Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16851–16868.
DOI: 10.3934/math.2022925
Received: 09 May 2022
Revised: 28 June 2022
Accepted: 04 July 2022
Published: 14 July 2022

Research article

Some sharp Sobolev inequalities on $BV(\mathbb{R}^n)$

Jin Dai* **and Shuang Mou**

School of Mathematics and Statistics, Shaanxi Normal University, Xi'an, 710119, China

* **Correspondence:** Email: daijin@snnu.edu.cn.

Abstract: In this paper, some sharp Sobolev inequalities on $BV(\mathbb{R}^n)$, the space of functions of bounded variation on $\mathbb{R}^n$, $n \geq 2$, are deduced through the L_p Brunn-Minkowski theory. We will prove that these inequalities can all imply the sharp Sobolev inequality on $BV(\mathbb{R}^n)$.

Keywords: Sobolev inequalities; functions of bounded variation; optimal constants
Mathematics Subject Classification: 46B20, 46E35, 52A21

1. Introduction

The sharp Sobolev inequality on the Sobolev space $W^{1,1}(\mathbb{R}^n)$, $n \geq 2$, going back to [10, 11, 30], states that

$$n\omega_n^{1/n}\|f\|_{1^*} \leq \||\nabla f|\|_1,$$

where $1^* = n/(n-1)$, $|\nabla f|$ is the Euclidean norm of the weak gradient of f, $\|f\|_p$ is the usual L^p norm of f in $\mathbb{R}^n$ and ω_k is the volume enclosed by the unit sphere S^{k-1} in $\mathbb{R}^k$. It is one of the fundamental inequalities in many branches of analysis and geometry. So far, the Sobolev inequality and relatives of the Sobolev inequality have been investigated intensively (see, for example, [1–8, 17, 21, 24, 27, 31, 37]).

Cianchi-Talenti [6, 31] established an extended version of the Sobolev inequality on $BV(\mathbb{R}^n)$, and the equality is actually attained. The relevant inequality states that if $f \in BV(\mathbb{R}^n)$, then we have

$$n\omega_n^{1/n}\|f\|_{1^*} \leq \|Df\|, \tag{1.1}$$

where $\|Df\|$ is the total variation of f and equality holds if and only if f is a multiple of the characteristic function of some ball.

In this work, for $p > 1$, the family of sharp Sobolev inequalities is established, which reads as follows:

$$\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}|u\cdot\sigma_f|^p d|Df|\right)^{-\frac{n}{p}} du \leq \left(\frac{1}{n}\right)^{n-1}\left(\frac{1}{c_{n-2,p}}\right)^{\frac{n}{p}}\|f\|_{1^*}^{-n}\|Df\|^{\frac{n(p-1)}{p}}, \tag{1.2}$$

for $f \in BV(\mathbb{R}^n)$. Here the symbol $\cdot$ is the standard Euclidean scalar product, du is the standard spherical Lebesgue measure, the vector valued Radon measure Df is the weak gradient of f, $|Df|$ is the variation measure of f and σ_f is the Radon-Nikodym derivative of Df with respect to $|Df|$ (cf. Section 3). On the other hand, through the L_p Brunn-Minkowski theory, we prove that *these inequalities can all imply the sharp Sobolev inequality* (1.1). In fact, this is a direct consequence of the Hölder inequality and the L_p Cauchy surface area (cf. Section 5).

In [38], Zhang proved the affine Sobolev-Zhang inequality on $C_c^1(\mathbb{R}^n)$, the space of C^1 functions of compact support on $\mathbb{R}^n$, $n \geq 2$, which states

$$\frac{1}{n}\int_{S^{n-1}} \|u \cdot \nabla f\|_1^{-n} du \leq \left(\frac{\omega_n}{2\omega_{n-1}}\right)^n \|f\|_{1^*}^{-n}. \tag{1.3}$$

While Zhang showed that the inequality (1.3) is stronger than the sharp Sobolev inequality on $C_c^1(\mathbb{R}^n)$ and is equivalent to the Petty projection inequality, which is a famous affine inequality in convex geometry and directly implies the isoperimetric inequality for convex bodies.

The inequality (1.3) is sharp, although equality is not attained on $C_c^1(\mathbb{R}^n)$ unless $f = 0$ a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$ (simply write $f = 0$ for this in the paper). But the characteristic functions of ellipsoids can be considered to be the virtual extremals. In [33], Wang proved an extended version of the affine Sobolev-Zhang inequality on the space of functions of bounded variation on $\mathbb{R}^n$, $BV(\mathbb{R}^n)$, and these characteristic functions turn into actual extremals. The extended inequality states that for $f \in BV(\mathbb{R}^n)$,

$$\frac{1}{n}\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \sigma_f| d|Df|\right)^{-n} du \leq \left(\frac{\omega_n}{2\omega_{n-1}}\right)^n \|f\|_{1^*}^{-n}. \tag{1.4}$$

It is proved that equality holds if and only if $f = \lambda 1_E$ for some $\lambda \in \mathbb{R}$ and some ellipsoid $E \subset \mathbb{R}^n$. Here 1_A denotes the characteristic function of $A \subset \mathbb{R}^n$.

Analogously, the extended affine Sobolev-Zhang inequality is stronger than the sharp Sobolev inequality on $BV(\mathbb{R}^n)$. And the inequality (1.4) is $GL(n)$ invariant while the Sobolev inequality is $O(n)$ invariant, where $GL(n)$ and $O(n)$ denote the general linear group and the orthogonal transformation group on $\mathbb{R}^n$, respectively. Particularly, Wang deduced the Petty projection inequality for sets of finite perimeter, which directly implies the isoperimetric inequality for sets of finite perimeter (see [33]).

In [27], Lutwak, Yang and Zhang had extended (1.3) to sharp affine L_p Sobolev inequalities on Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 < p < n$. The Sobolev affine energy of f is defined as

$$\mathcal{E}_p(f) = \left(\frac{1}{n}\int_{S^{n-1}} \|u \cdot \nabla f\|_p^{-n} du\right)^{-\frac{1}{n}},$$

for $1 < p < n$. About the related research of the energy $\mathcal{E}_p$, for example, please see [14, 15, 20, 22, 23].

If μ is an outer measure on $\mathbb{R}^n$, $f : \mathbb{R}^n \to [0, \infty]$ is μ-measurable and $E \subset \mathbb{R}^n$ is a μ-measurable set with $\mu(E) < \infty$, then we set

$$\fint_E f d\mu = \frac{1}{\mu(E)}\int_E f d\mu.$$

Now assuming $f \neq 0$, we can rewrite the inequality (1.4) as

$$\frac{1}{n}\int_{S^{n-1}} \left(\fint_{\mathbb{R}^n} |u \cdot \sigma_f| d|Df|\right)^{-n} du \leq \left(\frac{\omega_n}{2\omega_{n-1}}\right)^n \left(\frac{\|Df\|}{\|f\|_{1^*}}\right)^n. \tag{1.5}$$

For $p > 1$, using the Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u \cdot \sigma_f| d|Df| \le \left(\int_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \right)^{\frac{1}{p}},$$

where equality holds if and only if $|u \cdot \sigma_f| = constant$. Then (1.5) implies

$$\frac{1}{n} \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \right)^{-\frac{n}{p}} du \le \left(\frac{\omega_n}{2\omega_{n-1}} \right)^n \left(\frac{\|Df\|}{\|f\|_{1^*}} \right)^n. \tag{1.6}$$

But the inequality (1.6) is not sharp. The equality in (1.6) is never attained. In fact, $\lambda 1_E$ is the only extremal in (1.5) for any $\lambda \in \mathbb{R}$ and any ellipsoid E, while $|u \cdot \sigma_{\lambda 1_E}|$ is never equal to some constant indentically.

