an $\Re$-CMS and $\xi_{\Re}$ be a self-mapping, $\Re$-preserving, $\Re$-continuous and $\alpha_{\Re}\colon\mathfrak{B}\times\mathfrak{B}\to[0,\infty)$ be a function. Suppose that the below circumstances fulfill:

i) Suppose there exist $k\in(0,1)$ and a function $\Theta\in\Omega$ such that for all $\beta,\delta\in\mathfrak{B}$ with $\beta\Re\delta$,

$$\mathfrak{D}(\xi_{\Re}\beta,\ \xi_{\Re}\delta)\neq 0\Rightarrow\alpha_{\Re}(\beta,\delta)\Theta\big(\mathfrak{D}(\xi_{\Re}\beta,\ \xi_{\Re}\delta)\big)\le[\Theta(\mathfrak{D}(\beta,\delta))]^{k}. \tag{3.1}$$

ii) $\xi_{\Re}$ is $\alpha_{\Re}$-admissible.

iii) $\xi_{\Re}$ is $\Re$-continuous.

iv) There exists $\beta_0\in\mathfrak{B}$ such that $\beta_0\Re\xi_{\Re}\beta_0$ and $\alpha_{\Re}(\beta_0,\ \xi_{\Re}\beta_0)\ge 1$.

Then $\xi_{\Re}$ has a fixed point $e\in\mathfrak{B}$. Moreover, if for every two fixed points $e,\ f$ of $\xi_{\Re}$ we have $\alpha_{\Re}(e,f)\ge 1$, then the fixed point is unique.

Proof. Let $\beta_0\in\mathfrak{B}$ such that $(\forall\delta\in\mathfrak{B}\beta_0\Re\delta)$ or $(\forall\delta\in\mathfrak{B}\delta\Re\beta_0)$. By condition (iii), $\beta_0\Re\xi_{\Re}\beta_0$ or $\xi_{\Re}\beta_0\Re\beta_0$. For $\omega\in\mathbb{N}$, consider $\beta_\omega={\xi_{\Re}}^{\omega}\beta_o$. Assume that $\xi_{\Re}\beta_\omega=\xi_{\Re}\beta_{\omega+1}$ for some $\omega\in\mathbb{N}$. Then β_ω is a fixed point of $\xi_{\Re}$ and the proof is completed. Let $\xi_{\Re}\beta_\omega\neq\xi_{\Re}\beta_{\omega+1}$ for all $\omega\in\mathbb{N}$. Since $\xi_{\Re}$ is $\Re$-preserving, $(\xi_{\Re}\beta_\omega\Re\xi_{\Re}\beta_{\omega+1})$ or $(\xi_{\Re}\beta_{\omega+1}\Re\xi_{\Re}\beta_\omega)$. Hence, $\{\beta_\omega\}$ is an $\Re$-sequence. Again, by condition (ii),

$$\alpha_{\Re}(\beta_\omega,\ \xi_{\Re}\beta_\omega) = \alpha_{\Re}(\beta_\omega,\ \beta_{\omega+1}) \geq 1 \qquad \forall\omega \in \mathbb{N}. \tag{3.2}$$

From (3.1) and (3.2), we get

$$1 < \Theta(\mathfrak{D}(\beta_\omega, \beta_{\omega+1})) = \Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_\omega)$$

$$\leq \alpha_{\Re}(\beta_{\omega-1}, \beta_\omega)\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_\omega) \leq \left[\Theta\left(\mathfrak{D}(\beta_{\omega-1}, \beta_\omega)\right)\right]^k. \tag{3.3}$$

By (Θ_1), we have

$$\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) < \mathfrak{D}(\beta_{\omega-1}, \beta_\omega).$$

Hence, the sequence $\{\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\}$ is decreasing and $\{\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\}$ converges to a non-negative real number $r \geq 0$ such that

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) = r \text{ and } \mathfrak{D}(\beta_\omega, \beta_{\omega+1}) \geq r. \tag{3.4}$$

Then we prove that $r = 0$. Suppose that $r > 0$. Using (Θ_1), (3.3) and (3.4), we get

$$1 < \Theta(r) = \Theta\left(\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\right) \leq \left[\Theta\left(\mathfrak{D}(\beta_{\omega-1}, \beta_\omega)\right)\right]^k$$

$$\leq \cdots$$

$$\leq \left[\Theta\left(\mathfrak{D}(\beta_0, \beta_1)\right)\right]^{k^\omega} \forall\omega \in \mathbb{N}. \tag{3.5}$$

Letting $\omega \to \infty$ in (3.5), we get $\Theta(r) = 1$ and by using (Θ_2), we have $r = 0$. Therefore

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) = 0. \tag{3.6}$$

Assume that there are $\omega, \mathfrak{p} \in \mathbb{N}$ such that $\beta_\omega = \beta_{\omega+\mathfrak{p}}$. We must prove that $\mathfrak{p} = 1$. Assume that $\mathfrak{p} > 1$. Using (3.1) and (3.2), we get

$$\Theta\left(\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\right) = \Theta\left(\mathfrak{D}\left(\beta_{\omega+\mathfrak{p}}, \beta_{\omega+\mathfrak{p}+1}\right)\right) = \Theta\left(\mathfrak{D}\left(\xi_{\Re}\beta_{\omega+\mathfrak{p}-1}, \xi_{\Re}\beta_{\omega+\mathfrak{p}}\right)\right)$$

$$\leq \alpha_{\Re}\left(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\right)\Theta\left(\mathfrak{D}\left(\xi_{\Re}\beta_{\omega+\mathfrak{p}-1}, \xi_{\Re}\beta_{\omega+\mathfrak{p}}\right)\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}\left(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\right)\right)\right]^k. \tag{3.7}$$

Using (Θ_1), we get

$$\Theta\left(\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\right) < \mathfrak{D}\left(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\right)$$

and by (3.1), we obtain

$$\Theta\left(\mathfrak{D}\left(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\right)\right) = \Theta\left(\mathfrak{D}\left(\xi_{\Re}\beta_{\omega+\mathfrak{p}-2}, \xi_{\Re}\beta_{\omega+\mathfrak{p}-1}\right)\right)$$

$$\leq \alpha_{\Re}\left(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}\right)\Theta\left(\mathfrak{D}\left(\xi_{\Re}\beta_{\omega+\mathfrak{p}-2}, \xi_{\Re}\beta_{\omega+\mathfrak{p}-1}\right)\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}\left(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}\right)\right)\right]^k$$

$$< \mathfrak{D}\left(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\right). \tag{3.8}$$

By (Θ_1), we deduce

$$\mathfrak{D}(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}) < \mathfrak{D}(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}).$$

