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Research article

Second order evolutionary partial differential variational-like inequalities

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Abstract: This work is dedicated to the study of a second order evolutionary partial differential variational-like inequality in Banach space. We obtain the mild solution of our problem by applying the concept of strongly continuous cosine family of bounded linear operators, fixed point theorem for condensing multi-valued operators and measure of non-compactness. It is proved that the solution set of mixed variational-like inequalities is non-empty, bounded, closed and convex.

Keywords: second order evolution equation; variational-like inequality; cosine family; measure of non-compactness; mild solution
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1. Introduction

Differential variational inequality is a dynamical system that includes variational inequalities and ordinary differential equations. Differential variational inequalities plays an important role for formulating models involving both dynamics and inequality constraints. Aubin and Cellina [3] introduced the concept of differential variational inequality and after that it was studied by Pang and Stewart [25]. The partial differential variational inequalities was studied by Liu, Zeng and Motreanu [15] and shown that the solution set is compact and continuous. There are some obstacles in their work that constraint set necessarily be compact and only local boundary conditions are satisfied. Liu, Migórskii and Zeng [14] relaxed the conditions of [15] and proved the existence of partial differential variational inequality in non-compact setting. Properties of solution set like strong-weak upper semicontinuity and measurability was proved by them.

Differential variational inequalities are application oriented and have several applications in engineering and physical sciences, operation research, etc. In particular, they are applicable in electrical circuits with ideal diodes, economical dynamics, dynamic traffic network, functional problems, differential Nash games, control systems, etc., see for example [1, 2, 16–20, 23, 26, 27, 31].

Evolution equation can be explained as the differential law of the development (evolution) in time of a system. The evolution character of the equation make easier its numerical solution. Variational-like inequality is a generalized form of a variational inequalities and has many applications in operations research, optimization, convex mathematical programming, etc. On the other hand, many problems of engineering and applied sciences can be solved by using second order evolution equation, see for example [5, 6, 9, 10, 12, 13, 22, 28, 30, 32, 33].

2. Preliminaries and formulation of the problem

Throughout the paper, we assume $\widetilde{B_1}$ and $\widetilde{B_2}$ denote separable reflexive Banach spaces and $\widehat{K}(\neq \phi)$ be convex and closed subset of $\widetilde{B_1}$. We define some mapping below, that is,

$$\begin{aligned}
&\widetilde{\mathcal{F}} : [0,\mathcal{T}] \times \widetilde{B_2} \times \widetilde{B_2} \longrightarrow \mathcal{L}\left(\widetilde{B_1}, \widetilde{B_2}\right),\\
&\widetilde{\mathfrak{f}} : [0,\mathcal{T}] \times \widetilde{B_2} \times \widetilde{B_2} \longrightarrow \widetilde{B_2},\\
&\widetilde{\mathfrak{g}} : [0,\mathcal{T}] \times \widetilde{B_2} \times \widetilde{B_2} \longrightarrow \widetilde{B_2},\\
&\widetilde{A} : \widehat{K} \longrightarrow \widetilde{B_1^*},\\
&\eta : \widehat{K} \times \widehat{K} \longrightarrow \widetilde{B_1},\\
&\psi : \widetilde{K} \longrightarrow \mathbb{R} \cup \{+\infty\}, \text{ where } \mathcal{T} > 0.
\end{aligned}$$

Inspired by the above discussed work, in this paper, we introduce and study a second order evolutionary partial differential variational-like inequality in Banach spaces. We mention our problem below:

$$\begin{cases}
\mathfrak{y}''(x) = \widetilde{A}\mathfrak{y}(x) + \widetilde{\mathcal{F}}(x, \mathfrak{y}(x), \mathfrak{y}'(x))\widehat{\mathfrak{u}}(x) + \widetilde{\mathfrak{f}}(x, \mathfrak{y}(x), \mathfrak{y}'(x)), \text{ a.e. } x \in [0,\mathcal{T}],\\
\widehat{\mathfrak{u}}(x) \in Sol(\widetilde{K}, \widetilde{\mathfrak{g}}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) + \widetilde{\mathcal{A}}(\cdot), \psi), \text{ a.e. } x \in [0,\mathcal{T}],\\
\mathfrak{y}(0) = \mathfrak{y}_0, \mathfrak{y}'(0) = y_0.
\end{cases} \tag{2.1}$$

We also consider a variational-like inequality problem of finding $\widehat{\mathfrak{u}} : [0,\mathcal{T}] \to \widehat{K}$ such that

$$\langle \widetilde{\mathfrak{g}}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) + \widetilde{\mathcal{A}}(\widehat{\mathfrak{u}}(x)), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}(x))\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}(x)) \geq 0, \forall\, \widehat{\mathfrak{v}} \in \widehat{K}, \text{ a.e. } x \in [0,\mathcal{T}]. \tag{2.2}$$

The solution set of problem (2.2) is denoted by $Sol[(2.1)]$.

The mild solution of problem (2.1) is described by the following definition.

Definition 2.1. *A pair of function* $(\mathfrak{y}, \widehat{\mathfrak{u}})$ *such that* $\mathfrak{y} \in C^1([0,\mathcal{T}], \widetilde{B_2})$ *and* $\widehat{\mathfrak{u}} : [0,\mathcal{T}] \to \widehat{K}(\subset \widetilde{B_1})$ *measurable, called mild solution of problem* (2.1) *if*

$$\mathfrak{y}(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p, \mathfrak{y}(p), \mathfrak{y}'(p))\widehat{\mathfrak{u}}(p) + \widetilde{\mathfrak{f}}(p, \mathfrak{y}(p), \mathfrak{y}'(p))]dp,$$

where $x \in [0,\mathcal{T}]$ and $\widehat{\mathfrak{u}}(p) \in Sol(\widehat{K}, \widetilde{\mathfrak{g}}(p, \mathfrak{y}(p), \mathfrak{y}'(p)) + \mathcal{A}(\cdot, \psi)$. $R(x)$ will be defined in continuation. Here, $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.), \psi)$ denotes the solution set of mixed variational-like inequality (3.1). *If* $(\mathfrak{y}, \widehat{\mathfrak{u}})$ *is a mild solution of above assumed problem, then $\mathfrak{y}$ is said to be the mild trajectory and $\widehat{\mathfrak{u}}$ is called the variational control trajectory. Here $C^1([0,\mathcal{T}], \widetilde{B_2})$ denotes the Banach space of all continuous differentiable mappings $\mathfrak{y} : [0,\mathcal{T}] \to \widetilde{B_2}$ with norm*

$$\|\mathfrak{y}\|_{C^1} = \max\left\{ \max_{x\in[0,\mathcal{T}]} \|\mathfrak{y}(x)\|, \max_{x\in[0,\mathcal{T}]} \|\mathfrak{y}'(x)\| \right\},$$

and $L(\widetilde{B_2})$ denotes the Banach space of bounded linear operators from $\widetilde{B_2}$ into $\widetilde{B_2}$.

The subsequent part of this paper is organised in this way. In the next section, some definitions and results are defined, which will be used to achieve our goal. In Section 3, an existence result for variational-like inequalities is proved. Also, we have proved that $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.), \psi)$ is nonempty, closed and convex. The upper semicontinuity of the multi-valued mapping $\mathfrak{F} : [0,\mathcal{T}] \times \widetilde{B_2} \times \widetilde{B_2} \to \Pi_{bv}(\widehat{B_1})$ is discussed. In the last section, we have proved that the existence result for the mild solution of second order evolutionary partial differerntial variational-like inequalities under some appropriate conditions.

Let $\widehat{X_1}$ and $\widehat{X_2}$ are topological spaces. We shall use $\Pi(\widehat{X_2})$ to denote the family of all nonempty subsets of X_2, and

$\Pi_c(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is closed}\};$

$\Pi_b(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is bounded}\};$

$\Pi_{bc}(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is bounded and closed}\};$

$\Pi_{cv}(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is closed and convex}\};$

$\Pi_{bv}(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is bounded and convex}\};$

$\Pi_{kv}(\widehat{X_2}) := \{\widehat{D} \in \Pi(\widehat{X_2}) : \widehat{D} \text{ is compact and convex}\}.$

One parameter family $Q(x)$, where x is real number, of bounded linear operators from a Banach space $\widehat{B_2}$ into itself is called a strongly continuous cosine family if and only if

(1) $Q(x + p) + Q(x - p) = 2C(x)C(p), \forall x, p \in \mathbb{R}$,

(2) $Q(0) = I$, (I is the identity operator in $\widehat{B_2}$),

(3) $Q(x)w$ is continuous in x on $\mathbb{R}$ for every fixed $w \in \widehat{B_2}$.

We associate with the strongly continuous cosine family $Q(x)$ in $\widehat{B_2}$ the strongly continuous sine family $R(x)$, such that

$$R(x)W = \int_0^x Q(p)wdp, \quad w \in \widehat{B_2}, \ x \in \mathbb{R},$$

and the two sets

$$\begin{aligned} E_1 &= \{w \in \widehat{B_2} : Q(x)u \text{ is one time continuously differentiable in } x \text{ on } \mathbb{R}\}, \\ E_2 &= \{w \in \widehat{B_2} : Q(x)w \text{ is two times continuously differentiable in } x \text{ on } \mathbb{R}\}. \end{aligned}$$

The operator $A : D(A) \subset \widehat{B_2} \to \widehat{B_2}$ is the infinitesimal generator of a strongly continuous cosine family $Q(x)$, $x \in \mathbb{R}$ defined by $A(\mathfrak{y}) = d^2/dx^2 Q(0)\mathfrak{y}$ with $D(A) = E_2$.

