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Nodal solutions for the Kirchhoff-Schrödinger-Poisson system in $\mathbb{R}^3$

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Abstract: This paper is dedicated to studying the following Kirchhoff-Schrödinger-Poisson system:

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(|x|)u + \lambda\phi u = K(|x|)f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where V, K are radial and bounded away from below by positive numbers. Under some weaker assumptions on the nonlinearity f, we develop a direct approach to establish the existence of infinitely many nodal solutions $\{u_k^{b,\lambda}\}$ with a prescribed number of nodes k, by using the Gersgorin disc's theorem, Miranda theorem and Brouwer degree theory. Moreover, we prove that the energy of $\{u_k^{b,\lambda}\}$ is strictly increasing in k, and give a convergence property of $\{u_k^{b,\lambda}\}$ as $b \to 0$ and $\lambda \to 0$.

Keywords: nodal solutions; Kirchhoff type problem; Schrödinger-Poisson system; variational methods; disc's theorem
Mathematics Subject Classification: 35J20, 35J65

1. Introduction and main results

In this paper, we study the existence of infinitely many nodal solutions for the following Kirchhoff-Schrödinger-Poisson problem :

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(|x|)u + \lambda\phi u = K(|x|)f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where V, K, F satisfy the following assumptions:

(**V**) $V : [0, +\infty) \to \mathbb{R}$ is measurable with $V_0 := ess\inf_{[0,+\infty)} V > 0$;

(K) $K \in L^\infty([0,+\infty))$ with $ess\inf_{[0,+\infty)} K > 0$;
(F1) $f \in C(\mathbb{R};\mathbb{R})$, $f(t) = -f(-t)$, and $f(t) = o(t)$ as $t \to 0$;
(F2) there exists a constant $C_0 > 0$ and $p \in (4,6)$ such that for all $x \in \mathbb{R}^N$,

$$|f(t)| \le C_0\left(|t| + |t|^{p-1}\right), \quad \forall t \in \mathbb{R};$$

(F3) $\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^3} = +\infty$;
(F4) there exists $\theta \in (0,1)$ such that for all $x \in \mathbb{R}^N$, $t > 0$ and $\tau \in \mathbb{R}\backslash\{0\}$,

$$K(|x|)\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right]\text{sign}(1-t) + \theta V_0 \frac{\left|1-t^{-2}\right|}{\tau^2} \ge 0.$$

The nonlocal operator $(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u$ comes from the following Kirchhoff problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(x,u), \ x \in \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.2}$$

where $a, b > 0$. Problem (1.2) is related to is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u),$$

which was introduced by Kirchhoff [20]. After the pioneer work of Lions [23], lots of interesting results to problem (1.2) or similar problems were obtained in last decade. For example, by category theory, He and Zou [13] studied the existence of multiple positive solutions for (1.2). Moreover, they also studied the concentrated behavior of positive solution. Combining constraint variational methods and the quantitative deformation lemma, Shuai [28] studied the existence and asymptotic behavior of the least energy sign-changing solution to problem (1.2). Soon afterwards, under some more weak assumptions on f (like **(F1)**–**(F4)**), Tang and Cheng [33] improved and generalized some results obtained in [28]. For the investigation of stationary states of the Kirchhoff type equation and other non-local problems, we refer to [1, 9, 11–17, 27, 33, 38, 39] and reference therein.

When $a = 1, b = 0$, system (1.1) reduces to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = K(|x|)f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

For more details about its physical meaning, we refer the reader to [4, 31] and the references therein. Noting that system (1.3) involves a nonlocal term $\lambda\phi(x)u$, namely, it is not a pointwise identity. In view of this, more mathematical difficulties are waiting to be solved, which makes the study of system (1.3) more meaningful. In the last decade, much effort has been made to investigate system like (1.3) on the existence of positive solutions, ground state solutions, multiple solutions and sign-changing solution; see for examples [2,6,19,24,26,29,32,35,41] and the reference therein. In particular, when $K(|x|)f(u) = |u|^{p-1}u$ $(3 < p < 5)$, Wang and Zhou [35] proved that (1.3) had a least energy sign-changing solution via a constraint variational method combining the Brouwer degree theory. After that, Chen and Tang [6] consider (1.3) under a general nonlinearity $f(x,u)$ like **(F1)**–**(F4)**). Utilizing the Non-Nehari manifold method, the authors obtained a least energy sign-changing solution.