Continuing this process, we obtain

$$\mathfrak{D}(\beta_{\omega}, \beta_{\omega+1}) < \mathfrak{D}(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}) < \mathfrak{D}(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}) < \cdots < \mathfrak{D}(\beta_{\omega}, \beta_{\omega+1}),$$

which implies that $\mathfrak{p} = 1$ and that contradict our assumption. Therefore, $\mathfrak{p} = 1$. Now, we will prove that $\xi_{\mathfrak{R}}$ has a fixed point. We now examine that $\{\beta_{\omega}\}$ is a Cauchy $\mathfrak{R}$-sequence and we adopt conflicting that $\{\beta_{\omega}\}$ is not a Cauchy $\mathfrak{R}$-sequence. So there exists $\varepsilon > 0$ and we take two subsequences of $\{\beta_{\omega}\}$, which are $\{\beta_{\omega_k}\}$ and $\{\beta_{\sigma_k}\}$ with $\omega_k > \sigma_k > k$ for which,

$$\mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k}) \geq \varepsilon,\ \mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k-1}) < \varepsilon \ and\ \mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_{k-1}}) < \varepsilon. \qquad (3.9)$$

Using the triangular inequality, we derive

$$\varepsilon \leq \mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k}) \leq \mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k-1}) + \mathfrak{D}(\beta_{\sigma_k-1}, \beta_{\sigma_k}). \qquad (3.10)$$

Letting $k \to \infty$ in (3.11), using (3.10) and (3.6), we get

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k}) = \varepsilon. \qquad (3.11)$$

By using (3.1), there exists a positive integer k_0 such that

$$\mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k}) > 0 \qquad \forall \omega_k > \sigma_k > k \geq k_0,$$

$$\Theta(\varepsilon) \leq \Theta\left(\mathfrak{D}(\beta_{\omega_k+1}, \beta_{\sigma_k+1})\right) = \Theta\left(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega_k}, \xi_{\mathfrak{R}}\beta_{\sigma_k})\right)$$

$$\leq \alpha_{\mathfrak{R}}(\beta_{\omega_k}, \beta_{\sigma_k})\Theta\left(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega_k}, \xi_{\mathfrak{R}}\beta_{\sigma_k})\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}(\beta_{\omega_k}, \beta_{\sigma_k})\right)\right]^k = [\Theta(\varepsilon)]^k.$$

This is a contradiction, since $k \in (0,1)$, $\{\beta_{\omega}\}$ is a Cauchy $\mathfrak{R}$-sequence. Thus, there is $e \in \mathfrak{B}$ such that $\beta_{\omega} \to e$ as $\omega \to \infty$, then

$$e = lim_{\omega\to\infty}\beta_{\omega+1} = lim_{\omega\to\infty}\xi_{\mathfrak{R}}\beta_{\omega} = \xi_{\mathfrak{R}}e.$$

So e is a fixed point of $\xi_{\mathfrak{R}}$.

Now, assume that $\xi_{\mathfrak{R}}$ has two fixed points say $e \neq f$. Hence,

$$\mathfrak{D}(e, f) = \mathfrak{D}(\xi_{\mathfrak{R}}e, \xi_{\mathfrak{R}}f) \leq \alpha_{\mathfrak{R}}(e, f)\Theta\left(\mathfrak{D}(\xi_{\mathfrak{R}}e, \xi_{\mathfrak{R}}f)\right) \leq \left[\Theta\left(\mathfrak{D}(e, f)\right)\right]^k < \Theta\left(\mathfrak{D}(e, f)\right).$$

Which leads us to a contradiction. Thus, the fixed point is unique as required.

Theorem 3.7. Let $(\mathfrak{B}, \mathfrak{R}, \mathfrak{D})$ be an $\mathfrak{R}$-regular $\mathfrak{R}$-CMS and $\xi_{\mathfrak{R}}$ be a self-mapping, $\mathfrak{R}$-preserving and $\alpha_{\mathfrak{R}}: \mathfrak{B} \times \mathfrak{B} \to [0, \infty)$ be a function. Assume that the below situations hold:

(i) Assume that there exist $\Theta \in \Omega$ and $k \in (0,1)$ such that for all $\beta, \delta \in \mathfrak{B}$ with $\beta \Re \delta$,

$$\mathfrak{D}(\xi_{\Re}\beta, \xi_{\Re}\delta) \neq 0 \Rightarrow \alpha_{\Re}(\beta, \delta)\Theta(\mathfrak{D}(\xi\beta, \xi\delta) \leq [\Theta(\mathfrak{D}(\beta, \delta)]^{k}. \tag{3.12}$$

(ii) $\xi_{\Re}$ is $\alpha_{\Re}$-admissible.

(iii) There exists $\beta_0 \in \mathfrak{B}$ such that $\beta_0 \Re \xi_{\Re} \beta_0$ and $\alpha_{\Re}(\beta_0, \xi_{\Re}\beta_0) \geq 1$.

(iv) If $\{\beta_\omega\}$ is an $\Re$-sequence in $\mathfrak{B}$ such that $\alpha(\beta_\omega, \beta_{\omega+1}) \geq 1$ for all ω and $\beta_\omega \to \beta$, then there exists an $\Re$-subsequence $\{\beta_{\omega_k}\}$ of $\{\beta_\omega\}$ such that $\alpha(\beta_{\omega_k}, \beta) \geq 1$ for all k.

Then $\xi_{\Re}$ has a fixed point $e \in \mathfrak{B}$. Moreover, if for every two fixed points e, f of $\xi_{\Re}$ we have $\alpha_{\Re}(e,f) \geq 1$, then the fixed point is unique.

Proof. Let $\beta_0 \in \mathfrak{B}$ such that $(\forall \delta \in \mathfrak{B},\ \beta_0 \Re \delta)$ or $(\forall \delta \in \mathfrak{B},\ \delta \Re \beta_0)$. By condition (iii), $\beta_0 \Re \xi_{\Re} \beta_0$ or $\xi_{\Re}\beta_0 \Re \beta_0$. For $\omega \in \mathbb{N}$, consider $\beta_\omega = {\xi_{\Re}}^{\omega}\beta_o$. Assume $\xi_{\Re}\beta_\omega = \xi_{\Re}\beta_{\omega+1}$ for some $\omega \in \mathbb{N}$. Then β_ω is a fixed point of $\xi_{\Re}$ and the proof is completed. Let $\xi_{\Re}\beta_\omega \neq \xi_{\Re}\beta_{\omega+1}$ for all $\omega \in \mathbb{N}$. Since $\xi_{\Re}$ is $\Re$-preserving, $(\xi_{\Re}\beta_\omega \Re \xi_{\Re}\beta_{\omega+1})$ or $(\xi_{\Re}\beta_{\omega+1} \Re \xi_{\Re}\beta_\omega)$. Hence, $\{\beta_\omega\}$ is an $\Re$-sequence. By condition (i),

$$\alpha_{\Re}(\beta_\omega, \xi_{\Re}\beta_\omega) = \alpha_{\Re}(\beta_\omega, \beta_{\omega+1}) \geq 1 \qquad \forall \omega \in \mathbb{N}. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned} 1 < \Theta\mathfrak{D}\big((\beta_\omega, \beta_{\omega+1})\big) &= \Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_\omega)\big) \\ &\leq \alpha_{\Re}(\beta_{\omega-1}, \beta_\omega)\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_\omega)\big) \\ &\leq \big[\Theta\big(\mathfrak{D}(\beta_{\omega-1}, \beta_\omega)\big)\big]^{k}. \end{aligned} \tag{3.14}$$