Proposition 2.1. *[29] Let $Q(x),\ x \in \mathbb{R}$ be a strongly continuous cosine family in $\widehat{B_1}$. Then the following hold:*

(i) $Q(x) = Q(-x),\ \forall\ x \in \mathbb{R},$

(ii) $Q(p), R(p), Q(x),$ *and* $R(x)$ *commute* $\forall\ x,\ p \in \mathbb{R},$

(iii) $R(x+p) + Q(x-p) = 2R(x)Q(p),\ \forall\ x, p \in \mathbb{R},$

(iv) $R(x+p) = R(x)Q(p) + R(p)C(x),\ \forall\ x,\ p \in \mathbb{R},$

(v) $R(x) = -R(-x),\ \forall\ x \in \mathbb{R}.$

For furthure information related to the properties of the sine and cosine families, see [12,23,27] and references therein.

Definition 2.2. *[21] Let $\widehat{X_1},\ \widehat{X_2}$ are topological spaces. Then the multi-valued mapping $\widehat{F} : \widehat{X_1} \to \Pi(\widehat{X_2})$ is said to be:*

(i) *Upper semicontinuous (u.s.c., in short) at $x \in \widehat{X_1}$, if for each open set $\mathfrak{U} \subset \widehat{X_2}$ with $\widehat{F}(x) \subset \mathfrak{U}$, $\exists$ a neighbourhood $\mathcal{N}(x)$ of x such that*

$$\widehat{F}(\mathcal{N}(x)) := \underset{y\in\mathcal{N}(x)}{\widehat{F}(y)} \subset \mathfrak{U}.$$

If $\widehat{F}$ is u.s.c. $\forall\ x \in \widehat{X_1}$, then $\widehat{F}$ is said to be upper semicontinuous on $\widehat{X_1}$.

(ii) *Lower semicontinuous (l.s.c., in short) at $x \in \widehat{X_1}$ if, for each open set $\mathfrak{U} \subset \widehat{X_2}$ satisfying $\widehat{F} \cap \mathfrak{U} \neq \phi$, $\exists$ a neighbourhood $\mathcal{N}(x)$ of x such that $\widehat{F} \cap \mathfrak{U} \neq \phi\ \forall\ y \in \mathcal{N}(x)$. If $\widehat{F}$ is l.s.c. $\forall\ x \in \widehat{X_1}$, then $\widehat{F}$ is called lower semicontinuous on $\widehat{X_1}$.*

Proposition 2.2. *[21] Let $\widehat{F} : \widehat{X_1} \to \Pi(\widehat{X_2})$ be a multi-valued mapping, where $\widehat{X_1}$, and $\widehat{X_2}$ denote topological vector spaces. Then the following are equivalent:*

(i) *$\widehat{F}$ is upper semicontinuous,*

(ii) *the set*

$$\widehat{F}^{-}(C) = \{x \in \widehat{X_1} : \widehat{F}(x) \cap C \neq \phi\},$$

is closed in $\widehat{X_1}$, for each closed set $C \subset \widehat{X_2}$,

(iii) *the set*

$$\widehat{F}^{+}(C) = \{x \in \widehat{X_1} : \widehat{F}(x) \subset \mathfrak{U}\},$$

is open in $\widehat{X_1}$, for each open set $\mathfrak{U} \subset \widehat{X_2}$.

Proposition 2.3. *[4] Let $\Omega(\neq \phi)$ subset of Banach space $\widehat{X}$. Assume that the multi-valued mapping $\widehat{F} : \Omega \to \Pi(\widehat{X})$ is weakly compact and convex. Then, $\widehat{F}$ is strongly-weakly u.s.c. if and only if $\{x_n\} \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in \widehat{F}(x_n)$ implies $y_n \rightharpoonup y_0 \in \widehat{F}(x_0)$ up to a subsequence.*

Lemma 2.1. *[7] Let $\{x_n\}$ be a sequence such that $x_n \rightharpoonup \bar{x}$ in a normed space V. Then there is a sequence of combinations $\{y_n\}$ such that*

$$y_n = \sum_{i=n}^{\infty} \lambda_i x_i, \quad \sum_{i=n}^{\infty} \lambda_i = 1 \text{ and } \lambda_i \geq 0, \quad 1 \leq i \leq \infty,$$

which converges to $\bar{x}$ in norm.

Now we define the measurability of a multi-valued mapping, which is needed in the proof of existence of solution of second order evolutionary partial differential variational-like inequality problem (2.1).

Definition 2.3. *[11, 21]*

(*i*) *A multi-valued mapping $\widehat{F} : I \to \Pi(\widehat{X})$ is called measurable if for each open subset $\mathfrak{U} \subset \widehat{X}$ the set $\widehat{F}^{+}(\mathfrak{U})$ is measurable in $\mathbb{R}$.*
And

(*ii*) *the multi-valued mapping $\widehat{F} : I \to \Pi_{bc}(\widehat{X})$ is called strongly measurable if $\exists$ a sequence $\{\widehat{F}_n\}_{n=1}^{\infty}$ of step set-valued mappings such that*

$$\widehat{\mathcal{H}}(\widehat{F}(t), \widehat{F}_n(t)) \to 0, \quad \text{as } n \to \infty, t \in I \text{ a.e.},$$

here $\widehat{X}$ denotes Banach space, I be an interval of real numbers and $\widehat{\mathcal{H}}(.,.)$ denotes the Hausdorff metric on $\Pi_{bc}(\widehat{X})$.

Definition 2.4. *[11, 34] Let $\widehat{X}$ be Banach space and $(\mathbb{F}, \leq)$ be a partial ordered set. A function β : $\Pi_b(\widehat{X}) \to \mathbb{F}$ is called a measure of non compactness (MNC, for short) in $\widehat{X}$ if*

$$\beta(\overline{conv}O) = \beta(O) \quad \text{for every } O \in \Pi_b(\widehat{X}),$$

here $\overline{conv}O$ showing the closure of convex hull of O.

Definition 2.5. *[34] A measure of non compactness β is called*

(*i*) *monotone, if $O_0, O_1 \in \Pi_b(\widehat{X})$ and $O_0 \subseteq O_1$ implies $\beta(O_0) \leq \beta(O_1)$,*

(*ii*) *nonsingular, if $\beta(a \cup O) = \beta(O)$ $\forall$ $a \in \widehat{X}$ and $O \in \Pi_b(\widehat{X})$,*

(*iii*) *invariant with respect to union of compact set, if $\beta(K \cup O) = \beta(O)$ for each relatively compact set $K \subset \widehat{X}$ and $O \in \Pi_b(\widehat{X})$,*

(*iv*) *algebraically semiadditive, if $\beta(O_0 + O_1) \leq \beta(O_0) + \beta(O_1)$ for every O_0, $O_1 \in \Pi_b(\widehat{X})$,*

(*v*) *regular, if $\beta(O) = 0$ is equivalent to the relative compactness of O.*

A very famous example of measure of non compactness is the following Hausdorff measure of non compactness on $C([0, \mathcal{T}], \widehat{X})$ with $0 < \mathcal{T} < \infty$ calculated by the following formula:

$$\chi_T(O) = \frac{1}{2} \lim_{\delta \to 0} \sup_{x \in O} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|_{\widehat{X}}. \tag{2.3}$$

Here, $\chi_T(O)$ is said to be the modulus of equicontinuity of $O \subset C([0, \mathcal{T}], \widehat{X})$. Definition (2.4) is applicable on (2.3).

Definition 2.6. *[11] A multi-valued mapping $\widehat{F} : \widehat{K} \subset \widehat{X} \to \Pi(\widehat{X})$ is said to be condensing relative to measure of non compactness β (or β-condensing) if for each $O \subset \widehat{K}$, we have*

$$\beta(\widehat{F}(O)) \not\geq \beta(O).$$

That is not relatively compact.

Definition 2.7. *[8] A single valued mapping $T : \widehat{K} \to \widehat{X}^*$ is called relaxed η-α monotone if $\exists$ a mapping $\eta : \widehat{K} \times \widehat{K} \to \widehat{X}$ and a real-valued mapping $\alpha : \widehat{X} \to \mathbb{R}$, with $\alpha(tz) = t^p\alpha(z),\ \forall\ t > 0$, and $z \in \widehat{X}$, such that*

$$\langle Tx - Ty, \eta(x,y)\rangle \geq -\alpha(x-y),\ \forall\ x, y \in \widehat{K}, \tag{2.4}$$

where $p > 1$ is a constant.

Definition 2.8. *[8] A mapping $T : \widehat{K} \to \widehat{X}^*$ is called η-coercive with respect to ψ if $\exists\ x_0 \in \widehat{K}$ such that*

$$\underset{x\in K,\ \|x\|\to\infty}{lim\ inf} \frac{\langle T(x) - T(y), \eta(x,x_0)\rangle + \psi(x) - \psi(x_0)}{\|\eta(x,x_0)\|} \to +\infty. \tag{2.5}$$

Where $\eta : \widehat{K} \times \widehat{K} \to \widehat{X}$ be a mapping and $\psi : \widehat{X} \to \mathbb{R} \cup \{+\infty\}$ is proper convex lower semicontinuous function.