On the other hand, system (1.1) includes the well-known nonlinear Schrödinger equation

$$\begin{cases} -a\Delta u + V(|x|)u = K(|x|)f(u), \ x \in \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.4}$$

Problem of (1.4) has been extensively studied since 1970s. One can refer to the books [31, 36] and reference for the details and related results. In particular, Bartsch and Willem [3], Cao and Zhu [5] proved independently that, for every integer $k \geq 0$, there is a pair of solutions $u_k^{\pm}$ of (1.5), having precisely k nodes. They first obtain the solution of (1.1) in each annulus, including every ball and the complement of it, and then glue them by matching the normal derivative at each junction point. After that, many authors pay their attention to find nodal solutions of elliptical partial differential equations with non-local term. When $K(x)f(u) = |u|^{p-1}u$ $(3 < p < 5)$, Kim and Seok [19] construct a sign-changing solution of Schrödinger-Poisson system (1.3) with any prescribed number nodes. More precisely, due to the nonlocal term $\lambda\phi u$, the method used in [3, 5] could not apply to (1.3) directly. Hence, by changing (1.3) to another system and take advantage of Nehari method and gluing method in [3, 5], Kim and Seok obtained infinitely many sign-changing solutions. It is worth noting that Deng, Peng and Shuai [9] considered the Kirchhoff type problem (1.2), where $f \in C^1$ and satisfies the Nehari monotonicity condition

(**F5**) $\frac{f(t)}{|t|^3}$ is increasing on $(-\infty, 0) \cup (0, +\infty)$.

By using the Nehari method and gluing method as in [19], they proved (1.2) had a least energy sign-changing radial solution with any prescribed number nodes. It should be mentioned that the assumption (**F4**) is much weaker than (**F5**). As stated in [6], there are many functions satisfying (**F1**)–(**F4**)), but not (**F5**). For example, $f(t) = \left(|t|^3 + |t|^{3/2}\right)t, \forall t \in \mathbb{R}$. Very recently, Guo and Wu [12] considered (1.2) with assumption (**F1**)–(**F4**)), by using Non-Nehari method, matrix theory and Brouwer degree theory, they get a similar conclusion as in [9]. For the reader interested in nodal solutions for the elliptical partial differential equations, we would also like to refer [7, 10, 18] and references therein.

For system (1.1) contains bother nonlocal operator and nonlocal nonlinear term, the study of system (1.1) become technically complicated. In recent year, there were some scholars paying attention to system (1.1) or similar problems; see [8, 21, 37, 40] and the reference therein. Especially, when $K(x)f(u) = |u|^{p-1}u$ $(3 < p < 5)$, Deng and Yang [8] studied the nodal solution of (1.1) by using the gluing method. But they did not study the energy property and asymptotic behavior of the nodal solution. Motivated by the above results, it is natural to ask, under condition (**F1**)–(**F4**)), whether (1.1) had sign-changing solutions with any prescribed number nodes? If there exist such solution, how about its energy property and asymptotic behavior? To the best of our knowledge, this question is open for more general nonlinearities f, especially when $f \in C^0$.

Inspired by the work mentioned above, in this paper, we seek the nodal solutions to problem (1.1) under the assumption (**V**), (**K**) and (**F1**)–(**F4**). Before stating our main results, we give several definitions and some notations. Throughout this paper, we define

$$H := \left\{ u \in H_r^1\left(\mathbb{R}^3\right) \mid \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(|x|)u^2\right) dx < +\infty \right\},$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(|x|)u^2\right) dx.$$