By (Θ_1), we have

$$\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) < \mathfrak{D}(\beta_{\omega-1}, \beta_\omega).$$

Hence, the sequence $\{\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\}$ is decreasing and $\{\mathfrak{D}(\beta_\omega, \beta_{\omega+1})\}$ converges to a non-negative real number $r \geq 0$. We have

$$\lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) = r \text{ and } \mathfrak{D}(\beta_\omega, \beta_{\omega+1}) \geq r. \tag{3.15}$$

Then we prove that $r = 0$. Suppose that $r > 0$. Using (Θ_1), (3.14) and (3.15), we get

$$\begin{aligned} 1 < \Theta(r) = \Theta\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) &\leq \big[\Theta\big(\mathfrak{D}(\beta_{\omega-1}, \beta_\omega)\big)\big]^{k} \\ &\leq \cdots \\ &\leq \big[\Theta\big(\mathfrak{D}(\beta_0, \beta_1)\big)\big]^{k^{\omega}} \forall \omega. \in \mathbb{N}. \end{aligned} \tag{3.16}$$

Letting $\omega \to \infty$ in (3.16), we get $\Theta(r) = 1$ and by using (Θ_2) we have $r = 0$ and therefore,

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega, \beta_{\omega+1}) = 0. \tag{3.17}$$

Assume that there are $\omega, \mathfrak{p} \in \mathbb{N}$ such that $\beta_{\omega}, = \beta_{\omega+\mathfrak{p}}$. Then we prove that $\mathfrak{p} = 1$. Assume that $\mathfrak{p} > 1$. By (3.12) and (3.13), we deduce

$$\Theta\big(\mathfrak{D}(\beta_{\omega}, \beta_{\omega+1})\big) = \Theta\left(\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}}, \beta_{\omega+\mathfrak{p}+1}\big)\right) = \Theta\left(\mathfrak{D}\big(\xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-1}, \xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}}\big)\right)$$

$$\leq \alpha_{\mathfrak{R}}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big)\Theta\left(\mathfrak{D}\big(\xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-1}, \xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}}\big)\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big)\right)\right]^{k}. \tag{3.18}$$

Using (Θ_1), we obtain

$$\Theta\big(\mathfrak{D}(\beta_{\omega}\,, \beta_{\omega+1})\big) < \mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big)$$

and by using (3.12), we derive

$$\Theta\left(\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big)\right) = \Theta\left(\mathfrak{D}\big(\xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-2}, \xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-1}\big)\right)$$

$$\leq \alpha_{\mathfrak{R}}\big(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}\big)\Theta\left(\mathfrak{D}\big(\xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-2}, \xi_{\mathfrak{R}}\beta_{\omega+\mathfrak{p}-1}\big)\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}\big)\right)\right]^{k}$$

$$< \left(\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big)\right). \tag{3.19}$$

By (Θ_1),

$$\mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big) < \mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big).$$

Continuing this process, we obtain

$$\mathfrak{D}(\beta_{\omega}, \beta_{\omega+1}) < \mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-1}, \beta_{\omega+\mathfrak{p}}\big) < \mathfrak{D}\big(\beta_{\omega+\mathfrak{p}-2}, \beta_{\omega+\mathfrak{p}-1}\big) < \cdots < \mathfrak{D}(\beta_{\omega}, \beta_{\omega+1}), \tag{3.20}$$

which implies that $\mathfrak{p} = 1$ and that contradict our assumption. Therefore, $\mathfrak{p} = 1$. Now, we will prove that $\xi_{\mathfrak{R}}$ has a fixed point. We now verify that $\{\beta_{\omega}\}$ is a Cauchy $\mathfrak{R}$-sequence. We assume conflicting that $\{\beta_{\omega}\}$ is not a Cauchy $\mathfrak{R}$-sequence. Then there exists $\varepsilon > 0$ and we yield two subsequences of $\{\beta_{\omega}\}$ which are $\{\beta_{\omega_k}\}$ and $\{\beta_{\sigma_k}\}$ with $\omega_k > \sigma_k > k$ for which

$$\mathfrak{D}\big(\beta_{\omega_k}, \beta_{\sigma_k}\big) \geq \varepsilon\mathfrak{D}\big(\beta_{\omega_k}, \beta_{\sigma_k-1}\big) < \varepsilon. \tag{3.21}$$

Using the triangular inequality, we obtain

$$\varepsilon \leq \mathfrak{D}\big(\beta_{\omega_k}, \beta_{\sigma_k}\big) \leq \mathfrak{D}\big(\beta_{\omega_k}, \beta_{\sigma_k-1}\big) + \mathfrak{D}\big(\beta_{\sigma_k-1}, \beta_{\sigma_k}\big). \tag{3.22}$$

Letting $k \to \infty$in (3.22) and using (3.21) and (3.17), we obtain

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_{\omega_k},\beta_{\sigma_k}) = \varepsilon. \tag{3.23}$$

By using (3.12), there exists a positive integer k_0 such that

$$\mathfrak{D}(\beta_{\omega_k},\beta_{\sigma_k}) > 0 \qquad \forall\omega_k > \sigma_k > k \geq k_0.$$

So,

$$\Theta(\varepsilon) \leq \Theta\left(\mathfrak{D}(\beta_{\omega_k+1},\beta_{\sigma_k+1})\right) = \Theta\left(\mathfrak{D}(\xi_{\Re}\beta_{\omega_k},\xi_{\Re}\beta_{\sigma_k})\right)$$

$$\leq \alpha_{\Re}(\beta_{\omega_k},\beta_{\sigma_k})\Theta\left(\mathfrak{D}(\xi_{\Re}\beta_{\omega_k},\xi_{\Re}\beta_{\sigma_k})\right)$$

$$\leq \left[\Theta\left(\mathfrak{D}(\beta_{\omega_k},\beta_{\sigma_k})\right)\right]^k = [\Theta(\varepsilon)]^k,$$

which is a contradiction since $k \in (0,1)$. Thus, $\{\beta_\omega\}$ is a Cauchy $\Re$-sequence. Then there is $e \in \mathfrak{B}$ such that $\beta_\omega \to e$ as $\omega \to \infty$ and let $U = \{\omega \in \mathbb{N}: \xi_{\Re}\beta_\omega = \xi_{\Re}e\}$. Then we get the following two cases.

Case 1. Assume that $U = \infty$. Then there is a subsequence $\{\beta_{\omega_k}\}$ of $\{\beta_\omega\}$ such that $\beta_{\omega_{k+1}} = \xi_{\Re}\beta_{\omega_k} = \xi_{\Re}e,\ \forall k \in \mathbb{N}$. Recall that $\beta_\omega \to e$, so $e = \xi_{\Re}e$.