Theorem 2.1. *[11] Let $\widehat{X}$ be a Banach space and $\mathcal{M}$ its closed convex subset, then the fixed point set of β-condensing multi-valued mapping $\widehat{F} : \mathcal{M} \to \Pi_{kv}(\mathcal{M})$ is nonempty. That is $Fix\widehat{F} := \{x \in \mathcal{M} : x \in \widehat{F}(x)\} \neq \phi$. Where β is a nonsingular measure of non compactness defined on subsets of $\mathcal{M}$.*

3. Some properties of $Sol(K, \widehat{\mathfrak{w}} + \mathcal{A}(.), \psi)$

Let $\widehat{B_1}$ and $\widehat{B_2}$ are real reflexive Banach spaces and $\widehat{B_1}^*$ be the dual of $\widehat{B_1}$ and $\widehat{K}$ be a nonempty closed, convex subset of $\widehat{B_1}$.

We consider the following problem of finding $\widehat{\mathfrak{u}} \in \widehat{K}$ such that

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \geq 0,\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}, \tag{3.1}$$

where $\widehat{\mathfrak{w}} \in \widehat{B_1}^*$, $\mathcal{A} : \widehat{K} \to \widehat{B_1}^*$ and $\eta : \widehat{K} \times \widehat{K} \to \widehat{B_1}$. Problem (3.1) is called generalized mixed variational-like inequality. We prove the following lemma.

Lemma 3.1. *Suppose that the following conditions are satisfied:*

(I_1) $\mathcal{A} : \widehat{B_1} \to \widehat{B_1}^*$ *is an η-hemicontinuous and η-α monotone mapping;*

(I_2) $\psi : \widehat{B_1} \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex and lower semicontinuous;*

(I_3) *the mapping $\widehat{\mathfrak{u}} \to \langle \mathcal{A}\widehat{\mathfrak{z}}, \eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\rangle$ is convex, lower semicontinuous for fixed $\widehat{\mathfrak{v}}, \widehat{\mathfrak{z}} \in \widehat{K}$ and $\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{u}}) = 0,\ \forall\ \widehat{\mathfrak{u}} \in \widehat{K}$.*

Then $\widehat{\mathfrak{u}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.), \psi)$, if and only if $\widehat{\mathfrak{u}}$ is the solution of following inequality:

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}),\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.2}$$

Proof. Let $\widehat{\mathfrak{u}}$ is a solution of problem (3.1), then

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \geq 0.$$

Since $\mathcal{A}$ is relaxed η-α monotone, we have

$$\begin{aligned} & \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \\ = \ & \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \langle \mathcal{A}(\widehat{\mathfrak{v}}) - \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \\ \geq \ & \langle \mathcal{A}(\widehat{\mathfrak{v}}) - \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}), \ \forall \, \widehat{\mathfrak{v}} \in \widehat{K}. \end{aligned}$$

Hence, $\widehat{\mathfrak{u}}$ is the solution of inequality (3.2).
Conversely, let $\widehat{\mathfrak{u}} \in \widehat{K}$ be a solution of problem (3.2) and let $\widehat{\mathfrak{v}} \in \widehat{K}$ be any point $\psi(\widehat{\mathfrak{v}}) < \infty$. We define $\widehat{\mathfrak{v}}_s = (1 - s)\widehat{\mathfrak{u}} + s\widehat{\mathfrak{v}}, \quad s \in (0, 1)$, then due to convexity of $\widehat{K}$ $\widehat{\mathfrak{v}}_s \in K$. Since $\widehat{\mathfrak{v}}_s \in \widehat{K}$ is the solution of inequality (3.2), it follows from (I_1)–(I_3)

$$\begin{aligned} \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), \eta(\widehat{\mathfrak{v}}_s, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}_s) - \psi(\widehat{\mathfrak{u}}) &\geq \alpha(\widehat{\mathfrak{v}}_s - \widehat{\mathfrak{u}}) \\ \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), \eta((1 - s)\widehat{\mathfrak{u}} + s\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi((1 - s)\widehat{\mathfrak{u}} + s\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) &\geq \alpha((1 - s)\widehat{\mathfrak{u}} \\ &\quad + s\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}) \\ \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), (1 - s)\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{u}}) + s\eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + (1 - s)\psi(\widehat{\mathfrak{u}}) + s\psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) &\geq \alpha(s(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}})). \end{aligned}$$

Using (I_3), we have

$$\begin{aligned} \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), s\eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + s(\psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}})) &\geq s^p \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}) \\ \langle \widehat{\mathfrak{w}} + \mathcal{A}((1 - s)\widehat{\mathfrak{u}} + s\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + (\psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}})) &\geq s^{p-1} \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}), \end{aligned}$$

letting $s \to 0^+$, we get

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \geq 0, \ \forall \, \widehat{\mathfrak{v}} \in K.$$

□

Theorem 3.1. *Suppose that the conditions (I_1)–(I_3) are satisfied. Additionally, if the following conditions hold.*

(I_4) $\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}) + \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) = 0$,

(I_5) $\exists \ \widehat{\mathfrak{v}}_0 \in \widehat{K} \cap D(\psi)$ *such that*

$$\liminf_{\widehat{\mathfrak{u}} \in \widehat{K}, \ \|\widehat{\mathfrak{u}}\| \to \infty} \frac{\langle \mathcal{A}(\widehat{\mathfrak{u}}) - \mathcal{A}(\widehat{\mathfrak{v}}_0), \eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0) \rangle + \psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}}_0)}{\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\|} \longrightarrow +\infty. \tag{3.3}$$

Then, $Sol(K, \widehat{\mathfrak{w}} + \mathcal{A}(.), \ \psi) = \{\widehat{\mathfrak{u}} \in \widehat{K} : \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) \geq 0, \ \forall \ \widehat{\mathfrak{v}} \in K\} \neq \phi$, bounded, closed and convex, for $\widehat{\mathfrak{w}} \in \widehat{B_1}^*$.

Proof. Clearly, $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.), \ \psi) \neq \phi$, as $\widehat{\mathfrak{v}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.), \ \psi)$, for each $\widehat{\mathfrak{v}} \in \widehat{K}$.
Now, we have to show that $Sol(\widehat{K}, \widehat{\mathfrak{w}}+\mathcal{A}(.), \ \psi)$ is bounded. Suppose to contrary that $Sol(\widehat{K}, \widehat{\mathfrak{w}}+\mathcal{A}(.), \ \psi)$ is not bounded, then there exists a sequence $\{\widehat{\mathfrak{u}}_n\} \in Sol(\widehat{K}, \widehat{\mathfrak{w}}+\mathcal{A}(.), \ \psi)$ such that $\|\widehat{\mathfrak{u}}_n\|_{\widehat{B_1}} \to \infty$ as $n \to \infty$.

We can consider, $\forall\ n \in \mathbb{N}$, $\|\widehat{\mathfrak{u}}_n\| > n$. By η-coercive condition (3.3), $\exists$ a constant $M > 0$ and a mapping $\kappa : [0, \infty) \to [0, \infty)$ with $\kappa(k) \to \infty$ such that for every $\|\widehat{\mathfrak{u}}\|_{\widehat{B_1}} \geq M$,

$$\langle \mathcal{A}(\widehat{\mathfrak{u}}) - \mathcal{A}(\widehat{\mathfrak{v}}_0), \eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\rangle + \psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}}_0) \geq \kappa(\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}})\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}}.$$

Thus, if n is sufficiently large as $\kappa(n) > (\|\mathcal{A}(\widehat{\mathfrak{v}}_0)\| + \|\widehat{\mathfrak{w}}\|)$,

$$\begin{aligned} 0 &\leq \langle \mathcal{A}(\widehat{\mathfrak{u}}_n) + \widehat{\mathfrak{w}}, \eta(\widehat{\mathfrak{v}}_0, \widehat{\mathfrak{u}}_n)\rangle + \psi(\widehat{\mathfrak{v}}_0) - \psi(\widehat{\mathfrak{u}}_n) \\ &= \langle \mathcal{A}(\widehat{\mathfrak{u}}_n), \eta(\widehat{\mathfrak{v}}_0, \widehat{\mathfrak{u}}_n)\rangle + \langle \widehat{\mathfrak{w}}, \eta(\widehat{\mathfrak{v}}_0, \widehat{\mathfrak{u}}_n)\rangle + \psi(\widehat{\mathfrak{v}}_0) - \psi(\widehat{\mathfrak{u}}_n) \\ &= -\langle \mathcal{A}(\widehat{\mathfrak{u}}_n) - \mathcal{A}(\widehat{\mathfrak{v}}_0), \eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}}_0)\rangle + \psi(\widehat{\mathfrak{v}}_0) - \psi(\widehat{\mathfrak{u}}_n) + \langle \mathcal{A}(\widehat{\mathfrak{v}}_0), \eta(\widehat{\mathfrak{v}}_0, \widehat{\mathfrak{u}}_n)\rangle \\ &\quad + \langle \widehat{\mathfrak{w}}, \eta(\widehat{\mathfrak{v}}_0, \widehat{\mathfrak{u}}_n)\rangle \\ &\leq -\kappa(\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\|)\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}} + \|\mathcal{A}(\widehat{\mathfrak{v}}_0)\|.\|\eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}} + \|\widehat{\mathfrak{w}}\|.\|\eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}} \\ &= \|\eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}}\Big[- \kappa(\|\eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}}_0)\|_{\widehat{B_1}}) + \|\mathcal{A}(\widehat{\mathfrak{v}}_0)\| + \|\widehat{\mathfrak{w}}\|\Big] \\ &< 0. \end{aligned}$$

Which is not possible. Thus, $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ is bounded.