So, in this paper, we refine the constant $(\omega_n/(2\omega_{n-1}))^n$ in the inequality (1.6). We prove the sharp form of (1.6) (or the inequality (1.2)) in the following.

Theorem 1.1. *Let $f \in BV(\mathbb{R}^n)$, $f \neq 0$ and $p > 1$. Then*

$$\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \right)^{-\frac{n}{p}} du \le \left(\frac{1}{n} \right)^{n-1} \left(\frac{1}{c_{n-2,p}} \right)^{\frac{n}{p}} \left(\frac{\|Df\|}{\|f\|_{1^*}} \right)^n, \tag{1.7}$$

where equality holds whenever $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}$ and some ball $B \subset \mathbb{R}^n$. Moreover, for $p \neq n$, if the equality holds in (1.7), then $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}$ and some ball $B \subset \mathbb{R}^n$.

Here

$$c_{n-2,p} = \frac{\omega_{n+p-2}}{\omega_2 \omega_{n-2} \omega_{p-1}}, \tag{1.8}$$

$\omega_s = \pi^{s/2}/\Gamma(1 + s/2)$ for $s \ge 0$ and $\Gamma(\cdot)$ is the Gamma function. Note that if $p \to 1^+$ in (1.7), we can get the affine Sobolev inequality (1.5).

Like the extended affine Sobolev-Zhang inequality, the sharp inequality (1.7) can imply the sharp Sobolev inequality on $BV(\mathbb{R}^n)$ (cf. Section 5). However, these inequalities are only $O(n)$ invariant rather than $G(n)$ invariant.

Throughout the paper f is not equal to 0, a.e., with respect to the Lebesgue measure on $\mathbb{R}^n$ unless we give the particular remark.

In this paper, the main tool is the L_p Brunn-Minkowski theory in convex geometry. We will use the method of the *convexification* which has been used in [29, 33, 38]. For example, in [29], Lutwak-Yang-Zhang associated with each Sobolev function $f \in W^{1,1}(\mathbb{R}^n)$ an origin-symmetric convex body (compact convex set with non-empty interior) $\langle f \rangle$ by using the even Minkowski problem, which reads as follows: Given $f \in W^{1,1}(\mathbb{R}^n)$, there exists a unique origin-symmetric convex body $\langle f \rangle$ such that

$$\int_{S^{n-1}} g(u) dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\nabla f(x)) dx \tag{1.9}$$

for all even continuous functions g on $\mathbb{R}^n$ that are positively homogeneous of degree 1. Here $S(\langle f \rangle, \cdot)$ is the Alexandrov-Fenchel-Jessen surface area measure of $\langle f \rangle$ (cf. Section 2). We call $\langle f \rangle$ the Lutwak-Yang-Zhang (LYZ) body of f and call $\Phi : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_e^n$ (see below for its definition), $\Phi(f) = \langle f \rangle$, the

LYZ operator. It is shown in [33] that the LYZ operator can be extended to $BV(\mathbb{R}^n)$: Given $f \in BV(\mathbb{R}^n)$, there exists a unique origin-symmetric convex body $\langle f \rangle$ such that

$$\int_{S^{n-1}} g(u) dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\sigma_f) d|Df| \tag{1.10}$$

for all even continuous functions g on $\mathbb{R}^n$ that are positively homogeneous of degree 1.

In recent years, the LYZ operator $\langle \cdot \rangle$ has been studied and used widely in convex geometry and other mathematical areas. For example, Ludwig [25] established that each continuous affinely covariant Blaschke valuation on $W^{1,1}(\mathbb{R}^n)$ is the LYZ operator $\langle \cdot \rangle$. For more related information, please see [16, 34].

We state two key steps in order to prove Theorem 1.1. On the one hand, we will make use of the LYZ operator on $BV(\mathbb{R}^n)$ and the even L_p Minkowski problem. We associate each orgin-symmetric convex body K with the another orgin-symmetric convex body: Given $p > 1$, $p \neq n$ and $K \in \mathcal{K}_e^n$, the class of all orgin-symmetric convex bodies (cf. Section 2), there exists a unique convex body $\bar{K} \in \mathcal{K}_e^n$ such that

$$\int_{S^{n-1}} g(u) dS(K, u) = \int_{S^{n-1}} g(u) dS_p(\bar{K}, u) \tag{1.11}$$

for all continuous functions g on S^{n-1}. Here $S_p(\bar{K}, \cdot)$ is L_p surface area measure of $\bar{K}$. In fact, it is the obvious result by applying the even L_p Minkowski problem (cf. Section 4). Moreover, $\Psi : \mathcal{K}_e^n \to \mathcal{K}_e^n$, $\Psi(K) = \bar{K}$, is a bijection. On the other hand, the L_p projection body is introduced, which is the analogue of the projection body, for convex bodies. The L_p Petty projection inequality is used, which is the analogue of the Petty projection inequality, for convex bodies (cf. Section 5).

It follows from the Hölder inequality and the Cauchy surface area formula that the affine Sobolev-Zhang inequality implies the sharp Sobolev inequality. Similarly, the inequality (1.7) is stronger than the sharp Sobolev inequality on $BV(\mathbb{R}^n)$ by the Hölder inequality and the L_p Cauchy surface area formula (cf. Section 5).

2. Preliminaries from convex geometry

Our setting will be n-dimensional Euclidean space $\mathbb{R}^n$ where $n \geq 2$. The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. The standard Euclidean norm of the vector $x \in \mathbb{R}^n$ is denoted by $|x|$. The closed ball with center x and radius r is denoted by $B_r(x)$, and we write S^{n-1} for the topological boundary of the Euclidean unit ball $B_1(0)$. Let k be a non-negative integer. By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and $\mathcal{H}^n$ is equal to the n-dimensional Lebesgue measure. The scalar multiple of a set $A \subset \mathbb{R}^n$ is defined by

$$\lambda A = \{\lambda a : a \in A\}$$

for real numbers λ. We write $-A$ for $(-1)A$.

In this section, we collect some notations and basic facts about convex bodies (see, e.g., [12,13,32]).

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The class of convex bodies is denoted by $\mathcal{K}^n$. The class of convex bodies containing the origin in their interiors is denoted by $\mathcal{K}_o^n$. The set K is called origin-symmetric if $K = -K$. Let $\mathcal{K}_e^n$ denote the class of origin-symmetric convex bodies in $\mathbb{R}^n$. The set K is called symmetric if some translation of K is origin-symmetric. Each non-empty compact convex set K is uniquely determined by its support function h_K, defined by

$$h_K(x) = \sup\{x \cdot y : y \in K\}$$

for all $x \in \mathbb{R}^n$. Note that h_K is positively homogeneous of degree 1 and subadditive. Conversely, each function with these two properties is the support function of a unique compact convex set. For $K \in \mathcal{K}_e^n$, it is obviously ture that the support function of K is even, that is, $h_K(x) = h_K(-x)$ for all $x \in \mathbb{R}^n$.