Case 2. Assume $U < \infty$. Then there is $\omega_0 \in \mathbb{N}$ such that $\xi_{\Re}\beta_\omega \neq \xi_{\Re}e,\ \forall\omega \geq \omega_0$, in particular, $\beta_\omega \neq e$ and $\mathfrak{D}(\beta_\omega, e) > 0$ and also $\mathfrak{D}(\xi_{\Re}\beta_\omega,\xi_{\Re}e) > 0, \forall\omega \geq \omega_0$. Then we know that $(\beta_\omega\Re e)$ or $(e\Re\beta_\omega)\forall\omega \in \mathbb{N}$. So, we have

$$\alpha_{\Re}(\beta_\omega, e) \geq 1 \qquad \forall\omega \geq \omega_0$$

and we get

$$\alpha_{\Re}(\beta_\omega, e)\Theta(\mathfrak{D}(\xi_{\Re}\beta_\omega,\xi_{\Re}e)) \leq \left[\Theta(\mathfrak{D}(\beta_\omega, e))\right]^k \qquad \forall\omega \geq \omega_0.$$

Since

$$lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega, e) = 0,$$

by (Θ_2),

$$\lim_{\omega\to\infty}\Theta(\mathfrak{D}(\xi_{\Re}\beta_\omega,\xi_{\Re}e)) = 1,$$

which implies

$$\lim_{\omega\to\infty}(\mathfrak{D}(\xi_{\Re}\beta_\omega,\xi_{\Re}e)) = 0.$$

Thus, $\xi_{\Re}e = e$. Hence, e is the fixed point of $\xi_{\Re}$. Similarly, to the proof of Theorem 3.6, we can easily deduce that $\xi_{\Re}$ has a unique fixed point.

Theorem 3.8. Let $(\mathfrak{B},\ \Re,\ \mathfrak{D})$ be an $\Re$-CMS and $\xi_{\Re}: \mathfrak{B} \to \mathfrak{B}$ be a self-mapping, $\Re$-preserving, $\Re$-

continuous and $\alpha_{\mathfrak{R}}: \mathfrak{B} \times \mathfrak{B} \to [0, \infty)$ be a function. Assume that there exist $\Theta \in \Omega$ and $k \in (0,1)$ such that

$$\mathfrak{D}(\xi_{\mathfrak{R}}\beta, \xi_{\mathfrak{R}}\delta) \neq 0 \Rightarrow \Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta, \xi_{\mathfrak{R}}\delta)\big) \leq \big[\Theta\big(U(\beta, \delta)\big)\big]^k, \forall \beta, \delta \in \mathfrak{B} \text{ with } \beta\mathfrak{R}\delta \text{ and } k \in (0,1). \tag{3.24}$$

$$U(\beta, \delta) = \max\left\{\mathfrak{D}(\beta, \delta), \mathfrak{D}(\beta, \xi_{\mathfrak{R}}\beta), \mathfrak{D}(\delta, \xi_{\mathfrak{R}}\delta), \frac{\mathfrak{D}(\beta, \xi_{\mathfrak{R}}\beta)\mathfrak{D}(\delta, \xi_{\mathfrak{R}}\delta)}{1+\mathfrak{D}(\beta, \delta)}\right\}. \tag{3.25}$$

(i) $\xi_{\mathfrak{R}}$ is $\alpha_{\mathfrak{R}}$-admissible.
(ii) There exists $\beta_0 \in \mathfrak{B}$ such that $\beta_0\mathfrak{R}\xi_{\mathfrak{R}}\beta_0$ and $\alpha_{\mathfrak{R}}(\beta_0, \xi_{\mathfrak{R}}\beta_0) \geq 1$.

Then $\xi_{\mathfrak{R}}$ has a fixed point $e \in \mathfrak{B}$.

Proof. Let $\beta_0 \in \mathfrak{B}$ such that $(\forall \delta \in \mathfrak{B}\beta_0\mathfrak{R}\delta)$ or $(\forall \delta \in \mathfrak{B}\delta\mathfrak{R}\beta_0)$. By condition (ii), $\beta_0\mathfrak{R}\xi_{\mathfrak{R}}\beta_0$ or $\xi_{\mathfrak{R}}\beta_0\mathfrak{R}\beta_0$. For $\omega \in \mathbb{N}$, consider $\beta_\omega = {\xi_{\mathfrak{R}}}^{\omega}\beta_o$. Assume that $\xi_{\mathfrak{R}}\beta_\omega = \xi_{\mathfrak{R}}\beta_{\omega+1}$ for some $\omega \in \mathbb{N}$. Then β_ω is a fixed point of $\xi_{\mathfrak{R}}$ and the proof is completed. Let $\xi_{\mathfrak{R}}\beta_\omega \neq \xi_{\mathfrak{R}}\beta_{\omega+1}$ for all $\omega \in \mathbb{N}$. Since $\xi_{\mathfrak{R}}$ is $\mathfrak{R}$-preserving, $(\xi_{\mathfrak{R}}\beta_\omega\mathfrak{R}\xi_{\mathfrak{R}}\beta_{\omega+1})$ or $(\xi_{\mathfrak{R}}\beta_{\omega+1}\mathfrak{R}\xi_{\mathfrak{R}}\beta_\omega)$. Hence, $\{\beta_\omega\}$ is an $\mathfrak{R}$-sequence. By condition (i), for all $\omega \in \mathbb{N}$,

$$\alpha_{\mathfrak{R}}(\beta_\omega, \xi_{\mathfrak{R}}\beta_\omega) \geq 1, \qquad \alpha_{\mathfrak{R}}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_\omega) \geq 1.$$

So,

$$\begin{aligned}\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\big) &\leq \alpha_{\mathfrak{R}}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega)\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega)\big) \\ &\leq \big[\Theta\big(U(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega)\big)\big]^k. \end{aligned} \tag{3.26}$$

From (3.25),

$$\begin{aligned} U(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega) &= \max\left\{\begin{matrix}\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega), \mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1}), \mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_\omega), \\ \dfrac{\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1})\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_\omega)}{1+\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega)}\end{matrix}\right\} \\ &= \max\{\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1}), \mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1}), \mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_\omega)\} \\ &= \max\{\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega), \mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\}. \end{aligned} \tag{3.27}$$

If for some $\omega \in \mathbb{N}$,

$$U(\xi_{\mathfrak{R}}\beta_{\omega-1}, \xi_{\mathfrak{R}}\beta_\omega) = \mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1}),$$

then by (3.26)

$$\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\big) \leq \big[\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\big)\big]^k,$$

which implies that

$$\ln\big[\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\big)\big] \leq k\ln\big[\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega, \xi_{\mathfrak{R}}\beta_{\omega+1})\big)\big].$$

This is a contradiction to $k \in (0,1)$. By (3.27), one writes for all $\omega \in \mathbb{N}$,

$$U(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_{\omega}) = \mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_{\omega}),$$

and by (3.26)

$$\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) \leq \big[\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1}, \xi_{\Re}\beta_{\omega})\big)\big]^{k} \leq \big[\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega-2}, \xi_{\Re}\beta_{\omega-1})\big)\big]^{k^2}$$

$$\leq \cdots \leq \big[\Theta\big(\mathfrak{D}(\beta, \xi_{\Re}\beta)\big)\big]^{k^{\omega}}.$$