Now it remains to prove that $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ is closed.
Let $\{\widehat{\mathfrak{u}}_n\}$ be a sequence in $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ such that $\widehat{\mathfrak{u}}_n \to \widehat{\mathfrak{u}} \in \widehat{K}$. Then, $\forall\ n \in \mathbb{N}$

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{u}}_n), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}_n)\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}_n) \geq 0,\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.4}$$

From Lemmas (3.1) and (3.4) same as

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{u}}_n, \widehat{\mathfrak{v}})\rangle + \psi(\widehat{\mathfrak{u}}_n) - \psi(\widehat{\mathfrak{v}}) \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}_n),\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.5}$$

By using (I_4), we have

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}_n)\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}_n) + \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}_n) \leq 0,\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.6}$$

Which implies that

$$\lim_{n\to 0^+} sup\{\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}_n)\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}_n) + \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}_n)\} \leq 0,\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}, \tag{3.7}$$

as $\widehat{\mathfrak{u}} \to \langle \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle$, ψ and α are lower semicontinuous functions. From (3.7), we have

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}}) + \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}) \leq 0,\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}, \tag{3.8}$$

that is,

$$\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\rangle + \psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}}) \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}}),\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.9}$$

By Lemma 3.1, we get $\widehat{\mathfrak{u}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$, that is $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ is closed.
Lastly, we show that $Sol(\widehat{K}, \widehat{\mathfrak{w}} + G(.),\ \psi)$ is convex. For any $\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ and $s \in [0, 1]$, let $\widehat{\mathfrak{v}}_s = (1 - s)\widehat{\mathfrak{v}} + s\widehat{\mathfrak{u}}$. Since $\widehat{K}$ is convex, so that $\widehat{\mathfrak{u}}_s \in \widehat{K}$. Using (I_3) and letting $s \to 0^+$, we obtain

$$\begin{aligned} &\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), \eta(\widehat{\mathfrak{v}}_s, \widehat{\mathfrak{v}})\rangle + \psi(\widehat{\mathfrak{v}}_s) - \psi(\widehat{\mathfrak{v}})\rangle \\ &\qquad = \langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}}_s), \eta((1 - s)\widehat{\mathfrak{v}} + s\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}})\rangle + \psi((1 - s)\widehat{\mathfrak{v}} + s\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}}) \end{aligned}$$

$$
\begin{aligned}
&\le (1-s)\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}_s}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{v}})\rangle + s\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}_s}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle \\
&+ s(\psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}})) \\
&\le s[\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}_s}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\rangle + (\psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}}))] \\
&\le 0,
\end{aligned}
$$

that is,

$$
\langle \widehat{\mathfrak{w}} + \mathcal{A}(\widehat{\mathfrak{v}_s}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{v}_s})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{v}_s}) \ge 0.
$$

Hence, $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ is convex. □

Boundedness of $\widehat{\mathfrak{w}}$ implies that $Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ is bounded.

Theorem 3.2. *Suppose that all the conditions and mappings are same as considered in Theorem 3.1. Additionally,* $\forall\ \widehat{\mathfrak{w}} \in \overline{B}(n, \widehat{B_1}^*)$, $\exists$ *a constant* $M_n > 0$, *depending on n, such that*

$$
\|\widehat{\mathfrak{u}}\|_{\widehat{B_1}} \le M_n, \quad \forall\ \widehat{\mathfrak{u}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi), \tag{3.10}
$$

where $\overline{B}(n, \widehat{B_1}^*) = \{\widehat{\mathfrak{w}} \in \widehat{B_1}^* : \|\widehat{\mathfrak{w}}\|_{\widehat{B_1}^*} \le n\}$.

Proof. On contrary let us suppose that $\exists\ N_0 > 0$ and

$$
\underset{\widehat{\mathfrak{w}} \in \overline{B}(N_0, \widehat{B_1}^*)}{Sup} \left\{ \|\widehat{\mathfrak{u}}\|_{\widehat{B_1}} : \widehat{\mathfrak{u}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi) \right\} = +\infty.
$$

Therefore, $\exists\ \widehat{\mathfrak{w}_{\hat{k}}} \in \overline{B}(N_0, \widehat{B_1}^*)$ and $\widehat{\mathfrak{u}_{\hat{k}}} \in Sol(\widehat{K}, \widehat{\mathfrak{w}} + \mathcal{A}(.),\ \psi)$ with $\|\eta(\widehat{\mathfrak{u}_{\hat{k}}}, \widehat{\mathfrak{v}_0})\| > \hat{k}$ ($\hat{k} = 1, 2, 3, \cdots$). By η-coercivity assumption, $\exists$ a constant $M > 0$ such that $\forall\ \|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}_0})\| \ge M$ and a function $\kappa : [0, \infty) \to [0, \infty)$ with $\kappa(\hat{k}) \to \infty$ as $\hat{k} \to \infty$, we have

$$
\langle \mathcal{A}(\widehat{\mathfrak{u}}), \eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}_0})\rangle + \psi(\widehat{\mathfrak{u}}) - \psi(\widehat{\mathfrak{v}_0}) \ge \kappa(\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}_0})\|)\|\eta(\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}_0})\|_{\widehat{B_1}}.
$$

Thus, for $\hat{k} > M$ sufficiently large such that $\kappa(\hat{k}) > \frac{N_0 + \|\mathcal{A}(\widehat{\mathfrak{v}_0})\|}{\hat{k}}$, one has

$$
\begin{aligned}
0 &\le \langle \widehat{\mathfrak{w}_{\hat{k}}} + \mathcal{A}(\widehat{\mathfrak{u}_{\hat{k}}}), \eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\rangle + \psi(\widehat{\mathfrak{v}_0}) - \psi(\widehat{\mathfrak{u}_{\hat{k}}}) \\
&= \langle \widehat{\mathfrak{w}_{\hat{k}}}, \eta(\widehat{\mathfrak{v}_0}, u_{\hat{k}})\rangle - \langle \mathcal{A}(\widehat{\mathfrak{u}_{\hat{k}}}) - \mathcal{A}(\widehat{\mathfrak{v}_0}), \eta(\widehat{\mathfrak{u}_{\hat{k}}}, \widehat{\mathfrak{v}_0})\rangle + \langle \mathcal{A}(\widehat{\mathfrak{v}_0}), \eta(\widehat{\mathfrak{u}_{\hat{k}}}, \widehat{\mathfrak{v}_0})\rangle \\
&\quad + \psi(\widehat{\mathfrak{v}_0}) - \psi(\widehat{\mathfrak{u}_{\hat{k}}}) \\
&= \langle \widehat{\mathfrak{w}_{\hat{k}}}, \eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\rangle - [\langle \mathcal{A}(\widehat{\mathfrak{u}_{\hat{k}}}) - \mathcal{A}(\widehat{\mathfrak{v}_0}), \eta(\widehat{\mathfrak{u}_{\hat{k}}}, \widehat{\mathfrak{v}_0})\rangle + \psi(\widehat{\mathfrak{u}_{\hat{k}}}) - \psi(\widehat{\mathfrak{v}_0})] \\
&\quad + \langle \mathcal{A}(\widehat{\mathfrak{v}_0}), \eta(\widehat{\mathfrak{u}_{\hat{k}}}, \widehat{\mathfrak{v}_0})\rangle \\
&= \|\widehat{\mathfrak{w}_{\hat{k}}}\|_{\widehat{B_1}^*}\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| - r(\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\|)\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| + \|\mathcal{A}(\widehat{\mathfrak{v}_0})\|\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| \\
&= N_0\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| - r(\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\|)\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| + \|\mathcal{A}(\widehat{\mathfrak{v}_0})\|\|\eta(\widehat{\mathfrak{v}_0}, \widehat{\mathfrak{u}_{\hat{k}}})\| \\
&\le (N_0 + \|\mathcal{A}(\widehat{\mathfrak{v}_0})\|)\hat{k} - r(\hat{k}) < 0,
\end{aligned}
$$

which is a contradiction. Hence our supposition is wrong. □

Let $\widetilde{\mathfrak{g}} : [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2} \to \widehat{B_1}^*$ be the single valued mapping and a multi-valued mapping $\mathfrak{F} : [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2} \to \Pi(\widehat{K})$ is defined as follows:

$$
\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) := \left\{ \widehat{\mathfrak{u}} \in \widehat{K} : \widehat{\mathfrak{u}} \in Sol(\widehat{K}, \widetilde{\mathfrak{g}}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) + \mathcal{A}(.),\ \psi) \right\}.
$$

It follows from Theorem 3.1 that $\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$ is nonempty, bounded, closed and convex that is, $\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) \in \Pi_{bcv}(\widehat{B_1})\ \forall\ (x, \mathfrak{y}(x), \mathfrak{y}'(x)) \in [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2}$.

Theorem 3.3. *Suppose that all the conditions and mappings are same as considered in Theorem 3.1 and the mapping* $\widetilde{g} : [0,\mathcal{T}] \times \widehat{B_2} \times \widehat{B_2} \to \widehat{B_1}^*$ *is bounded and continuous, then the following assertions hold:*

(*i*) $\mathfrak{F}$ *is strongly weakly u.s.c.;*

(*ii*) $x \to \mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$ *is measurable* $\forall\ \mathfrak{y}, \mathfrak{y}' \in \widehat{B_2}$*;*

(*iii*) *for every bounded subset* $\Omega^* = \Omega_1 \times \Omega_2$ *of* $C^1\Big([0,\mathcal{T}], \widehat{B_2} \times \widehat{B_2}\Big)$, $\exists$ *a constant* M_{Ω^*} *such that*

$$\|\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))\| := sup\{\|\widehat{\mathfrak{u}}\|_{\widehat{B_1}} : \widehat{\mathfrak{u}} \in \mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))\} \leq M_{\Omega^*},\ \forall\ \ x \in [0,\mathcal{T}] \tag{3.11}$$

and $(\mathfrak{y}, \mathfrak{y}') \in \Omega^*$.