Let K be a convex body and $\nu : \partial K \to S^{n-1}$ the generalized Gauss map (ν is set-valued), where ∂K is the topological boundary of K. Note that $\nu(x)$ is the set of all outer unit normal vectors at boundary point x of K. For each Borel set $\omega \subset S^{n-1}$, the inverse spherical image $\nu^{-1}(\omega)$ of ω is the set of all boundary points of K which have an outer unit normal vector belonging to the set ω. It is easy to check that $\nu^{-1}(\omega)$ is measurable. Associated with each convex body K a Borel measure $S(K, \cdot)$ on S^{n-1} called the surface area measure of K, is defined by

$$S(K, \omega) = \mathcal{H}^{n-1}(\nu^{-1}(\omega)),$$

for each Borel set $\omega \subset S^{n-1}$.

The mixed volume $V_1(K, L)$ of two convex bodies K and L is defined by

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(K, u). \tag{2.1}$$

In this paper, V and S stand for the volume functional and the surface area functional on $\mathcal{K}^n$, respectively. It is easy to see both

$$V_1(K, K) = V(K) = \mathcal{H}^n(K)$$

and

$$nV_1(K, B_1(0)) = S(K) = \mathcal{H}^{n-1}(\partial K).$$

A fundamental inequality which will be used is the first Minkowski inequality.

Lemma 2.1. *([13, p. 101]) If $K, L \in \mathcal{K}^n$, then*

$$V_1(K, L) \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}, \tag{2.2}$$

where equality holds if and only if K and L are positive homothetic.

Here K and L are positive homothetic, if there exist $\lambda > 0$ and $x \in \mathbb{R}^n$ such that

$$L = \lambda K + x = \{\lambda y + x : y \in K\}.$$

For $K \in \mathcal{K}_o^n$, the polar body K° of K is defined by

$$K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

A formula of the volume of the polar body K° is that

$$V(K^\circ) = \frac{1}{n} \int_{S^{n-1}} h_K^{-n} du. \tag{2.3}$$

The projection body of $K \in \mathcal{K}^n$ is the convex body whose support function is defined by

$$h_{\Pi K}(x) = \frac{1}{2}\int_{S^{n-1}} |u \cdot x| dS(K,u), \quad x \in \mathbb{R}^n.$$

Note that

$$h_{\Pi K}(v) = \mathcal{H}^{n-1}(K|v^{\perp}),$$

where $v \in S^{n-1}$, $K|v^{\perp}$ is orthogonal projection of K onto the linear subspace orthogonal to v and $\mathcal{H}^{n-1}(K|v^{\perp})$ is the volume of $K|v^{\perp}$ in the $(n-1)$-dimensional linear subspace. For convenience, $(\Pi K)^{\circ}$ is denoted by $\Pi^{\circ} K$.

Next, some notations of the L_p Brunn-Minkowski theory are introduced.

Given $K \in \mathcal{K}_o^n$ and $p \in \mathbb{R}$, a Borel measure $S_p(K,\cdot)$ on S^{n-1} called the L_p surface area measure of K, is defined by

$$S_p(K,\omega) = h_K^{1-p} S(K,\omega)$$

for each Borel set $\omega \subset S^{n-1}$. Obviously, $S_1(K,\cdot) = S(K,\cdot)$. We denote the total measure $S_p(K, S^{n-1})$ of the L_p surface area measure of K by $S_p(K)$.

For $p \geq 1$, the L_p mixed volume $V_p(K,L)$ of two convex bodies $K, L \in \mathcal{K}_o^n$ is defined by

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(u) dS_p(K,u). \tag{2.4}$$

Note that, particularly, $V_p(K,K) = V(K)$.

For $p \geq 1$, the L_p projection body of $K \in \mathcal{K}_o^n$ is the convex body whose support function is defined by

$$h^p_{\Pi_p K}(x) = \frac{1}{c_{n-2,p} n\omega_n}\int_{S^{n-1}} |u \cdot x|^p dS_p(K,u), \quad x \in \mathbb{R}^n.$$

Here $c_{n-2,p}$ is consistent with (1.8). The normalization of $\Pi_p K$ is such that $\Pi_p B_1(0) = B_1(0)$; therefore, $\Pi_1 = \omega_{n-1}^{-1}\Pi \neq \Pi$. Similarly, we write $\Pi_p^{\circ} K$ for $(\Pi_p K)^{\circ}$.

3. The space $BV(\mathbb{R}^n)$

In this section, we review some basic notations and facts about functions of bounded variation on $\mathbb{R}^n$ (see [9]).

Throughout this paper, $C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ stands for the class of the compactly supported continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $L^p(\mathbb{R}^n)$ contains all Lebesgue measurable functions f with

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty.$$

Definition 3.1. *A function $f \in L^1(\mathbb{R}^n)$ has bounded variation if*

$$\sup\Big\{\int_{\mathbb{R}^n} f \operatorname{div}\phi dx : \phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), |\phi| \leq 1\Big\} < \infty.$$

We write

$$BV(\mathbb{R}^n)$$

to denote the space of functions of bounded variation.

Here div denotes the divergence operator.

Theorem 3.2. *([9, p. 194, Structure Theorem for BV functions]) Assume that $f \in BV(\mathbb{R}^n)$. Then there exist a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function*

$$\sigma_f : \mathbb{R}^n \to \mathbb{R}^n$$

such that

(1) $|\sigma_f(x)| = 1$ μ-a.e., and

(2) for all $\phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f \operatorname{div} \phi dx = -\int_{\mathbb{R}^n} \phi \cdot \sigma_f d\mu.$$

Hence, we will write $|Df|$ for the measure μ, and $Df := \sigma_f |Df|$. While

$$\sigma_f(x) = \lim_{r \to 0^+} \frac{Df(B_r(x))}{|Df|(B_r(x))}$$

for $x \in \mathbb{R}^n$ a.e., with respect to $|Df|$.

For example, each Sobolev function on $\mathbb{R}^n$ has bounded variation, that is,

$$W^{1,1}(\mathbb{R}^n) \subset BV(\mathbb{R}^n).$$

4. The Minkowski problem and the LYZ operator

If μ is the surface area measure $S(K, \cdot)$ of the convex body $K \in \mathcal{K}^n$, then

$$\int_{S^{n-1}} u d\mu(u) = o, \tag{4.1}$$

where o is the origin. And it is clear that

$$\mu \text{ cannot be concentrated on any great subsphere of } S^{n-1}. \tag{4.2}$$

Conversely, the Minkowski problem was proposed, which reads as follows:

Find necessary and sufficient conditions on a finite Borel measure μ on the unit sphere S^{n-1} so that μ is the surface area measure of a convex body $K \in \mathcal{K}^n$.

Fortunately, (4.1) and (4.2) are also sufficient in order that μ be the surface area measure of a convex body $K \in \mathcal{K}^n$. Moreover, we have uniqueness, that is, the K is unique up to translations.

In recent years, for various Minkowski problems, a large number of related results have been obtained (see, e.g., [18, 19, 28, 35, 36, 39]).