So, we have

$$1 \leq \Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) \leq \big[\Theta\big(\mathfrak{D}(\beta, \xi_{\Re}\beta)\big)\big]^{k^{\omega}}, \quad \forall \omega \in \mathbb{N}. \tag{3.28}$$

Letting $\omega \to \infty$ in (3.28), we deduce

$$\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) \to 1.$$

Then from (Θ_2),

$$\lim_{\omega\to\infty}\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1}) = 0.$$

By (Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$lim_{\omega\to\infty}\frac{\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) - 1}{[\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})]^{r}} = l.$$

Assume that $l \in (0,\infty)$. In this case, let $u = \frac{l}{2}$. With the help of limit's definition, there exists $\omega_0 \in \mathbb{N}$, such that

$$\left|\frac{\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) - 1}{[\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})]^{r}} - l\right| \leq u, \quad \forall \omega \geq \omega_0.$$

This implies that

$$\frac{\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) - 1}{[\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})]^{r}} \geq l - u = u, \quad \forall \omega \geq \omega_0.$$

Then,

$$\omega[\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})]^{r} \leq B\omega[\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})) - 1], \quad \forall \omega \geq \omega_0,$$

where $B = {}^{1}/_{u}$.

Now, suppose that $l = \infty$ and $u > 0$ is a random positive number. With the help of limit's definition, there exists $\omega_0 \in \mathbb{N}$ such that

$$\left|\frac{\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\big) - 1}{[\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})]^{r}} - l\right| \geq u, \quad \forall \omega \geq \omega_0,$$

which implies

$$\omega[\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1})]^r \le B\omega[\Theta(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1}))-1], \quad \forall\omega\ge\omega_0,$$

where $B = {}^1\!/_u$. In all cases, there exists $B>0$ such that

$$\omega[\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1})]^r \le B\omega[\Theta(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1}))-1], \quad \forall\omega\ge\omega_0,$$

$$lim_{\omega\to\infty}\omega[\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1})]^r = 0. \tag{3.29}$$

So there exists $\omega_1\in\mathbb{N}$ such that

$$\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+1}) \le \frac{1}{\omega^{1/r}} \quad \forall\omega\ge\omega_1. \tag{3.30}$$

We take $\beta_\omega\ne\xi_{\mathfrak{R}}\beta_\sigma$ for every $\omega,\sigma\in\mathbb{N}$ with $\omega\ne\sigma$ and

$$\begin{aligned}&\Theta(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+2}))\\ &\le \alpha_{\mathfrak{R}}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\Theta(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1}))\\ &\le \left[\Theta\big(U(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\big)\right]^k.\end{aligned} \tag{3.31}$$

$$U(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1}) = \max\left\{\begin{matrix}\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1}),\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1}),\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega+1},\xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega+1}),\\ \frac{\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega-1})\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega+1},\xi_{\mathfrak{R}}\xi_{\mathfrak{R}}\beta_{\omega+1})}{1+\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})}\end{matrix}\right\}. \tag{3.32}$$

We know that Θ is non-decreasing, and so we get from (3.31) and (3.32),

$$\Theta(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+2})) \le \left[\max\left\{\begin{matrix}\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\big),\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_\omega)\big),\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega+1},\xi_{\mathfrak{R}}\beta_{\omega+2})\big),\\ \frac{\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_\omega)\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega+1},\xi_{\mathfrak{R}}\beta_{\omega+2})}{1+\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})}\end{matrix}\right\}\right]^k.$$

That is,

$$\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta_{\omega+2})\big) \le \left[\max\left\{\begin{matrix}\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\big),\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_\omega)\big),\\ \Theta\big(\mathfrak{D}(\xi\beta_{\omega+1},\xi\beta_{\omega+2})\big)\end{matrix}\right\}\right]^k. \tag{3.33}$$

Let I be the set of $\omega\in\mathbb{N}$ such that

$$A_\omega = \max\left\{\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\big),\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\,\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_\omega)\big),\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega+1},\xi_{\mathfrak{R}}\beta_{\omega+2})\big)\right\}$$

$$= \Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta_{\omega-1},\xi_{\mathfrak{R}}\beta_{\omega+1})\big).$$

If $|I|<\infty$, then there exists $\omega_3\in\mathbb{N}$ such that for every $\omega\ge\omega_3$,

$$\max\{\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega+1})),\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega})),\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega+1},\xi_{\Re}\beta_{\omega+2}))\}$$

$$=\max\{\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega+1})),\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega+1},\xi_{\Re}\beta_{\omega+2}))\}.$$

In this case, we get from (3.33),

$$\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2}))\leq[\max\{\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega+1})),\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega+1},\xi_{\Re}\beta_{\omega+2}))\}]^{k}.$$

Letting $\omega\to\infty$ in the above inequality and using (3.29), we deduce

$$\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2}))\to 1 \text{ as } \omega\to\infty.$$

If $|I|=\infty$, then we can find a sequence of $\{A_{\omega}\}$ so that

$$A_{\omega}=\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega+1})) \text{ for } \omega \text{ large enough.}$$

In this case, we derive from (3.33),

$$1<\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2}))\leq[\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-1},\xi_{\Re}\beta_{\omega+1}))]^{k}\leq[\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega-2},\xi_{\Re}\beta_{\omega}))]^{k^{2}}$$
$$\leq\cdots\leq[\Theta(\mathfrak{D}(\beta_{0},\xi_{\Re}\beta_{2}))]^{k^{\omega}}$$

for ω large enough.

Letting $\omega\to\infty$, we get

$$\Theta(\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2}))\to 1 \text{ as } \omega\to\infty. \tag{3.34}$$

Using (Θ_{2}), we obtain

$$\lim_{\omega\to\infty}\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2})=0,$$

and by the condition (Θ_{3}), there exists $\omega_{2}\in\mathbb{N}$ such that

$$\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2})\leq\frac{1}{\omega^{1/r}}\qquad\forall\omega>\omega_{2}. \tag{3.35}$$

Let $\omega_{3}=\max\{\omega_{0},\omega_{1}\}$. Then we consider two cases.