Proof. (**i**) Let $C \subset \widehat{B_1}$ be any weakly closed subset of $\widehat{B_1}$, suppose that $\{(x_n, \mathfrak{y}_n, \mathfrak{y}'_n)\} \subset [0,\mathcal{T}] \times \widehat{B_2} \times \widehat{B_2}$ such that $(x_n, \mathfrak{y}_n, \mathfrak{y}'_n) \to (x^*, \mathfrak{y}^*, \mathfrak{y}^{*\prime})$ in $[0,\mathcal{T}] \times \widehat{B_2} \times \widehat{B_2}$ with $(x_n, \mathfrak{y}_n, \mathfrak{y}_n') \in \mathfrak{F}^{-1}(C) := \{(x, \mathfrak{y}, \mathfrak{y}') \mid C \cap \mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}') \neq \phi\}$. Therefore, for any $n \in \mathbb{N}$, there exists $\widehat{\mathfrak{u}_n} \in C \cap \mathfrak{F}(x_n, \mathfrak{y}_n, \mathfrak{y}_n')$ such that

$$\langle \widetilde{\mathfrak{g}}(x_n, \mathfrak{y}_n, \mathfrak{y}_n') + \mathcal{A}(\widehat{\mathfrak{u}_n}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}_n})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}_n}) \geq 0,\ \forall\, \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.12}$$

By Lemma 3.1, (3.12) is equivalent to

$$\langle \widetilde{\mathfrak{g}}(x_n, \mathfrak{y}_n, \mathfrak{y}_n') + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}_n})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{v}_n}) \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}_n}),\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.13}$$

Which implies that,

$$\lim_{n\to 0^+} sup\{\langle \widetilde{\mathfrak{g}}(x_n, \mathfrak{y}_n, \mathfrak{y}_n') + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}_n})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{v}_n})\} \geq \lim_{n\to 0^+} sup\{\alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}_n})\},\quad \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.14}$$

Since $\widetilde{\mathfrak{g}}$ is continuous. Therefore, by Theorem 3.3, it implies that $\{\widehat{\mathfrak{u}_n}\}$ is bounded. Hence, by reflexivity of $\widehat{B_1}$, we can suppose that $\widehat{\mathfrak{u}_n} \to \widehat{\mathfrak{u}^*} \in C$ in $\widehat{B_1}$.
From (3.14), we get

$$\langle \widetilde{\mathfrak{g}}(x^*, \mathfrak{y}^*, \mathfrak{y}^{*\prime}) + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}^*})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}^*}) \geq \alpha(\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}^*}),\ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}. \tag{3.15}$$

Using Lemma 3.1, we have

$$\langle \widetilde{\mathfrak{g}}(x^*, \mathfrak{y}^*, \mathfrak{y}^{*\prime}) + \mathcal{A}(\widehat{\mathfrak{v}}), \eta(\widehat{\mathfrak{v}}, \widehat{\mathfrak{u}^*})\rangle + \psi(\widehat{\mathfrak{v}}) - \psi(\widehat{\mathfrak{u}^*}) \geq 0,\ \ \forall\ \widehat{\mathfrak{v}} \in \widehat{K}.$$

It follows from weakly closeness of C that

$$(x^*, \mathfrak{y}^*, \mathfrak{y}^{*\prime}) \in \mathfrak{F}^{-1}(C) := \{(x, \mathfrak{y}, \mathfrak{y}') : C \cap \mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}') \neq \phi\}.$$

Hence, $\mathfrak{F}$ is strongly weakly u.s.c..
(**ii**) Define a set

$$L_\lambda := \{x \in [0,\mathcal{T}]; d(v, \mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))) > \lambda\},\ \forall\ (\mathfrak{y}, \mathfrak{y}') \in \widehat{B_2} \times \widehat{B_2},\ \widehat{\mathfrak{v}} \in \widehat{B_1}.$$

Now we will show that L_λ is an open set for all $\lambda \geq 0$. For this let $\{x_n\} \subset (L_\lambda)^c = [0, \mathcal{T}] \backslash L_\lambda$ be a sequence with $x_n \to x$. Then $\forall\ n \in \mathbb{N}$, we have $d(v, \mathfrak{F}(x_n, \mathfrak{y}, \mathfrak{y}')) \leq \lambda$. As for every $(x, \mathfrak{y}, \mathfrak{y}') \in [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2}$, the multi-valued mapping $\mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}')$ is bounded, closed and convex by Theorem 3.1, we get $\forall\ n \in \mathbb{N}$, $\widehat{\mathfrak{u}_n} \in \mathfrak{F}(x_n, \mathfrak{y}, \mathfrak{y}')$ such that $\|\widehat{\mathfrak{v}} - \widehat{\mathfrak{u}_n}\| \leq \lambda$. By Theorem 3.3, $\{\widehat{\mathfrak{u}_n}\}$ is bounded, so we may assume that $\widehat{\mathfrak{u}_n} \rightharpoonup \widehat{\mathfrak{u}} \in \widehat{K}$. By (i), $\widehat{\mathfrak{u}} \in \mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$. Hence, we obtain

$$d(v, \mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}')) \leq \|\widehat{\mathfrak{u}} - \widehat{\mathfrak{v}}\|_{\widehat{B_1}} = \liminf_{n\to\infty} \|\widehat{\mathfrak{u}_n} - \widehat{\mathfrak{v}}\|_{\widehat{B_1}} \leq \lambda,$$

that is $x \in (L_\lambda)^c$, thus $[0, \mathfrak{F}] \setminus L_\lambda$ is closed. Hence, L_λ is open, consequently L_λ is measurable. By [24, Proposition 6.2.4], the mapping $x \mapsto \mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}')$ is measurable $\forall\ (\mathfrak{y}, \mathfrak{y}') \in \widehat{B_2} \times \widehat{B_2}$.
(iii) As $\widetilde{\mathfrak{g}}$ is bounded. Therefore

$$\widetilde{\mathfrak{g}}_{\Omega^*} := \{\widetilde{\mathfrak{g}}(x, \mathfrak{y}(x), \mathfrak{y}'(x)) :\ x \in [0, \mathcal{T}] \text{ and } (\mathfrak{y}, \mathfrak{y}') \in \Omega^*\},$$

is also bounded in $\widehat{B_1}$ for every bounded subset Ω^* of $C^1\Big([0, \mathcal{T}], \widehat{B_2} \times \widehat{B_2}\Big)$. Then, by Theorem 3.3, $\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$ is bounded, $\forall\ x \in [0, \mathcal{T}]$ and $(\mathfrak{y}, \mathfrak{y}') \in \Omega^*$. Hence, $\exists$ a constant $M_{\Omega^*} > 0$ such that 3.11 holds. □

4. Main results

Before proving our main result, we mention that by Theorem 3.3, $\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$ is measurable and $\widehat{B_1}$ is a separable Banach space. Hence, by [21, Theorem 3.17] $\mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x))$ possess a measurable selection ξ such that $\xi \in L^\infty\Big([0, \mathcal{T}]; \widehat{B_1}\Big) \subset L^2\Big([0, \mathcal{T}], \widehat{B_1}\Big)\ \forall\ (\mathfrak{y}, \mathfrak{y}') \in C^1\Big([0, \mathcal{T}], \widehat{B_2} \times \widehat{B_2}\Big)$. So

$$P_{\mathfrak{F}}(\mathfrak{y}, \mathfrak{y}') := \Big\{\xi \in L^2([0, \mathcal{T}], \widehat{B_1}) \mid \xi(t) \in \mathfrak{F}(x, \mathfrak{y}(x), \mathfrak{y}'(x)),\ \ a.e.,\ x \in [0, \mathcal{T}]\Big\}, \tag{4.1}$$

is well defined $\forall\ (\mathfrak{y}, \mathfrak{y}') \in C^1([0, \mathcal{T}], \widehat{B_2} \times \widehat{B_2})$.