In [33], the author introduced the extended LYZ body $\langle f \rangle$ of $f \in BV(\mathbb{R}^n)$ by using the solution of the classical Minkowski problem.

Definition 4.1. *([33]) For $f \in BV(\mathbb{R}^n)$ which is not 0, the LYZ body is defined to be the origin-symmetric convex body $\langle f \rangle$, such that*

$$\int_{S^{n-1}} g(u) dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\sigma_f) d|Df| \tag{4.3}$$

for every $g : \mathbb{R}^n \to \mathbb{R}$ that is even, continuous and positively 1-homogeneous.

We can write a convex body K using its characteristic function 1_K. And the characteristic function 1_K has bounded variation. Then, we have $\mathcal{K}^n \subset BV(\mathbb{R}^n)$ in this sense. Hence, the LYZ operator $\Phi : BV(\mathbb{R}^n) \to \mathcal{K}_e^n$, $\Phi(f) = \langle f \rangle$ can be regarded as a operator from $BV(\mathbb{R}^n)$ to $BV(\mathbb{R}^n)$.

Now, we collect some properties of the LYZ operator on $BV(\mathbb{R}^n)$.

Lemma 4.2. *([33]) For $t \in \mathbb{R}^+$ and $f \in BV(\mathbb{R}^n)$, we have $\langle tf \rangle = t^{\frac{1}{n-1}} \langle f \rangle$, $\langle -f \rangle = \langle f \rangle$.*

Lemma 4.3. *([33]) Given $K \in \mathcal{K}_e^n$, if $K + x$ is the translation of K with respect to $x \in \mathbb{R}^n$, we have $\langle 1_{K+x} \rangle = K$.*

Lemma 4.4. *([33]) Let $f \in BV(\mathbb{R}^n)$. Then*

$$V(\langle f \rangle)^{\frac{n-1}{n}} \geq \|f\|_{\frac{n}{n-1}}, \tag{4.4}$$

where there is equality if and only if f is a multiple of the characteristic function of a symmetric convex body.

We can rewrite (4.4) as

$$\|1_{\langle f \rangle}\|_{\frac{n}{n-1}} \geq \|f\|_{\frac{n}{n-1}}.$$

Thus, Lemma 4.4 guarantees that the $L^{\frac{n}{n-1}}$ norm of $f \in BV(\mathbb{R}^n)$ is increased by the LYZ operator, while the LYZ operator keeps

$$\int_{\mathbb{R}^n} g(\sigma_f) d|Df| = \int_{\mathbb{R}^n} g(\sigma_{1_{\langle f \rangle}}) d|D1_{\langle f \rangle}| \tag{4.5}$$

for every $g : \mathbb{R}^n \to \mathbb{R}$ that is even, continuous and positively 1-homogeneous.

Next, we introduce the L_p Minkowski problem. Using the solution of the even L_p Minkowski problem, we can get a bijective operator $\Psi : \mathcal{K}_e^n \to \mathcal{K}_e^n$ (see Theorem 4.7).

Analogously, the L_p Minkowski problem is that for $p \in \mathbb{R}$ find necessary and sufficient conditions on a finite Borel measure μ on the unit sphere S^{n-1} so that μ is the L_p surface area measure of a convex body $K \in \mathcal{K}^n$.

Particularly, the even L_p Minkowski problem is of great interest, which reads as follows:

Find necessary and sufficient conditions on an even finite Borel measure μ on the unit sphere S^{n-1} so that μ is the even L_p surface area measure of an origin-symmetric convex body K.

Here, the measure μ is even if $\mu(-\omega) = \mu(\omega)$ for each Borel set $\omega \subset S^{n-1}$. For $p > 0$, the problem has been solved. We have the following result.

Lemma 4.5. *([32, p. 498]) Let $p > 0$ and $p \neq n$. Let μ be an even finite Borel measure on S^{n-1} which cannot be concentrated on any great subsphere of S^{n-1}. Then, there exists $K \in \mathcal{K}_e^n$ such that $S_p(K, \cdot) = \mu$.*

Moreover, for suitable p, the solution of the even L_p Minkowski problem is unique, which holds true by

Lemma 4.6. *([32, p. 494]) Let $p > 1$, $p \neq n$, and $K, L \in \mathcal{K}_o^n$. If*

$$S_p(K, \cdot) = S_p(L, \cdot),$$

then $K = L$.

Using the solution of the even L_p Minkowski problem, we associate each orgin-symmetric convex body K with the another orgin-symmetric convex body, which reads as follows:

Theorem 4.7. *Given $p > 1$, $p \neq n$ and $K \in \mathcal{K}_e^n$, there exists a unique convex body $\bar{K} \in \mathcal{K}_e^n$ such that*

$$\int_{S^{n-1}} g(u)dS(K,u) = \int_{S^{n-1}} g(u)dS_p(\bar{K},u) \tag{4.6}$$

for all continuous functions g on S^{n-1}. Moreover, the operator $\Psi : \mathcal{K}_e^n \to \mathcal{K}_e^n$, $\Psi(K) = \bar{K}$ is a bijection.

Proof. Let $K \in \mathcal{K}_e^n$, $p > 1$ and $p \neq n$. The surface area measure $S(K,\cdot)$ of K satisfies the necessary and sufficient conditions of the even L_p Minkowski problem. Using Lemmas 4.5 and 4.6, then there exists a unique origin-symmetric convex body $\bar{K}$ such that

$$S(K,\cdot) = S_p(\bar{K},\cdot).$$

Thus, we have

$$\int_{S^{n-1}} g(u)dS(K,u) = \int_{S^{n-1}} g(u)dS_p(\bar{K},u)$$

for all continuous functions g on S^{n-1}.

If $K, K_1 \in \mathcal{K}_e^n$ and $\Psi(K) = \Psi(K_1) = \bar{K}$, then we have

$$S(K,\cdot) = S_p(\bar{K},\cdot) = S(K_1,\cdot).$$

By the uniqueness the solution of the Minkowski problem, $K = K_1$ up to some translation. Since K and K_1 are origin-symmetric, $K = K_1$. Thus Ψ is bijective. □

Now, we deduce some properties of the operator Ψ.

It follows from the homogeneity of Hausdorff measures and the homogeneity of support functions that $S_p(\lambda K,\cdot) = \lambda^{n-p} S_p(K,\cdot)$ for $p \geq 1$, $\lambda > 0$ and $K \in \mathcal{K}_o^n$.

The following lemma states that $\Psi(B)$ is also a ball with center at o for a ball $B \in \mathcal{K}_e^n$.