The embedding $H \hookrightarrow H_r^1\left(\mathbb{R}^3\right)$ is continuous due to the condition (V). Then the embedding $H \hookrightarrow L^q\left(\mathbb{R}^3\right)$ is compact for $2 < q < 6$, due to the well known result of Strauss [30]. Besides, as is well known from the Lax-Milgram theorem, for any $u \in H^1\left(\mathbb{R}^3\right)$, there exists a unique

$$\phi_u = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy \in D^{1,2}\left(\mathbb{R}^3\right)$$

solving the equation $-\Delta\phi = u^2$. Then the energy functional associated with system (1.1) is well-defined

$$I_{b,\lambda}(u) = \frac{1}{2}\|u\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{\lambda}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}dxdy - \int_{\mathbb{R}^3} K(|x|)F(u)dx, \tag{1.5}$$

where $F(u) = \int_0^u f(\tau)d\tau$. Moreover, $I_{b,\lambda} \in C^1(H,\mathbb{R})$, due to the growth condition (**F1**), (**F2**) and the Hardy-Littlewood-Sobolev inequality, see [22, Theorem 4.3]. Hence, for any $u, \varphi \in H$, we have

$$\begin{aligned} \left\langle I'_{b,\lambda}(u), \varphi\right\rangle = & \int_{\mathbb{R}^3} (a\nabla u\nabla\varphi + V(|x|)u\varphi)dx + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u\nabla\phi dx \\ & + \lambda\int_{\mathbb{R}^3} \phi_u u\varphi dx - \int_{\mathbb{R}^3} K(|x|)f(u)\varphi dx. \end{aligned} \tag{1.6}$$

Clearly, critical points of $I_{b,\lambda}$ are the weak solutions for nonlocal problem (1.1). A necessary condition for $u \in H$ to be a critical point of $I_{b,\lambda}$ is that $\left\langle I'_{b,\lambda}(u), u\right\rangle = 0$. This necessary condition defines the Nehari manifold

$$\mathcal{N} = \left\{u \in H\backslash\{0\} : \left\langle I'_{b,\lambda}(u), u\right\rangle = 0\right\},$$

and the corresponding ground state energy is

$$c_0 = \inf_{\mathcal{N}} I_{b,\lambda}.$$

Meanwhile, for each $k \in \mathbb{N}_+$, for each positive integer k, we denote by $\boldsymbol{r}_k = (r_1, \ldots, r_k) \in \mathbb{R}^k$ with $0 = r_0 < r_1 < \cdots < r_k < r_{k+1} = \infty$, and define a Nehari type set

$$\begin{aligned} \mathcal{N}_k = \{\, u \in H : \exists\, \boldsymbol{r}_k \text{ such that } (-1)^{i+1}u_i > 0 \text{ in } B_i, u_i = 0 \text{ on } \partial B_i \\ \text{and } I'_{b,\lambda}(u)u_i = 0,\ \forall 1 \le i \le k+1\right\}, \end{aligned} \tag{1.7}$$

where $B_1 = \left\{x \in \mathbb{R}^N : 0 \le |x| < r_1\right\}$, $B_i := \left\{x \in \mathbb{R}^N : r_{i-1} < |x| < r_i\right\}$, $u_i = u_{B_i}$ and χ_{B_i} is the characteristic function on B_i. Clearly, $u = \sum_{i=1}^{k+1} u_i$ and $\mathcal{N}_k$ consists of nodal functions with precisely k nodes. We consider the level

$$c_k = \inf_{\mathcal{N}_k} I_{b,\lambda} \tag{1.8}$$

Our first main result of the paper can be stated as follows.

Theorem 1.1. *Under the assumptions* (**V**), (**K**), (**F1**)–(**F4**)*, for every integer* $k \ge 0$*, there exists a radial solution* u_k *of* (1.1)*, which changes sign exactly k-times.*

The proof of Theorem 1.1 is based on the argument presented in [19] regarding the Nehari method and gluing method (see [3, 5]). However, comparing to [19], the contribution of this work is greatly relax the assumptions on f and subsequently deal with the difficulties it brings. The main difficult is when the nonlinearity term f does not satisfy Nehari monotonicity condition (**F5**), one can not use the method in [19] to prove the set $\mathcal{N}_k$ is nonempty. Meanwhile, due to $f \notin C^1$, it is impossible to prove the differentiability of the Nehari set $\mathcal{N}_k$ and the Lagrangian multiplier principle dose not work. To overcome this difficulty, we use the Non-Nehari method in [6] and the Gersgorin disc's Theorem (see [12]) to prove $\mathcal{N}_k$ is nonempty. Besides, taking advantage of Brouwer degree theory and Miranda theorem, we obtain the least energy sign-changing solution of (1.1) with exactly k nodes for any $k \geq 0$.