Lemma 4.1. *Suppose that $(I_1) - (I_4)$ hold and $\widetilde{g} : [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2} \to \widehat{B_1}^*$ is bounded and continuous. Then, multi-valued mapping $P_{\mathfrak{F}}$ is strongly upper semicontinuous.*

Proof. Let $\{\mathfrak{y}_n, \mathfrak{y}'_n\} \subset C^1([0, \mathcal{T}], \widehat{B_2} \times \widehat{B_2})$ with $(\mathfrak{y}_n,\ \mathfrak{y}'_n) \to (\mathfrak{y}_0, \mathfrak{y}'_0)$ in $C^1([0, \mathcal{T}], \widehat{B_2} \times \widehat{B_2})$ and $\xi_n \in P_{\mathfrak{F}}(\mathfrak{y}_n, \mathfrak{y}'_n)$ for $n \in \mathbb{N}$. Now, we need to prove that $\exists$ a subsequence of $\{\xi_n\}$, such that $\xi_n \to \xi_0 \in P_{\mathfrak{F}}(\mathfrak{y}_0, \mathfrak{y}'_0)$.
Indeed, (I_5) confirms that the sequence $\{\xi_n\}$ is bounded in $L^2([0, \mathcal{T}], \widehat{B_1})$. Therefore, we can suppose $\xi_n \to \xi_0$ weakly in $L^2([0, T], \widehat{B_1})$. By Lemma 2.1, there is $\{\xi\}$, a finite combination of the $\{\xi_i : i \geq n\}$ with $\bar{\xi}_n \to \xi_0$ converges strongly in $L^2([0, \mathcal{T}], \widehat{B_1})$.
Since $\mathfrak{F}$ is strongly weakly upper semicontinuous and $(\mathfrak{y}_n, \mathfrak{y}'_n) \to (\mathfrak{y}_0, \mathfrak{y}'_0) \in C^1([0, \mathcal{T}], \widehat{B_2})$, therefore for every weak neighborhood $\mathcal{Y}_x$ of $\mathfrak{F}(x, \mathfrak{y}_0(x), \mathfrak{y}'_0(x))$ there exists a strong neighborhood

$$\mathfrak{F}(x, \mathfrak{y}, \mathfrak{y}') \subset \mathcal{Y}_x, \quad \forall\ (\mathfrak{y}, \mathfrak{y}') \in \mathcal{U}_x.$$

Which shows that $\xi \in P_{\mathfrak{F}}(\mathfrak{y}_0, \mathfrak{y}'_0)$. Thus, by Proposition 2.3, $P_{\mathfrak{F}}$ is strongly upper semi continuous. □

We also need the following assumptions for achieving the goal.

(I_6) $\widetilde{\mathfrak{g}} : [0, \mathcal{T}] \times \widehat{B_2} \times \widehat{B_2} \to \widehat{B_1}^*$ is continuous and bounded;

(I_7) $\widetilde{\mathcal{F}}(.,\mathfrak{y},.):[0,\mathcal{T}]\to\mathcal{L}(\widehat{B_1},\widehat{B_2})$, $\widetilde{\mathcal{F}}(.,.,\mathfrak{y}'):[0,\mathcal{T}]\to\mathcal{L}(\widehat{B_1},\widehat{B_2})$ are measurable for all $\mathfrak{y},\mathfrak{y}'\in\widehat{B_2}$ and $\widetilde{\mathcal{F}}(x,.,.):\widehat{B_2}\to\mathcal{L}(\widehat{B_2},\widehat{B_1})$ is continuous for a.e. $x\in[0,\mathcal{T}]$, where $\mathcal{L}(\widehat{B_1},\widehat{B_2})$ denotes the class of bounded linear operators from $\widehat{B_1}$ to $\widehat{B_2}$, and there exists $\rho_{\widetilde{\mathcal{F}}}\in L^2([0,\mathcal{T}],\mathbb{R}_+)$ and a non-decreasing continuous mapping $\gamma_{\widetilde{\mathcal{F}}}:[0,\infty)\to[0,\infty)$ such that

$$\|\widetilde{\mathcal{F}}(x,\mathfrak{y}(x),\mathfrak{y}'(x))\|\le\rho_{\widetilde{\mathcal{F}}}(x)\gamma_{\widetilde{\mathcal{F}}}\Big(\|\mathfrak{y}(x)\|_{\widehat{B_2}}+\|\mathfrak{y}'(x)\|_{\widehat{B_2}}\Big),\ \forall\ (x,\mathfrak{y},\mathfrak{y}')\in[0,\mathcal{T}]\times\widehat{B_2}\times\widehat{B_2}.$$

(I_8) $\widetilde{\mathfrak{f}}(.,\mathfrak{y},.)$, $\widetilde{\mathfrak{f}}(.,.,\mathfrak{y}'):[0,\mathcal{T}]\to\widehat{B_2}$ are measurable for all $\mathfrak{y},\mathfrak{y}'\in\widehat{B_2}$ and there exists $\rho_{\widetilde{\mathfrak{f}}}\in L^2\Big([0,\mathcal{T}],\mathbb{R}_+\Big)$ such that for $x\in[0,\mathcal{T}]$ $\widetilde{\mathfrak{f}}(x,.,.):\widehat{B_2}\to\widehat{B_2}$ satisfies

$$\begin{cases}\|\widetilde{\mathfrak{f}}(x,\mathfrak{y},\dagger)-\widetilde{\mathfrak{f}}(x,\mathfrak{y}',\dagger)\|\le\rho_{\widetilde{\mathfrak{f}}}(x)\|\mathfrak{y}-\mathfrak{y}'\|_{\widehat{B_2}},\\ \|\widetilde{\mathfrak{f}}(x,\mathfrak{y},\dagger)-\widetilde{\mathfrak{f}}(x,\mathfrak{y},\dagger')\|\le\rho_{\widetilde{\mathfrak{f}}}(x)\|\dagger-\dagger'\|_{\widehat{B_2}},\\ \|\widetilde{\mathfrak{f}}(x,0,0)\|\le\rho_{\widetilde{\mathfrak{f}}}(x).\end{cases}\tag{4.2}$$

The following result ensures the existence of solution of problem (2.1).

Theorem 4.1. *Under the assumptions* (I_1)–(I_8)*, if the following inequalities hold*

$$\liminf_{\hat{k}\to\infty}\Big[\frac{\gamma_{\widetilde{\mathcal{F}}}(\hat{k})}{\hat{k}}\|\rho_{\widetilde{\mathcal{F}}}(x)\|M_{\|\widehat{\mathfrak{G}}\|}+\|\rho_{\widetilde{\mathfrak{f}}}(x)\|_{L^2}+\frac{\|\mathfrak{y}_0\|+\|y_0\|}{\hat{k}\mathcal{T}^{1/2}}\Big]<\frac{1}{\delta\mathcal{T}^{1/2}},\tag{4.3}$$

$$\|Q(x_1)-Q(x_2)\|\le\|x_1-x_2\| \text{ and } \|R(x_1)-R(x_2)\|\le\|x_1-x_2\|,\tag{4.4}$$

where

$$\delta=max\Big\{\sup_{x\in J}\|Q(x)\|_{L(\widehat{B_2})},\ \ \sup_{x\in J}\|R(x)\|_{L(\widehat{B_2})}\Big\},$$

and $M_{\|\widehat{\mathfrak{G}}\|}>0$ *is a constant stated in Theorem 3.2, then, the problem* (2.1) *has at least one mild solution* $(\mathfrak{y},\widehat{\mathfrak{u}})$.

Proof. We define the multi-valued mapping $\Gamma:C^1([0,\mathcal{T}],\widehat{B_2})\to\Pi(C^1([0,\mathcal{T}],\widehat{B_2}))$ such that

$$\begin{aligned}\Gamma(\mathfrak{y}):=\Big\{y\in C^1([0,\mathcal{T}],\widehat{B_2})\Big|\ y(x)=Q(x)\mathfrak{y}_0+R(x)y_0+\int_0^x R(x-p)\Big[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)\\ +\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\Big]dp,\ x\in[0,\mathcal{T}],\ \xi\in P_{\widetilde{\mathfrak{F}}}(x)\Big\},\end{aligned}\tag{4.5}$$

where $P_{\widetilde{\mathfrak{F}}}$ is defined in (4.1). Our aim is to show that $Fix(\Gamma)\neq\phi$.

Step-I. $\Gamma(\mathfrak{y})\in\Pi_{cv}\Big(C^1([0,\mathcal{T}],\widehat{B_2})\Big)$ for each $\mathfrak{y}\in C^1([0,\mathcal{T}],\widehat{B_2})$.

Clearly, $\Gamma(\mathfrak{y})$ is convex for every $\mathfrak{y}\in C^1([0,\mathcal{T}],\widehat{B_2})$ due to the convexity of $P_{\widetilde{\mathfrak{F}}}(\mathfrak{y})$.

Since for each $y\in\Gamma(\mathfrak{y})$, we can choose $\xi\in P_{\widetilde{\mathfrak{F}}}(\mathfrak{y})$ such that

$$\begin{aligned}y(x)&=Q(x)\mathfrak{y}_0+R(x)y_0+\int_0^x R(x-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)\\ &\quad+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp,\end{aligned}$$

which implies that,

$$
\begin{aligned}
\|y(x)\| &\le \|\mathfrak{y}_0 Q(x)\| + \|y_0 R(x)\| + \Big\| \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) \\
&\quad +\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp\Big\| \\
&\le \delta\|\mathfrak{y}_0\| + \delta\|y_0\| + \delta\Big[\int_0^x \|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)\|dp \\
&\quad + \int_0^x \|\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|dp\Big].
\end{aligned}
$$