Lemma 4.8. *Let $K \in \mathcal{K}_e^n$. If $K = rB_1(0)$ for some $r > 0$, then $\Psi(K) = \bar{K} = r^{\frac{n-1}{n-p}} B_1(0)$.*

Proof. By the definition and the homogeneity of the surface area measure, we have

$$\int_{S^{n-1}} g(u)dS(rB_1(0),u) = \int_{S^{n-1}} g(u)r^{n-1}dS(B_1(0),u)$$

for all continuous functions g on S^{n-1}. Since $h_{B_1(0)} = 1$, $S_p(L,\cdot) = h_L^{1-p} S(L,\cdot)$ for some $L \in \mathcal{K}_o^n$ and $S_p(\lambda K,\cdot) = \lambda^{n-p} S_p(K,\cdot)$, we have

$$\begin{aligned}
\int_{S^{n-1}} g(u)r^{n-1}dS(B_1(0),u) &= \int_{S^{n-1}} g(u)r^{n-1}dS_p(B_1(0),u) \\
&= \int_{S^{n-1}} g(u)(r^{\frac{n-1}{n-p}})^{n-p}dS_p(B_1(0),u) \\
&= \int_{S^{n-1}} g(u)dS_p(r^{\frac{n-1}{n-p}} B_1(0),u).
\end{aligned}$$

Then, we have

$$\int_{S^{n-1}} g(u)dS(rB_1(0),u) = \int_{S^{n-1}} g(u)dS_p(r^{\frac{n-1}{n-p}}B_1(0),u)$$

for all continuous functions g on S^{n-1}. That is $\Psi(rB_1(0)) = r^{\frac{n-1}{n-p}}B_1(0)$. □

Since the LYZ operator increases the $L^{\frac{n}{n-1}}$ norm of a function which has bounded variation, similarly, the Ψ operator changes the volume of a origin-symmetric convex body. We have the following inequality.

Theorem 4.9. *Let $K \in \mathcal{K}_e^n$, $p > 1$, $p \neq n$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$V(\bar{K})^{\frac{n-p}{np}} \geq V(K)^{\frac{n-1}{n}}\Big(\frac{S(K)}{n}\Big)^{-\frac{1}{q}}, \tag{4.7}$$

where equality holds if and only if K is a Euclidean ball with center at o.

Remark 4.10. *Note that $S(K) = S_p(\bar{K})$. The inequality (4.7) can be rewritten as*

$$\Big(\frac{nV(\bar{K})}{S_p(\bar{K})}\Big)^{\frac{n-p}{np}} \geq \Big(\frac{nV(K)}{S(K)}\Big)^{\frac{n-1}{n}}.$$

Proof. Let $K, L \in \mathcal{K}_e^n$, $p > 1$ and $p \neq n$. By the Hölder inequality, we have

$$\int_{S^{n-1}} h_L(u)dS(K,u) \leq \Big(\int_{S^{n-1}} h_L^p(u)dS(K,u)\Big)^{1/p} S(K)^{1/q}, \tag{4.8}$$

where equality holds if and only if $L = B_r(0)$ for some $r > 0$. Since

$$\int_{S^{n-1}} h_L^p(u)dS(K,u) = \int_{S^{n-1}} h_L^p(u)dS_p(\bar{K},u),$$

by Theorem 4.7, we have

$$\int_{S^{n-1}} h_L(u)dS(K,u) \leq \Big(\int_{S^{n-1}} h_L^p(u)dS_p(\bar{K},u)\Big)^{1/p} S(K)^{1/q}. \tag{4.9}$$

Setting $L = \bar{K}$ in the inequality (4.9), we get

$$nV_1(K,\bar{K}) \leq (nV(\bar{K}))^{1/p} S(K)^{1/q}. \tag{4.10}$$

It follows from the equality condition in (4.8) and Lemma 4.8 that the equality holds in (4.10) if and only if $K = B_r(0)$ for some $r > 0$. Using Lemma 2.1, we have

$$\begin{aligned} (nV(\bar{K}))^{\frac{1}{p}} S(K)^{\frac{1}{q}} &\geq nV_1(K,\bar{K}) \\ &\geq nV(K)^{\frac{n-1}{n}} V(\bar{K})^{\frac{1}{n}}. \end{aligned} \tag{4.11}$$

Then

$$V(\bar{K})^{\frac{n-p}{np}} \geq V(K)^{\frac{n-1}{n}}\Big(\frac{S(K)}{n}\Big)^{-\frac{1}{q}}.$$

If $K = rB_1(0)$ for some $r > 0$, then $\bar{K} = r^{\frac{n-1}{n-p}} B_1(0)$. By the equality condition in (4.10) and the equality condition in Lemma 2.1, we have

$$V(\bar{K})^{\frac{n-p}{np}} = V(K)^{\frac{n-1}{n}} \Big(\frac{S(K)}{n}\Big)^{-\frac{1}{q}}.$$

If $K \neq rB_1(0)$ for each $r > 0$, the equality does not hold in the inequality (4.10). Then

$$V(\bar{K})^{\frac{n-p}{np}} > V(K)^{\frac{n-1}{n}} \Big(\frac{S(K)}{n}\Big)^{-\frac{1}{q}}.$$

Thus, we have

$$V(\bar{K})^{\frac{n-p}{np}} = V(K)^{\frac{n-1}{n}} \Big(\frac{S(K)}{n}\Big)^{-\frac{1}{q}},$$

if and only if K is a ball with center at o. □

As a consequence of Lemma 4.4 and Theorem 4.9, we obtain:

Corollary 4.11. *Let $f \in BV(\mathbb{R}^n)$, $f \neq 0$, $p > 1$, $p \neq n$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f\|_{\frac{n}{n-1}} \Big(\frac{\|Df\|}{n}\Big)^{-\frac{1}{q}} \leq V(\langle \bar{f} \rangle)^{\frac{n-p}{np}},$$

where equality holds if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B.

Proof. Let $f \in BV(\mathbb{R}^n)$, $f \neq 0$, $p > 1$, $p \neq n$ and $\frac{1}{p} + \frac{1}{q} = 1$. Using Theorem 4.9 and Lemma 4.4, we have

$$\begin{aligned} V(\langle \bar{f} \rangle)^{\frac{n-p}{np}} &\geq V(\langle f \rangle)^{\frac{n-1}{n}} \Big(\frac{S(\langle f \rangle)}{n}\Big)^{-\frac{1}{q}} \\ &\geq \|f\|_{\frac{n}{n-1}} \Big(\frac{S(\langle f \rangle)}{n}\Big)^{-\frac{1}{q}}. \end{aligned}$$

It follows from $\|Df\| = S(\langle f \rangle)$ that

$$\|f\|_{\frac{n}{n-1}} \Big(\frac{\|Df\|}{n}\Big)^{-\frac{1}{q}} \leq V(\langle \bar{f} \rangle)^{\frac{n-p}{np}}.$$

Let $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B. Using the equality conditions in Theorem 4.9 and Lemma 4.4, we get

$$\|f\|_{\frac{n}{n-1}} \Big(\frac{\|Df\|}{n}\Big)^{-\frac{1}{q}} = V(\langle \bar{f} \rangle)^{\frac{n-p}{np}}.$$

Now, assume that

$$\|f\|_{\frac{n}{n-1}} \Big(\frac{\|Df\|}{n}\Big)^{-\frac{1}{q}} = V(\langle \bar{f} \rangle)^{\frac{n-p}{np}}$$

for $f \in BV(\mathbb{R}^n)$ and $f \neq 0$. Thus

$$\|f\|_{\frac{n}{n-1}} \Big(\frac{\|Df\|}{n}\Big)^{-\frac{1}{q}} = V(\langle f \rangle)^{\frac{n-1}{n}} \Big(\frac{S(\langle f \rangle)}{n}\Big)^{-\frac{1}{q}} = V(\langle \bar{f} \rangle)^{\frac{n-p}{np}}.$$