Case1. If $\sigma>2$ is odd, then $\sigma=2L+1, L\geq 1$and using (3.30), for all $\omega\geq\omega_{3}$, we get

$$\begin{aligned}\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+\sigma})&\leq\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+1})+\mathfrak{D}(\xi_{\Re}\beta_{\omega},\xi_{\Re}\beta_{\omega+2})\\&\quad+\cdots+\mathfrak{D}(\xi_{\Re}\beta_{\omega+2L},\xi_{\Re}\beta_{\omega+2L+1})\\&\leq\frac{1}{\omega^{1/r}}+\frac{1}{(\omega+1)^{1/r}}+\cdots+\frac{1}{(\omega+2L)^{1/r}}.\end{aligned}$$

Case2. If $\sigma>2$ is even, then $\sigma=2L,\ L\geq 1$and using (3.30) and (3.35) $\forall\omega\geq\omega_{3}$, we get

$$(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+\sigma}) \leq \mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+2}) + \mathfrak{D}(\xi_{\Re}\beta_{\omega+2}, \xi_{\Re}\beta_{\omega+3}) + \cdots + \mathfrak{D}(\xi_{\Re}\beta_{\omega+2L-1}, \xi_{\Re}\beta_{\omega+2L})$$

$$\leq \frac{1}{\omega^{1/r}} + \frac{1}{(\omega+2)^{1/r}} + \cdots + \frac{1}{(\omega+2L-1)^{1/r}} \leq \sum_{i=\omega}^{\infty} \frac{1}{i^{1/r}}.$$

In both cases, we obtain

$$\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+\sigma}) \leq \sum_{i=\omega}^{\infty} \frac{1}{i^{1/r}}, \forall \omega \geq \omega_3 \text{ and } \sigma \geq 1.$$

From the convergence of the series $\sum \frac{1}{i^{1/r}}$ (since $\frac{1}{r} > 1$), we obtain that $\{\xi_{\Re}\beta_{\omega}\}$ is a Cauchy $\Re$-sequence. Since $(\mathfrak{B}, \Re, \mathfrak{D})$ is an $\Re$-CMS, there is $\beta^* \in \mathfrak{B}$ such that $\xi_{\Re}\beta_{\omega} \to \beta^*$ as $\omega \to \infty$ and we can suppose that $\xi_{\Re}\beta^* \neq \beta^*$. Assume that $\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*) > 0$. Using (3.24), we get

$$\Theta\big(\mathfrak{D}(\xi_{\Re}\beta_{\omega+1}, \xi_{\Re}\beta^*)\big) \leq \left[\Theta\big(U(\xi_{\Re}\beta_{\omega}, \beta^*)\big)\right]^{K}, \qquad \forall \omega \in \mathbb{N},$$

where

$$U(\xi_{\Re}\beta_{\omega}, \beta^*) = \max\left\{\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \beta^*), \mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1}), \mathfrak{D}(\beta^*, \xi_{\Re}\beta_{\omega}), \frac{\mathfrak{D}(\xi_{\Re}\beta_{\omega}, \xi_{\Re}\beta_{\omega+1})\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)}{1 + \mathfrak{D}(\xi_{\Re}\beta_{\omega}, \beta^*)}\right\}.$$

Letting $\omega \to \infty$, we obtain

$$\Theta\big(\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)\big) \leq \left[\Theta\big(\mathfrak{D}(\beta^*, \beta^*)\big)\right]^{K} < \Theta\big(\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)\big).$$

Therefore, $\beta^* = \xi_{\Re}\beta^*$. It is a contradiction to the hypothesis that $\xi_{\Re}$ does not have a periodic point. Thus, $\xi_{\Re}$ has a periodic point β^* of period q. Assume that the set of fixed-points of $\xi_{\Re}$ is empty. Then we have $q > 1$ and $\beta^* \neq \xi_{\Re}\beta^*$. Using (1), we deduce

$$\Theta\big(\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)\big) = \Theta\left(\mathfrak{D}\big({\xi_{\Re}}^{q}\beta^*, {\xi_{\Re}}^{q+1}\beta^*\big)\right)$$

$$\leq \alpha_{\Re}\big({\xi_{\Re}}^{q-1}\beta^*, {\xi_{\Re}}^{q}\beta^*\big)\Theta\left(\mathfrak{D}\big({\xi_{\Re}}^{q}\beta^*, {\xi_{\Re}}^{q+1}\beta^*\big)\right)$$

$$\leq \left[\Theta\big(\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)\big)\right]^{k^q} < \Theta\big(\mathfrak{D}(\beta^*, \xi_{\Re}\beta^*)\big).$$

It is a contradiction. Thus, the set of fixed-point of $\xi_{\Re}$ is non-empty, that is, $\xi_{\Re}$ has at least one fixed-point. Now, presume that $u, \beta^* \in \mathfrak{B}$ are two fixed-points of $\xi_{\Re}$ and

$$(u\Re\beta^*) \text{ or } (\beta^*\Re u), \qquad \text{so } (\xi_{\Re}u\Re\xi_{\Re}\beta^*) \text{ or } (\xi_{\Re}\beta^*\Re\xi_{\Re}u).$$

Then

$$\mathfrak{D}(\beta^*, u) = \mathfrak{D}(\xi_{\Re}\beta^*, \xi_{\Re}u) > 0.$$

Using (3.24), we obtain

$$\Theta\big(\mathfrak{D}(\beta^*,u)\big)=\Theta\big(\mathfrak{D}(\xi_{\mathfrak{R}}\beta^*,\xi_{\mathfrak{R}}u)\big)\le\big[\Theta\big(\mathfrak{D}(\beta^*,u)\big)\big]^k<\Theta\big(\mathfrak{D}(\beta^*,u)\big),$$

which is a contradiction. Then $\xi_{\mathfrak{R}}$ has only one fixed point.

Example 3.9. Consider $\mathfrak{B}=(-2,0]$ and

$$\mathfrak{D}(\beta,\delta)=\begin{cases}0, & \text{if } \beta=\delta\\ \max\{\beta,\delta\}, & \text{otherwise.}\end{cases}\qquad \forall\,\beta,\delta\in\mathfrak{B}.$$

Take $\beta\mathfrak{R}\delta\Leftrightarrow\beta+\delta\ge 0$. Then $(\mathfrak{B},\mathfrak{R},\mathfrak{D})$is an $\mathfrak{R}$-MS but it is not a metric space. For this, let $\beta=-1$ and $\delta=-\frac{1}{2}$, then $\mathfrak{D}(\beta,\delta)=\max\left\{-1,-\frac{1}{2}\right\}=-1$, does not belong to $[0,+\infty)$.

Define the function $\alpha_{\mathfrak{R}}:\mathfrak{B}\times\mathfrak{B}\to[0,\infty)$ by

$$\alpha_{\mathfrak{R}}(\beta,\delta)=\begin{cases}1, & \text{if } \beta,\delta\in[0,2]\\ e^{-\min\{\beta,\delta\}} & \text{if } \beta,\delta\in(0,-2)\\ 0, & \text{otherwise.}\end{cases}$$

Define the mapping $\xi_{\mathfrak{R}}:\mathfrak{B}\to\mathfrak{B}$ by

$$\xi_{\mathfrak{R}}(\beta)=\begin{cases}1 & \text{if } \beta\in\left[\frac{-1}{2},\frac{1}{2}\right]\cup\{1\},\\ \dfrac{\min\{1,\beta\}}{1+\max\{1,\beta\}} & \text{if } \beta\in\left(-2,\frac{-1}{2}\right)\cup\left(\frac{1}{2},2\right]\setminus\{1\}.\end{cases}$$

Then $(\mathfrak{B},\mathfrak{R},\mathfrak{D})$ is an $\mathfrak{R}$-CMS, but it is not a CMS. Here, we show that it is not a CMS. For this, assume $\beta_\omega=\frac{1}{\omega}-2$ is a Cauchy sequence, letting limit as $\omega\to+\infty$ then $\{\beta_\omega\}$ converges to -2. Hence, it is not a CMS that is clear from the definition of completeness.