Using (I_7) and (I_8) and applying Hölder's inequality,

$$
\begin{aligned}
\|y(x)\| &\le \delta\|\mathfrak{y}_0\| + \delta\|y_0\| + \delta\Big[\int_0^x \|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)\|dp \\
&\quad + \int_0^x \|\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|dt\Big], \\
&= \delta(\|\mathfrak{y}_0\| + \|y_0\|) + \delta\Big[\int_0^x \rho_{\widetilde{\mathcal{F}}}(p)\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}\| + \|\mathfrak{y}'\|)M_{\|\widehat{\mathfrak{g}}\|}dp \\
&\quad + \int_0^x \rho_{\widetilde{\mathfrak{f}}}(p)(1 + \|\mathfrak{y}\| + \|\mathfrak{y}'\|)dp\Big] \\
&= \delta(\|\mathfrak{y}_0\| + \|y_0\|) + \delta\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}\| + \|\mathfrak{y}'\|)M_{\|\widehat{\mathfrak{g}}\|}\int_0^x \rho_{\widetilde{\mathcal{F}}}(p)dp \\
&\quad +\delta(1 + \|\mathfrak{y}\| + \|\mathfrak{y}'\|)\int_0^x \rho_{\widetilde{\mathfrak{f}}}(p)dp, \\
&= \delta\Big(\|\mathfrak{y}_0\| + \|y_0\| + \gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}\| + \|\mathfrak{y}'\|)M_{\|\widehat{\mathfrak{g}}\|}\|\rho_{\widetilde{\mathcal{F}}}\|\mathcal{T}^{1/2} \\
&\quad +(1 + \|\mathfrak{y}\| + \|\mathfrak{y}'\|)\|\rho_{\widetilde{\mathfrak{f}}}\|\mathcal{T}^{1/2}\Big) \\
&= \delta\mathcal{T}^{1/2}\Big[\frac{\|\mathfrak{y}_0\| + \|y_0\|}{\mathcal{T}^{1/2}} + \gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}\| + \|\mathfrak{y}'\|)M_{\|\widehat{\mathfrak{g}}\|}\|\rho_{\widetilde{\mathcal{F}}}\| \\
&\quad +(1 + \|\mathfrak{y}\| + \|\mathfrak{y}'\|)\|\rho_{\widetilde{\mathfrak{f}}}\|\Big].
\end{aligned}
$$

Hence, $\Gamma(\mathfrak{y})$ is bounded in $C^1([0,\mathcal{T}],\widehat{B_2})$ for each $\mathfrak{y} \in C^1([0,\mathcal{T}],\widehat{B_2})$.

Next we shall prove that $\Gamma(\mathfrak{y})$ is a collection of equicontinuous mappings $\forall\ \mathfrak{y} \in C^1([0,\mathcal{T}],\widehat{B_2})$.

$$
\begin{aligned}
\|y(x_2) - y(x_1)\|_{\widehat{B_2}} &= \Big\|\mathfrak{y}_0 Q(x_2) + y_0 R(x_2) + \int_0^{x_2} R(x_2 - p)\Big[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) \\
&\quad +\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\Big]dp - \mathfrak{y}_0 Q(x_1) - y_0 R(x_1) \\
&\quad - \int_0^{x_1} R(x_1 - t)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) + \widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp\Big\|_{\widehat{B_2}}
\end{aligned}
$$

$$
\le \|\mathfrak{y}_0\|\|Q(x_2) - Q(x_1)\| + \|y_0\|\|R(x_2) - R(x_1)\|
$$

$$
\begin{aligned}
&+\Big\|\int_0^{x_2} R(x_2-t)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp \\
&-\int_0^{x_1} R(x_1-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp \\
&+\int_0^{x_1} R(x_2-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp \\
&-\int_0^{x_1} R(x_2-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp\Big\| \\
\leq\ & \|\mathfrak{y}_0\|\|x_2-x_1\|+\|y_0\|\|x_2-x_1\|+\int_{x_1}^{x_2}\Big\|R(x_2-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p), \\
&\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp\Big\|+\int_0^{x_1}\Big\|(R(x_2-p) \\
&-R(x_1-p))[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]\Big\|dp \\
=\ & (\|\mathfrak{y}_0\|+\|y_0\|)\|x_2-x_1\|+I_1+I_2,
\end{aligned}
\tag{4.6}
$$

where $I_1 = \int_{x_1}^{x_2}\Big\|R(x_2-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp\Big\|,$

and

$$
\begin{aligned}
I_2 =\ & \int_0^{x_1}\Big\|(R(x_2-p)-R(x_1-p))[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) \\
&+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]\Big\|dp.
\end{aligned}
$$

Applying Hölder's inequality, we have

$$
\begin{aligned}
I_1 \leq\ & \int_{x_1}^{x_2}\|R(x_2-p)\|\|[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p)\|dp \\
&+\int_{x_1}^{x_2}\|R(x_2-p)\|\|\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]\|dp \\
\leq\ & \int_{x_1}^{x_2}\delta M_{\|\mathfrak{G}\|}\rho_{\widetilde{\mathcal{F}}}(p)\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)dp \\
&+\int_{x_1}^{x_2}\delta\gamma_{\widetilde{\mathfrak{g}}}(1+\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)dp, \\
=\ & \delta M_{\|\mathfrak{G}\|}\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)\|\rho_{\widetilde{\mathcal{F}}}(p)\|(x_2-x_1)^{1/2} \\
&+\delta\gamma_{\widetilde{\mathfrak{f}}}(1+\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)(x_2-x_1)^{1/2} \\
=\ & \delta(x_2-x_1)^{1/2}\Big[M_{\|\mathfrak{G}\|}\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)\|\rho_{\widetilde{\mathcal{F}}}(p)\| \\
&+\gamma_{\widetilde{\mathfrak{f}}}(1+\|\mathfrak{y}(p)\|+\|\mathfrak{y}'(p)\|)\Big]\to 0 \text{ as } x_1\to x_2.
\end{aligned}
\tag{4.7}
$$

Further by Proposition 2.2 and (4.4) and Hölder's inequality, we have

$$
\begin{aligned}
I_2 =\ & \int_0^{x_1}\Big\|(R(x_2-p)-R(x_1-p))[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) \\
&+\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]\Big\|dp
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\| \int_0^{x_1} [R(p)(Q(x_2)-Q(x_1)) + Q(p)(R(x_1)-R(x_2))] \\
&\quad \times[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) + \widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]\Big\| dp \\
&\le \int_0^{x_1} \|R(p)\|\|Q(x_1)-Q(x_2)\|\|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) \\
&\quad +\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|dp + \int_0^{x_1} \|Q(p)\|\|R(x_1)-R(x_2)\| \\
&\quad \|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi(p) + \widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|dp \\
&\le \int_0^{x_1} \delta\|x_1-x_2\|\Big[\|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|\|\xi(p)\| \\
&\quad +\|\widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|\Big]dp + \int_0^{x_1} \delta\|x_1-x_2\| \\
&\quad \times[\|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\| + \|\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\|]dp \\
&\le 2\delta(\|x_1-x_2\|)\int_0^{x_1} [M_{\|\mathfrak{G}\|}\rho_{\widetilde{\mathcal{F}}}(p)\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}(p)\| + \|\mathfrak{y}'(p)\|) \\
&\quad +\rho_{\widetilde{\mathfrak{f}}}(p)\gamma_{\widetilde{\mathfrak{f}}}(1+\|\mathfrak{y}(p)\| + \|\mathfrak{y}'(p)\|)]dp, \\
&\le 2\delta\|x_1-x_2\|\Big[M_{\|\mathfrak{G}\|}\|\rho_{\widetilde{\mathcal{F}}}(p)\|\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}(p)\| + \|\mathfrak{y}'(p)\|) \\
&\quad +\gamma_{\widetilde{\mathcal{F}}}(1+\|\mathfrak{y}(p)\| + \|\mathfrak{y}'(p)\|)\Big]x^{1/2} \to 0 \text{ as } x_1 \to x_2. \qquad (4.8)
\end{aligned}
$$

From (4.6)–(4.8), we have

$$
\|y(x_2) - y(x_1)\|_{\widehat{B_2}} \longrightarrow 0, \text{ as } x_1 \to x_2.
$$

Hence, $\Gamma(\mathfrak{y})$ is equicontinuous, $\forall\ \mathfrak{y} \in C^1([0,\mathcal{T}],\widehat{B_2})$. By Arzela-Ascoli theorem [34], we obtained that $\Gamma(\mathfrak{y})$ is relatively compact $\forall\ \mathfrak{y} \in C^1([0,\mathcal{T}],\widehat{B_2})$.
Now, we have to check that $\Gamma(\mathfrak{y})$ is closed in $C^1([0,\mathcal{T}],\widehat{B_2})$ $\forall\ \mathfrak{y} \in C^1([0,\mathcal{T}],\widehat{B_2})$.
Let $\{y_n\} \subset \Gamma(\mathfrak{y})$ is a sequence with $y_n \to y^*$ in $C^1([0,\mathcal{T}];\widehat{B_2})$ as $n \to \infty$. Hence, there exist a sequence $\{\xi_n\} \subset P_{\mathfrak{F}}(\mathfrak{y})$ such that

$$
y_n(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi_n(p) + \widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp,
$$

$x \in [0,\mathcal{T}]$. By (*iii*) of Theorem 3.3, it follows that the sequence $\{\xi_n\}$ is weakly relatively compact. Since $P_{\mathfrak{F}}(\mathfrak{y})$ is upper semicontinuity (see Lemma 4.1), we may assume $\xi_n \to \xi^* \in P_{\mathfrak{F}}(\mathfrak{y})$ in $L^2([0,\mathcal{T}],\widehat{B_1})$, where $\xi^* \in P_{\mathfrak{F}}(\mathfrak{y})$. On the other hand, by strongly continuity of $Q(x)$ and $R(x)$ for $x > 0$, we have

$$
y^*(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))\xi^*(p) + \widetilde{\mathfrak{f}}(p,\mathfrak{y}(p),\mathfrak{y}'(p))]dp,
$$

$x \in [0,\mathcal{T}]$. Which implies that $y^* \in \Gamma(\mathfrak{y})$, that is $\Gamma(\mathfrak{y}) \in \Pi_{cv}(C^1[0,\mathcal{T}],\widehat{B_2})$.