By Theorem 4.9 and Lemma 4.4, we have $\langle f \rangle = rB_1(0)$ for some $r > 0$ and $f = \lambda 1_{K+x}$ for some $K \in \mathcal{K}_e^n$ and some $x \in \mathbb{R}^n$. Using Lemmas 4.2 and 4.3, we get

$$rB_1(0) = \langle f \rangle = \langle \lambda 1_{K+x} \rangle = \lambda^{\frac{1}{n-1}} \langle 1_{K+x} \rangle = \lambda^{\frac{1}{n-1}} K.$$

Thus, $K = (r/\lambda^{\frac{1}{n-1}})B_1(0)$. Then, we get $f = \lambda 1_B$ for the ball $B = (r/\lambda^{\frac{1}{n-1}})B_1(0) + x$. □

Remark 4.12. *Although the $L^{\frac{n}{n-1}}$ norm of $f \in BV(\mathbb{R}^n)$ is variant under the operator LYZ and the operator Ψ, the variation measure $|Df|$ satisfies some invariance. We have*

$$\int_{\mathbb{R}^n} g(\sigma_f)d|Df| = \int_{S^{n-1}} g(u)dS(\langle f\rangle, u) = \int_{S^{n-1}} g(u)dS_p(\langle \bar{f}\rangle, u), \tag{4.12}$$

for every $g : \mathbb{R}^n \to \mathbb{R}$ that is even, continuous and positively 1-homogeneous.

5. Sharp Sobolev inequalities on $BV(\mathbb{R}^n)$

In this section, we prove Theorem 1.1 and show that the inequality (1.7) implies the Sobolev inequality on $BV(\mathbb{R}^n)$.

We will use the L_p Petty projection inequality, which is proved by Lutwak-Yang-Zhang in [26]. It is the L_p analogue of the Petty projection inequality.

Theorem 5.1. *([26] or [32, p. 575]) For $1 < p < \infty$ and for $K \in \mathcal{K}_o^n$,*

$$V(K)^{(n-p)/p}V(\Pi_p^\circ K) \leq \omega_n^{n/p},$$

with equality if and only if K is an origin-symmetric ellipsoid.

Now, we prove Theorem 1.1.

Proof. Let $f \in BV(\mathbb{R}^n)$, $f \neq 0$, $p > 1$, $p \neq n$ and $\frac{1}{p} + \frac{1}{q} = 1$. By (2.3) and Remark 4.12, we calculate

$$\begin{aligned}
V(\Pi_p^\circ\langle\bar{f}\rangle) &= \frac{1}{n}\int_{S^{n-1}} h_{\Pi_p\langle\bar{f}\rangle}^{-n}du \\
&= \frac{1}{n}\int_{S^{n-1}}\Big(\frac{1}{c_{n-2,p}n\omega_n}\int_{S^{n-1}}|u\cdot v|^p dS_p(\langle\bar{f}\rangle, v)\Big)^{-\frac{n}{p}}du \\
&= \frac{1}{n}\int_{S^{n-1}}\Big(\frac{1}{c_{n-2,p}n\omega_n}\int_{\mathbb{R}^n}|u\cdot\sigma_f|^p d|Df|\Big)^{-\frac{n}{p}}du \\
&= (c_{n-2,p}n\omega_n)^{\frac{n}{p}}\frac{1}{n}\int_{S^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\sigma_f|^p d|Df|\Big)^{-\frac{n}{p}}du.
\end{aligned} \tag{5.1}$$

Using Corollary 4.11, we have

$$\|f\|_{\frac{n}{n-1}}^n\Big(\frac{\|Df\|}{n}\Big)^{-\frac{n}{q}} \leq V(\langle\bar{f}\rangle)^{\frac{n-p}{p}}, \tag{5.2}$$

where equality holds if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B. Thus, it follows from (5.2) and (5.1) that

$$\int_{S^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\sigma_f|^p d|Df|\Big)^{-\frac{n}{p}}du \leq V(\langle\bar{f}\rangle)^{\frac{n-p}{p}}V(\Pi_p^\circ\langle\bar{f}\rangle)\|f\|_{\frac{n}{n-1}}^{-n}\|Df\|^{\frac{n}{q}}\Big(\frac{1}{n}\Big)^{n-1}\Big(\frac{1}{c_{n-2,p}\omega_n}\Big)^{\frac{n}{p}},$$

i.e.,

$$\int_{S^{n-1}}\Big(⨍_{\mathbb{R}^n}|u\cdot\sigma_f|^p d|Df|\Big)^{-\frac{n}{p}}du \leq \frac{V(\langle\bar{f}\rangle)^{\frac{n-p}{p}}V(\Pi_p^\circ\langle\bar{f}\rangle)}{\omega_n^{n/p}}\Big(\frac{\|Df\|}{\|f\|_{1^*}}\Big)^n\Big(\frac{1}{n}\Big)^{n-1}\Big(\frac{1}{c_{n-2,p}}\Big)^{\frac{n}{p}}, \tag{5.3}$$

where $1^* = n/(n-1)$ and equality holds if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B. Applying Theorem 5.1, we see

$$\frac{V(\langle \bar{f} \rangle)^{\frac{n-p}{p}} V(\Pi_p^\circ \langle \bar{f} \rangle)}{\omega_n^{n/p}} \leq 1, \tag{5.4}$$

with equality if and only if $\langle \bar{f} \rangle$ is an origin-symmetric ellipsoid. Then, it directly follows from (5.4) and (5.3) that

$$\int_{S^{n-1}} \left(\fint_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \right)^{-\frac{n}{p}} du \leq \left(\frac{1}{n}\right)^{n-1} \left(\frac{1}{c_{n-2,p}}\right)^{\frac{n}{p}} \left(\frac{\|Df\|}{\|f\|_{1^*}}\right)^n. \tag{5.5}$$

If $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B, then $\langle \bar{f} \rangle = \lambda^{\frac{1}{n-p}} r^{\frac{n-1}{n-p}} B_1(0)$, where r is the radius of B. Hence, the equality holds in (5.4), so the equality holds in (5.5).

If the equality holds in (5.5), then the equality holds in (5.3) and (5.2). So $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B.

In summary, the equality holds in (5.5) if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B.

Now, let $p \to n$ in the inequality (5.5). It follows from the dominated convergence theorem that

$$\int_{S^{n-1}} \left(\fint_{\mathbb{R}^n} |u \cdot \sigma_f|^n d|Df| \right)^{-1} du \leq \left(\frac{1}{n}\right)^{n-1} \left(\frac{1}{c_{n-2,n}}\right) \left(\frac{\|Df\|}{\|f\|_{1^*}}\right)^n. \tag{5.6}$$

Since

$$\int_{S^{n-1}} \left(\fint_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \right)^{-\frac{n}{p}} du = \left(\frac{1}{n}\right)^{n-1} \left(\frac{1}{c_{n-2,p}}\right)^{\frac{n}{p}} \left(\frac{\|Df\|}{\|f\|_{1^*}}\right)^n$$

with $p \neq n$ and $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball B, we have

$$\int_{S^{n-1}} \left(\fint_{\mathbb{R}^n} |u \cdot \sigma_f|^n d|Df| \right)^{-1} du = \left(\frac{1}{n}\right)^{n-1} \left(\frac{1}{c_{n-2,n}}\right) \left(\frac{\|Df\|}{\|f\|_{1^*}}\right)^n$$

by $p \to n$ for $f = \lambda 1_B$. □

Now, we prove that the inequality (1.7) or (5.5) is stronger than the sharp Sobolev inequality on $BV(\mathbb{R}^n)$.