If $\delta\mathfrak{R}\beta\Leftrightarrow\delta+\beta\ge 0$ then it is easy to realize that $\xi_{\mathfrak{R}}\delta\mathfrak{R}\xi_{\mathfrak{R}}\beta\Leftrightarrow\xi_{\mathfrak{R}}\delta+\xi_{\mathfrak{R}}\beta\ge 0$. So, $\xi_{\mathfrak{R}}$ is $\mathfrak{R}$-preserving.

Assume $\{\beta_\omega\}$ is an $\mathfrak{R}$-sequence convergent to β. Then

$$\lim_{\omega\to\infty}\mathfrak{D}(\beta_\omega,\beta)=\lim_{\omega\to\infty}\begin{cases}0, & \text{if } \beta=\delta=0\\ \max\{\beta_\omega,\beta\}, & \text{otherwise}\end{cases}.$$

Then clearly, this implies that

$$\lim_{\omega\to\infty}\mathfrak{D}(\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta)=\lim_{\omega\to\infty}\begin{cases}0, & \text{if } \xi_{\mathfrak{R}}\beta=\xi_{\mathfrak{R}}\delta=0\\ \max\{\xi_{\mathfrak{R}}\beta_\omega,\xi_{\mathfrak{R}}\beta\}, & \text{otherwise}\end{cases}.$$

It shows that $\xi_{\mathfrak{R}}$ is $\mathfrak{R}$-continuous.

Also, $\xi_{\mathfrak{R}}$ is $\alpha_{\mathfrak{R}}$-admissible, but not an α-admissible mapping. Here, we show that it is not α-admissible. For this, assume that $\mathfrak{B}$ is not an $\mathfrak{R}$-set and we take $\beta=-1$ and $\delta=-\frac{1}{2}$. Then,

$$\alpha\left(-1,-\frac{1}{2}\right)=e^{-\min\left\{-1,-\frac{1}{2}\right\}}=e^1>1,$$

and

$$\alpha\left(\xi(-1),\xi\left(-\frac{1}{2}\right)\right)=\alpha\left(-\frac{1}{2},1\right)=0\ngeq 1.$$

Given $\Theta:(0,\infty)\to(1,\infty)$ as $\Theta(t)=e^{t}$.

Note that $\xi_{\Re}$ does not fulfill to be an α-Θ-contraction, but it verifies all the conditions of the $\alpha_{\Re}$-$\Theta_{\Re}$-contraction. Take $\beta=-\frac{3}{2}$ and $\delta=-1$. Then $\alpha\left(-\frac{3}{2},-1\right)=e^{\frac{3}{2}}$. Also,

$$\alpha\left(-\frac{3}{2},-1\right)e^{\left(\mathfrak{D}\left(-\frac{3}{4},-\frac{1}{2}\right)\right)}=(e^{\frac{3}{2}})e^{-\frac{1}{2}}=e\nleq e^{k\,\mathfrak{D}\left(-\frac{3}{2},-1\right)}=e^{-k}.$$

So $\xi_{\Re}$ is not an $\alpha-\Theta$-contraction, but $\xi_{\Re}$ is an $\alpha_{\Re}$-$\Theta_{\Re}$-contraction for each $k\in[\frac{1}{2},1)$. Clearly, if there exists $\beta_0\in\mathfrak{B}$ such that $\beta_0\Re\xi_{\Re}\beta_0$, then $\alpha_{\Re}(\beta_0,\xi_{\Re}\beta_0)\geq 1$. Hence, all conditions of Theorem 3.6 are fulfilled and $\xi_{\Re}$ has a fixed point $e=1$.

4. Application

Within this part, we apply Theorem 3.6 to investigate the existence and uniqueness of a solution of a nonlinear fractional differential equation (see [17]) given by

$$d_{\pi}^{\gamma}\beta(t)=f\big(t,\beta(t)\big)(t\in(0,1),\qquad\gamma\in(1,2]),$$

with boundary conditions

$$\beta(0)=0,\beta'(0)=I\qquad I\in(0,1),$$

where d_{π}^{γ} means the Caputo fractional derivative of order γ, which is given as

$$d_{\pi}^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t(t-s)^{n-\gamma-1}f^{n}(s)ds\qquad(n-1<\gamma<n,n=[\gamma]+1),$$

and $f:[0,1]\times\mathbb{R}\to\mathbb{R}^{+}$ is a continuous function. We consider $\mathfrak{B}=C([0,1],\mathbb{R})$, from $[0,1]$ into $\mathbb{R}$ with supremum $|\beta|=\operatorname{Sup}_{t\in[0,1]}|\beta(t)|$.

The Riemann-Liouville fractional integral of order γ (see [18]) is given by

$$I^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f(s)ds\qquad(\gamma>0).$$

Firstly, we give the reasonable form of a nonlinear fractional differential equation and then inquest the existence of a solution by the fixed-point theorem. Now, we assume the below fractional differential equations

$$d_{\pi}^{\gamma}\beta(t)=f\big(t,\beta(t)\big)(t\in(0,1),\gamma\in(1,2]),\tag{4.1}$$

with the integral boundary conditions

$$\beta(0)=0, \qquad \beta'(0)=I \qquad \big(I\in(0,1)\big),$$

where

i. $f:[0,1]\times\mathbb{R}\to\mathbb{R}^+$ is a continuous function.

ii. $\beta(t):[0,1]\to\mathbb{R}$ is continuous,

so that

$$|f(t,\beta)-f(t,\delta)|\le L|\beta-\delta|,$$

for all $t\in[0,1]$ and for all $\beta,\delta\in\mathfrak{B}$ such that $\beta(t)-\delta(t)\ge 0$, L is a constant with $LЛ<1$ where

$$Л=\frac{1}{\Gamma(\gamma+1)}+\frac{2k^{\gamma+1}\Gamma(\gamma)}{(2-k^2)\Gamma(\gamma+1)}.$$

Here, $\xi_{\mathfrak{R}}$ is $\alpha_{\mathfrak{R}}$-admissible. Also, there exists $\beta_0(t)\in\mathfrak{B}$ such that $\beta_0(t)\mathfrak{R}\xi_{\mathfrak{R}}\beta_0(t)$ and $\alpha_{\mathfrak{R}}\big(\beta_0(t),\xi_{\mathfrak{R}}\beta_0(t)\big)\ge 1$. Then the differential equation (4.1) has a unique solution.