Step-II. The multi-valued mapping Γ is closed.
For this assume $\mathfrak{y}_n \to \mathfrak{y}^*$ and $y_n \to y^*$ in $C^1([0,\mathcal{T}],\widehat{B_2})$ with $y_n \in \Gamma(\mathfrak{y}_n)$ $\forall\ n \in \mathbb{N}$. We need to prove that

$y^* \in \Gamma(\mathfrak{y}^*)$. From the definition of multi-valued map Γ, we may take $\xi_n \in P_{\mathfrak{F}}(\mathfrak{y}_n)$ $\forall$ $n \in \mathbb{N}$ such that

$$y_n(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p, \mathfrak{y}_n(p), \mathfrak{y}_n{}'(p))\xi_n(p) + \widetilde{\mathfrak{f}}(p, \mathfrak{y}_n(p), \mathfrak{y}_n{}'(p))]dp, \quad x \in [0, \mathcal{T}]. \tag{4.9}$$

With the help of Theorem 3.3 and Lemma 4.1, we may consider that $\xi_n \rightharpoonup \xi^* \in P_{\mathfrak{F}}(\mathfrak{y}^*)$. By using, I_8 we get that $\widetilde{\mathfrak{f}}(., \mathfrak{y}_n(.), \mathfrak{y}_n'(.)) \to \widetilde{\mathfrak{f}}(., \mathfrak{y}^*, \mathfrak{y}^{*\prime})$ in $L^2([0, \mathcal{T}], \widehat{B_2})$.
By using the continuity of $\widetilde{\mathcal{F}}(x, ., .)$ and strongly continuity of $Q(x), R(x)$ for $x > 0$, we obtain from (4.9) that

$$y^*(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p, \mathfrak{y}^*(p), \mathfrak{y}^{*\prime}(p))\xi^*(p) + \widetilde{\mathfrak{f}}(p, \mathfrak{y}^*(p), \mathfrak{y}^{*\prime}(p))]dp, \quad x \in [0, \mathcal{T}],$$

and $\xi^* \in P_{\mathfrak{F}}(\mathfrak{y})$. Thus $\mathfrak{y}^* \in \Gamma(\mathfrak{y}^*)$.

Step-III. Γ is χ_T condensing.
Let $\mathcal{D} \subset \Pi_b(C^1([0, \mathcal{T}], \widehat{B_2}))$. Therefore, $\mathcal{D}$ is not relatively compact subset of $C^1([0, \mathcal{T}], \widehat{B_2})$. For $\mathcal{D}$, we need to prove that $\chi_{\mathcal{T}}((\mathcal{D})) \nleq \chi_{\mathcal{T}}(\Gamma((\mathcal{D})))$. Since $\mathcal{D}$ is bounded subset of $C^1([0, \mathcal{T}], \widehat{B_2})$, then by applying the same technique as in Step-I, we may prove that $\Gamma((\mathcal{D}))$ is relatively compact, that is, $\chi_{\mathcal{T}}(\mathcal{D}) = 0$. Hence, $\chi_{\mathcal{T}}((\mathcal{D})) \le \chi_{\mathcal{T}}(\Gamma(\mathcal{D})) = 0$ implies that $\mathcal{D}$ is relatively compact by regularity of χ_T, we conclude that Γ is $\chi_{\mathcal{T}}$-condensing.

Step-IV. $\exists$ a constant $M_{\mathfrak{R}} > 0$ such that

$$\Gamma(\bar{B}_{M_{\mathfrak{R}}} \subset \bar{B}M_{\mathfrak{R}}) := \{\mathfrak{y} \in C^1([0, \mathcal{T}], \widehat{B_2}) : \|\mathfrak{y}\|_C \le M_{\mathfrak{R}}\} \subset C^1([0, \mathcal{T}], \widehat{B_2}). \tag{4.10}$$

Let us assume that $\forall$ $k > 0$, $\exists$ two sequences $\{\mathfrak{y}_k\}$ and $\{y_k\}$ such that
$\|\mathfrak{y}_k\|_{C^1([0,\mathcal{T}],\widehat{B_2})}$, $\|\mathfrak{y}_k'\|_{C^1([0,\mathcal{T}],\widehat{B_2})} \le k/2$ and $y_k \in \Gamma(\mathfrak{y}_k)$ such that $\|y_k\| > 0$. Hence, there is $\xi_k \in P_{\mathfrak{F}}(\mathfrak{y}_k)$ such that

$$y_k(x) = Q(x)\mathfrak{y}_0 + R(x)y_0 + \int_0^x R(x-p)[\widetilde{\mathcal{F}}(p, \mathfrak{y}_k(p), \mathfrak{y}_k{}'(p))\xi_k(p) + \widetilde{\mathfrak{f}}(p, \mathfrak{y}_k(p), \mathfrak{y}_k{}'(p))]dp, \quad x \in [0, \mathcal{T}].$$

Using Hölder's inequality, for every $x \in [0, \mathcal{T}]$, we have

$$\begin{aligned}
\|\mathfrak{y}_k(x)\| &\le \|Q(x)\|\|\mathfrak{y}_0\| + \|R(x)\|\|y_0\| \\
&\quad + \int_0^x \|R(x-p)\|\|[\widetilde{\mathcal{F}}(p, \mathfrak{y}_k(p), \mathfrak{y}_k{}'(p))\xi_k(p) + \widetilde{\mathfrak{f}}(p, \mathfrak{y}_k(p), \mathfrak{y}_k{}'(p))]\|dp, \\
&= \delta(\|\mathfrak{y}_0\| + \|y_0\|) + \int_0^x \delta[\gamma_{\widetilde{\mathcal{F}}}(\|\mathfrak{y}_k\| + \|\mathfrak{y}_k{}'\|)\rho_{\widetilde{\mathcal{F}}}(p)M_{\|\widetilde{\mathfrak{f}}\|}]dp \\
&\quad + \int_0^x \delta\gamma_{\widetilde{\mathfrak{f}}}(1 + \|\mathfrak{y}_k\| + \|\mathfrak{y}_k'\|)\rho_{\widetilde{\mathfrak{f}}}(p)dp \\
&\le \delta(\|\mathfrak{y}_0\| + \|y_0\|) + \int_0^x \delta[\gamma_{\widetilde{\mathcal{F}}}(k)\rho_{\widetilde{\mathcal{F}}}(p)M_{\|\widetilde{\mathfrak{G}}\|}]dp
\end{aligned}$$

$$
\begin{aligned}
& + \int_0^x \delta\gamma_{\tilde{\mathfrak{f}}}(1+k)\rho_{\tilde{\mathfrak{f}}}(p)dp \\
\le\ & \delta(\|\mathfrak{y}_0\| + \|y_0\|) + \delta\gamma_{\widetilde{\mathcal{F}}}(k)\|\rho_{\widetilde{\mathcal{F}}}(x)\|M_{\|\widehat{\mathfrak{g}}\|}\mathcal{T}^{1/2} \\
& +\delta\gamma_{\tilde{\mathfrak{f}}}(1+\kappa)\|\rho_{\tilde{\mathfrak{f}}}(x)\|\mathcal{T}^{1/2} \\
=\ & \delta\mathcal{T}^{1/2}\Big[\gamma_{\widetilde{\mathcal{F}}}(k)\|\rho_{\widetilde{\mathcal{F}}}(x)\|M_{\|\widehat{\mathfrak{g}}\|} + \gamma_{\tilde{\mathfrak{f}}}(1+k)\|\rho_{\tilde{\mathfrak{g}}}(x)\| + \frac{\|\mathfrak{y}_0\| + \|y_0\|}{\mathcal{T}^{1/2}}\Big],
\end{aligned}
$$

we obtain by using (4.9),

$$
\begin{aligned}
1 \ &\le\ \liminf_{k\to\infty} \frac{\|y_k\|_{C^1([0,\mathcal{T}],\widehat{B_2})}}{k} \\
&\le\ \liminf_{k\to\infty}\Big[\gamma_{\widetilde{\mathcal{F}}}(k)\|\rho_{\widetilde{\mathcal{F}}}(x)\|M_{\|\widehat{\mathfrak{g}}\|}\mathcal{T}^{1/2} + \gamma_{\tilde{\mathfrak{f}}}(1+k)\|\rho_{\tilde{\mathfrak{f}}}(x)\|\mathcal{T}^{1/2} + (\|\mathfrak{y}_0\| + \|y_0\|)\Big] \\
&<\ 1,
\end{aligned}
$$

which is a contradiction. Therefore there exists $M_{\mathfrak{R}}$ such that (4.10) holds.
Thus, all requirements of Theorem 2.1 are fulfilled. This implies that $Fix\Gamma \neq \phi$ in $\overline{B}_{M_{\mathfrak{R}}}$. Therefore, (SOEPDVLI) has at least one mild solution $(\mathfrak{y}, \widehat{\mathfrak{u}})$. □

5. Conclusions

In this paper, a second order evolutionary partial differential variational-like inequality problem is introduced and studied in a Banach space, which is much more general than the considered by Liu-Migórski-Zeng [14], Li-Huang-O'Regan [13] and Wang-Li-Li-Huang [33] etc. We investigate suitable conditions to prove an existence theorem for our problem by using the theory of strongly continuous cosine family of bounded linear operator, fixed point theorem for condensing set-valued mapping and the theory of measure of non-compactness.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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