The sharp Sobolev inequality on $BV(\mathbb{R}^n)$ states that

Corollary 5.2. *Let $f \in BV(\mathbb{R}^n)$. Then*

$$n\omega_n^{1/n} \|f\|_{1^*} \leq \|Df\|, \tag{5.7}$$

where equality holds if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball $B \subset \mathbb{R}^n$.

Firstly, we deduce the L_p Cauchy surface area formula, which is a direct consequence by the following lemma.

Lemma 5.3. *Let $K, L \in \mathcal{K}_o^n$ and $p \geq 1$. Then*

$$V_p(L, \Pi_p K) = V_p(K, \Pi_p L). \tag{5.8}$$

Proof. Using (2.4) and Fubini's theorem, we directly calculate

$$\begin{aligned} V_p(L, \Pi_p K) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_p K}(u) dS_p(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{c_{n-2,p} n\omega_n}\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v) dS_p(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{c_{n-2,p} n\omega_n}\int_{S^{n-1}} |u\cdot v|^p dS_p(L,u) dS_p(K,v) \\ &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_p L}(v) dS_p(K,v) \\ &= V_p(K, \Pi_p L). \end{aligned}$$

□

The L_p Cauchy surface area formula is the following.

Theorem 5.4. *Let $p \geq 1$ and $K \in \mathcal{K}^n_o$. Then*

$$S_p(K) = \frac{1}{c_{n-2,p} n\omega_n}\int_{S^{n-1}} \left(\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v)\right) du.$$

Proof. Let $p \geq 1$ and $K \in \mathcal{K}^n_o$. Setting $L = B_1(0)$ in the equality (5.8), we have

$$V_p(B_1(0), \Pi_p K) = V_p(K, \Pi_p B_1(0)).$$

By $\Pi_p B_1(0) = B_1(0)$, then

$$V_p(B_1(0), \Pi_p K) = V_p(K, B_1(0)).$$

Thus, we can compute

$$\begin{aligned} S_p(K) &= nV_p(K, B_1(0)) \\ &= nV_p(B_1(0), \Pi_p K) \\ &= \int_{S^{n-1}} h^p_{\Pi_p K}(u) dS_p(B_1(0), u) \\ &= \int_{S^{n-1}} h^p_{\Pi_p K}(u) du \\ &= \frac{1}{c_{n-2,p} n\omega_n}\int_{S^{n-1}} \left(\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v)\right) du. \end{aligned}$$

□

Remark 5.5. *Let $p = 1$ in Theorem 5.4 and note that $2\omega_2\omega_{n-2} = n\omega_n$. We get the classical Cauchy surface area formula, which reads as follows: If $K \in \mathcal{K}^n$, then*

$$S(K) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp) du.$$

Now, we prove Corollary 5.2 by using Theorem 1.1 for fixed $p > 1$ with $p \neq n$.

Proof. Let $f \in BV(\mathbb{R}^n)$. If $f = 0$, then the inequality (1.1) holds trivially.

Assume $f \neq 0$ and $p > 1$. From Remark 4.12 and Theorem 5.4, we have

$$\begin{aligned} c_{n-2,p} n\omega_n &= \int_{S^{n-1}} \Big(\frac{1}{S_p(\langle \bar{f} \rangle)} \int_{S^{n-1}} |u \cdot v|^p dS_p(\langle \bar{f} \rangle, v) \Big) du \\ &= \int_{S^{n-1}} \Big(\frac{1}{\|Df\|} \int_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big) du \\ &= \int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big) du. \end{aligned} \tag{5.9}$$

We set $\beta = \frac{n}{n+p}$, that is, $-\frac{p}{n} + \frac{1}{\beta} = 1$. It follows from the Hölder inequality that

$$\int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big) du \geq \Big(\int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big)^{-\frac{n}{p}} du \Big)^{-\frac{p}{n}} (n\omega_n)^{(n+p)/n},$$

where equality holds if and only if

$$⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| = constant,$$

that is, $\Pi_p \langle \bar{f} \rangle$ is a Euclidean ball with center at o. Thus, combining with (5.9), we see that

$$\int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big)^{-\frac{n}{p}} du \geq \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} n\omega_n.$$

Now, using Theorem 1.1, we calculate

$$\begin{aligned} \Big(\frac{1}{n}\Big)^{n-1} \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} \Big(\frac{\|Df\|}{\|f\|_{1^*}} \Big)^n &\geq \int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big)^{-\frac{n}{p}} du \\ &\geq \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} n\omega_n, \end{aligned}$$

that is,

$$\|Df\| \geq n\omega_n^{1/n} \|f\|_{1^*}.$$

If $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball $B \subset \mathbb{R}^n$, then the equality holds in the inequality (1.7) and $\Pi_p \langle \bar{f} \rangle$ is a Euclidean ball with center at o. Thus,

$$\begin{aligned} \Big(\frac{1}{n}\Big)^{n-1} \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} \Big(\frac{\|Df\|}{\|f\|_{1^*}} \Big)^n &= \int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big)^{-\frac{n}{p}} du \\ &= \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} n\omega_n, \end{aligned}$$

i.e.,

$$\|Df\| = n\omega_n^{1/n} \|f\|_{1^*}.$$

If $p \neq n$ and $f \neq \lambda 1_B$ for all $\lambda \in \mathbb{R}^n$ and all Euclidean balls $B \subset \mathbb{R}^n$, then

$$\Big(\frac{1}{n}\Big)^{n-1} \Big(\frac{1}{c_{n-2,p}} \Big)^{\frac{n}{p}} \Big(\frac{\|Df\|}{\|f\|_{1^*}} \Big)^n > \int_{S^{n-1}} \Big(⨍_{\mathbb{R}^n} |u \cdot \sigma_f|^p d|Df| \Big)^{-\frac{n}{p}} du.$$

Thus,

$$\|Df\| > n\omega_n^{1/n}\|f\|_{1^*}.$$

In summary, for $p \neq n$ we have that

$$\|Df\| = n\omega_n^{1/n}\|f\|_{1^*}$$

if and only if $f = \lambda 1_B$ for some $\lambda \in \mathbb{R}^n$ and some Euclidean ball $B \subset \mathbb{R}^n$. □

6. Conclusions

In this work, we establish a family of new Sobolev inequalities on $BV(\mathbb{R}^n)$, and we prove that each one in the family can imply the classical Sobolev inequality with the sharp constant on $BV(\mathbb{R}^n)$ which is one of the most important inequality in analysis. Our approach is the L_p Brunn-Minkowski theory in convex geometry. We use the Lutwak-Yang-Zhang operator so that inequalities of BV functions relate to inequalities of convex bodies. Then, we establish a family of inequalities of convex bodies. As a consequence, we achieve the goal.

Acknowledgments

The work of the authors was supported by the Recruitment Program for Young Professionals of China and the Fundamental Research Funds for the Central Universities (Grant No. GK202101008) and Postgraduate Innovation Team Project of Shaanxi Normal University (No. TD2020008Z).