Proof. We take the below $\mathfrak{R}$ relation on $\mathfrak{B}$:

$$\beta(t)\mathfrak{R}\delta(t) \text{ iff } \beta(t)+\delta(t)\ge 0 \text{ for all } t\in[0,1].$$

The given function $\mathfrak{D}(\beta,\delta)=\operatorname*{Sup}_{t\in[0,1]}|\beta(t)-\delta(t)|\ \forall\beta,\delta\in\mathfrak{B}$ is an $\mathfrak{R}$-CM. We define a mapping $\xi_{\mathfrak{R}}:\mathfrak{B}\to\mathfrak{B}$ by

$$\xi_{\mathfrak{R}}\beta(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f\big(s,\beta(s)\big)ds+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\Big(\int_0^s(s-m)^{\gamma-1}f\big(m,\beta(m)\big)dm\Big)ds, \quad (4.2)$$

for all $t\in[0,1]$. Equation (4.1) has a solution a function $\beta\in\mathfrak{B}$ iff $\beta(t)=\xi_{\mathfrak{R}}\beta(t)$ for all $t\in[0,1]$. For the purpose to check the existence of a fixed point of $\xi_{\mathfrak{R}}$, we are going to examine that $\xi_{\mathfrak{R}}$ is $\mathfrak{R}$-preserving, an $\mathfrak{R}$-contraction and $\mathfrak{R}$-continuous.

Let for all $t\in[0,1]$ so that $\beta(t)\mathfrak{R}\delta(t)$, which means that $\beta(t)+\delta(t)\ge 0$, and clearly from Eq (4.2),

$$\xi_{\mathfrak{R}}\beta(t)+\xi_{\mathfrak{R}}\delta(t)\ge 0.$$

This implies that

$$\xi_{\mathfrak{R}}\beta(t)\mathfrak{R}\xi_{\mathfrak{R}}\delta(t).$$

Hence, $\xi_{\mathfrak{R}}$ is $\mathfrak{R}$-preserving. For all $t\in[0,1]$ and $\beta(t)\mathfrak{R}\delta(t)$, we get

$$\xi_{\mathfrak{R}}\beta(t)-\xi_{\mathfrak{R}}\delta(t)$$

$$=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f\big(s,\beta(s)\big)ds+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\left(\int_0^s(s-m)^{\gamma-1}f\big(m,\beta(m)\big)dm\right)ds$$

$$-\Big[\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f\big(s,\delta(s)\big)ds+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\Big(\int_0^s(s-m)^{\gamma-1}f\big(m,\delta(m)\big)dm\Big)ds\Big].$$

Next, we show that $\xi_{\mathfrak{R}}$ is an $\mathfrak{R}$-contraction. For $t\in[0,1]$ so that $\beta(t)\mathfrak{R}\delta(t)$, we obtain

$$|\xi_{\Re}\beta(t)-\xi_{\Re}\delta(t)|$$

$$=\left|\begin{array}{l}\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f\big(s,\beta(s)\big)ds+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\left(\int_0^s(s-m)^{\gamma-1}f\big(m,\beta(m)\big)dm\right)ds-\\ \frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f\big(s,\delta(s)\big)ds+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\left(\int_0^s(s-m)^{\gamma-1}f\big(m,\delta(m)\big)dm\right)ds\end{array}\right|$$

$$\leq\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\big|f\big(s,\beta(s)\big)-f\big(s,\delta(s)\big)\big|ds$$

$$+\frac{2t}{(2-k^2)\Gamma(\gamma)}\int_0^k\left(\int_0^s(s-m)^{\gamma-1}\big|f\big(m,\beta(m)\big)-f\big(m,\delta(m)\big)\big|dm\right)ds$$

$$\leq\frac{L|\beta-\delta|}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}ds+\frac{2L|\beta-\delta|}{\Gamma(\gamma)}\int_0^k\left(\int_0^s(s-m)^{\gamma-1}dm\right)ds$$

$$\leq\frac{L|\beta-\delta|}{\Gamma(\gamma+1)}+\frac{2k^{\gamma+1}L|\beta+\delta|\Gamma(\gamma)}{(2-k^2)\Gamma(\gamma+2)}$$

$$\leq L|\beta-\delta|\left(\frac{1}{\Gamma(\gamma+1)}+\frac{2k^{\gamma+1}\Gamma(\gamma)}{(2-k^2)\Gamma(\gamma+2)}\right)=LЛ|\beta-\delta|.$$

From the fact $LЛ<1$. Let us take $\Theta(t)=e^{te^t},\forall t>0$. Then

$$\alpha_{\Re}\big(\beta(t),\delta(t)\big)\Theta(d(\xi_{\Re}\beta,\xi_{\Re}\delta)$$

$$=\alpha_{\Re}\big(\beta(t),\delta(t)\big)e^{(d(\xi_{\Re}\beta,\xi_{\Re}\delta))e^{d(\xi_{\Re}\beta,\xi_{\Re}\delta)}}$$

$$\leq\alpha_{\Re}\big(\beta(t),\delta(t)\big)e^{(LЛd(\beta,\delta))e^{LЛd(\beta,\delta)}}$$

$$\leq\alpha_{\Re}\big(\beta(t),\delta(t)\big)e^{(kd(\beta,\delta))e^{kd(\beta,\delta)}}$$

$$\leq\left[e^{(d(\beta,\delta))e^{d(\beta,\delta)}}\right]^k=[\Theta(d(\beta,\delta)]^k,$$

where $k=LЛ$ and $k\in(0,1)$. This implies that $\xi_{\Re}$ is an $\Re$-contraction.

Suppose $\{\beta_n\}$ is an $\Re$-sequence in $\mathfrak{B}$ such that $\{\beta_n\}$ converge to $\beta\in\mathfrak{B}$. Because $\xi_{\Re}$ is $\Re$-preserving, $\{\beta_n\}$ is an $\Re$-sequence for each $n\in\mathbb{N}$. Because $\xi_{\Re}$ is an $\Re$-contraction, we have

$$\alpha_{\Re}\big(\beta(t),\delta(t)\big)\Theta\left(d\big(\xi_{\Re}\beta_n(t),\xi_{\Re}\beta(t)\big)\right)\leq[\Theta(d\big(\beta_n(t),\beta(t)\big)]^k.$$

As $\lim_{n\to\infty}d\big(\beta_n(t),\beta(t)\big)=0$ for all $\tau>0$, then it is clear that

$$\lim_{n\to\infty}d\big(\xi_{\Re}\beta_n(t),\xi_{\Re}\beta(t)\big)=0.$$

Hence, $\xi_{\Re}$ is $\Re$-continuous. Thus, all circumstances of Theorem 3.6 are fulfilled. This implies that $\beta(t)$ is the fixed point of $\xi_{\Re}$.

5. Conclusions

In this manuscript, the notion of the concept of $\alpha_{\Re}$-$\Theta_{\Re}$-contractions is introduced and some fixed-point results are proved in the sense of $\Re$-CMSs by using an $\alpha_{\Re}$-$\Theta_{\Re}$-contraction. Some constructive examples and applications to the fractional differential equation are also imparted. This work can also be extended in the sense of $\Re$-extended metric spaces, $\Re$-controlled metric spaces, $\Re$-double controlled metric spaces, $\Re$-triple controlled metric spaces, and many other structures.

Conflict of interest

The authors declare that they have no competing interests regarding the publication of this paper.

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