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

1. T. Aubin, Problèmes isopérimétriques et espaces de Sobolev (French), *J. Differ. Geom.*, **11** (1976), 573–598. https://doi.org/10.4310/jdg/1214433725
2. T. Aubin, Y.Y. Li, On the best Sobolev inequality, *J. Math. Pure. Appl.*, **78** (1999), 353–387. https://doi.org/10.1016/S0021-7824(99)00012-4
3. D. Bakry, M. Ledoux, Lévy-Gromov's isoperimetric inequality for an infinite dimensional diffusion generator, *Invent. Math.*, **123** (1996), 259–281. https://doi.org/10.1007/s002220050026
4. W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, *Ann. Math.*, **138** (1993), 213–242. https://doi.org/10.2307/2946638
5. W. Beckner, M. Pearson, On sharp Sobolev embedding and the logarithmic Sobolev inequality, *Bull. Lond. Math. Soc.*, **30** (1998), 80–84. https://doi.org/10.1112/S0024609397003901
6. A. Cianchi, A quantitative Sobolev inequality in *BV*, *J. Funct. Anal.*, **237** (2006), 466–481. https://doi.org/10.1016/j.jfa.2005.12.008
7. D. Cordero-Erausquin, B. Nazaret, C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities, *Adv. Math.*, **182** (2004), 307–332. https://doi.org/10.1016/S0001-8708(03)00080-X

8. O. Druet, Optimal Sobolev inequalities of arbitrary order on Riemannian compact manifolds, *J. Funct. Anal.*, **159** (1998), 217–242. https://doi.org/10.1006/jfan.1998.3264
9. L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of fuctions*, 2 Eds., CRC Press Taylor and Francis Group, Boca Raton, 2015.
10. H. Federer, W. Fleming, Normal and integral currents, *Ann. Math.*, **72** (1960), 458–520. https://doi.org/10.2307/1970227
11. W. Fleming, R. Rischel, An integral formula for total gradient variation, *Arch. Math.*, **11** (1960), 218–232. https://doi.org/10.1007/BF01236935
12. R. J. Gardner, *Geometric tomography*, Springer-Verlag, Berlin, 2006.
13. P. M. Gruber, *Convex and discrete geometry*, Springer-Verlag, Berlin Heidelberg, 2007.
14. C. Haberl, F. E. Schuster, Asymmetric affine L_p Sobolev inequalities, *J. Funct. Anal.*, **257** (2009), 641–658. https://doi.org/10.1016/j.jfa.2009.04.009
15. C. Haberl, F. E. Schuster, J. Xiao, An asymmetric affine Pólya-Szegö principle, *Math. Ann.*, **352** (2012), 517–542. https://doi.org/10.1007/s00208-011-0640-9
16. Q. Huang, A. Li, Optimal Sobolev norms in the affine class, *J. Math. Anal. Appl.*, **436** (2016), 568–585. http://dx.doi.org/10.1016/j.jmaa.2015.11.063
17. Q. Huang, A. Li, The L_p Gagliardo-Nirenberg-Zhang inequality, *Adv. Appl. Math.*, **113** (2020), 101971. https://doi.org/10.1016/j.aam.2019.101971
18. Y. Huang, E. Lutwak, D. Yang, G. Zhang, Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems, *Acta Math.*, **216** (2016), 325–388. https://doi.org/10.1007/s11511-016-0140-6
19. Y. Huang, E. Lutwak, D. Yang, G. Zhang, The L_p-Aleksandrov problem for L_p-integral curvature, *J. Differ. Geom.*, **110** (2018), 1–29. https://doi.org/10.4310/jdg/1536285625
20. P. Kniefacz, F. E. Schuster, Sharp Sobolev inequalities via projection averages, *J. Geom. Anal.*, **31** (2021), 7436–7454. https://doi.org/10.1007/s12220-020-00544-6
21. Y. Y. Li, M. Zhu, Sharp Sobolev inequalities involving boundary terms, *Geom. Funct. Anal.*, **8** (1998), 59–87. https://doi.org/10.1007/s000390050048
22. Y. Lin, The affine Orlicz Pólya-Szegö principle on $BV(\Omega)$, *Calc. Var. Partial Dif.*, **58** (2019). https://doi.org/10.1007/s00526-019-1622-0
23. Y. Lin, D. Xi, Affine Orlicz Pólya-Szegö principles and their equality cases, *Int. Math. Res. Not.*, **9** (2021), 7159–7204. https://doi.org/10.1093/imrn/rnz061
24. M. Ludwig, J. Xiao, G. Zhang, Sharp convex Lorentz-Sobolev inequalities, *Math. Ann.*, **350** (2011), 169–197. https://doi.org/10.1007/s00208-010-0555-x
25. M. Ludwig, Valuations on Sobolev spaces, *Amer. J. Math.*, **134** (2012), 827–842. https://doi.org/10.1353/ajm.2012.0019
26. E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, *J. Differ. Geom.*, **56** (2000), 111–132. https://doi.org/10.4310/jdg/1090347527
27. E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities, *J. Differ. Geom.*, **62** (2002), 17–38. https://doi.org/10.4310/jdg/1090425527

28. E. Lutwak, D. Yang, G. Zhang, On the L_p-Minkowski problem, *Trans. Amer. Math. Soc.*, **356** (2004), 4359–4370. https://doi.org/10.1090/S0002-9947-03-03403-2

29. E. Lutwak, D. Yang, G. Zhang, Optimal Sobolev norms and L^p Minkowski problem, *Int. Math. Res. Not.*, **2006** (2006), 62987. https://doi.org/10.1155/IMRN/2006/62987

30. V. G. Maz'ya, Classes of domains and imbedding theorems for function spaces, *Dokl. Akad. Nauk. SSSR*, **133** (1960), 527–530.

31. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pur. Appl.*, **110** (1976), 353–372. https://doi.org/10.1007/BF02418013

32. R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, 2 Eds., Cambridge Univ., 2014.

33. T. Wang, The affine Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$, *Adv. Math.*, **230** (2012), 2457–2473. https://doi.org/10.1016/j.aim.2012.04.022

34. T. Wang, Semi-valuations on $BV(\mathbb{R}^n)$, *Indiana Univ. Math. J.*, **63** (2014), 1447–1465. Available from: `http://www.jstor.org/stable/24904282.`

35. T. Wang, On the discrete functional L_p Minkowski problem, *Int. Math. Res. Not.*, **20** (2015), 10563–10585. https://doi.org/10.1093/imrn/rnu256

36. G. Xiong G. J. Xiong, The Orlicz Minkowski problem for the electrostatic p-capacity, *Adv. Appl. Math.*, **137** (2022), 102339. https://doi.org/10.1016/j.aam.2022.102339

37. S. T. Yau, *Sobolev inequality for measure space*, Internat. Press, Cambridge, MA, 1997.

38. G. Zhang, The affine Sobolev inequality, *J. Differ. Geom.*, **53** (1999), 183–202. https://doi.org/10.4310/jdg/1214425451

39. B. Zhu, S. Xing, D. Ye, The dual Orlicz-Minkowski problem, *J. Geom. Anal.*, **28** (2018), 3829–3855. https://doi.org/10.1007/s12220-018-0002-x

AIMS Press