Similar to [9], our another aim in the present paper is to show that the energy of u_k obtained in Theorem 1.1 is strictly increasing in k.

Theorem 1.2. *If the assumptions of Theorem* 1.1 *hold, then the energy of u_k is strictly increasing in k, i.e.,*

$$I_{b,\lambda}(u_{k+1}) > I_{b,\lambda}(u_k), \quad \text{for any } k \geq 0.$$

Moreover, $I_{b,\lambda}(u_k) > (k+1)I_{b,\lambda}(u_0)$, where u_0 is the solution u_k with $k = 0$ in Theorem 1.1.

Note that u_k obtained in Theorem 1.1 depends on b, λ. Thus, we denote it by $u_k^{b,\lambda}$ and analyze the convergence properties of $u_k^{b,\lambda}$ as $b \to 0^+$ and $\lambda \to 0^+$.

Theorem 1.3. *Under the assumptions of Theorem 1.1, for any sequence $\{b_n\}$ and $\{\lambda_n\}$ with $b_n \to 0^+$ and $\lambda_n \to 0^+$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$ and $\{\lambda_n\}$, such that the solution $u_k^{b_n,\lambda_n}$ obtained in Theorem 1.1, converge to w_k strongly in H as $n \to \infty$, where w_k is a least energy radial solution of* (1.4) *which changes sign exactly k-times.*

The paper is organized as follows. In Section 2, we first present variational framework and and modify the energy functional (1.1) to the functional corresponding to a system of $(k+1)$-equations. Besides, we give some useful Lemmas in Matrix theory. In Section 3, we devote to find a nontrivial critical point of the modified functional. In Section 4, we obtain a solution of (1.1) with k-nodes by using critical point obtained in Section 2 as a building block, henceforth complete the proof of Theorem 1.1. Finally, in Section 4, we prove Theorems 1.2 and 1.3.

2. Preliminaries

In this section, we outline the variational framework for problem (1.1) and modify the energy functional to another functional, which corresponds to a system of $(k+1)$-equations. In addition, we will also give some results from matrix theory, which is important in our proofs.

For each integer k, we define

$$\Gamma_k = \left\{\boldsymbol{r}_k = (r_1, \ldots, r_k) \in \mathbb{R}^k \mid 0 := r_0 < r_1 < \cdots < r_{k+1} := \infty\right\},$$

and for each $\boldsymbol{r}_k \in \Gamma_k$, set

$$\begin{aligned}
B_1^{\boldsymbol{r}_k} &= \left\{x \in \mathbb{R}^3 : 0 \leq |x| < r_1\right\},\\
B_i^{\boldsymbol{r}_k} &= \left\{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i\right\}, \quad \text{for } i = 2, 3, \cdots, k\\
B_{k+1}^{\boldsymbol{r}_k} &= \left\{x \in \mathbb{R}^3 : |x| \geq r_k\right\}.
\end{aligned}$$

Obviously, $B_1^{\boldsymbol{r}_k}$ is a ball, $B_2^{\boldsymbol{r}_k}, \cdots, B_k^{\boldsymbol{r}_k}$ are annuli and $B_{k+1}^{\boldsymbol{r}_k}$ is the complement of a ball. Fix an element $\boldsymbol{r}_k = (r_1, \cdots, r_k) \in \boldsymbol{\Gamma}_k$ and thereby a family of annuli $\left\{B_i^{\boldsymbol{r}_k}\right\}_{i=1}^{k+1}$, we write

$$\mathcal{H}^{\boldsymbol{r}_k} = H_1^{\boldsymbol{r}_k} \times \cdots \times H_{k+1}^{\boldsymbol{r}_k},$$

where

$$H_i^{\boldsymbol{r}_k} := \left\{u \in H_0^1\left(B_i^{\boldsymbol{r}_k}\right) \mid u(x) = u(|x|), u(x) = 0 \text{ if } x \notin B_i^{\boldsymbol{r}_k}\right\},$$

for $i = 1, \cdots, k+1$. Apparently, $H_i^{\boldsymbol{r}_k}$ is Hilbert space with the norm

$$\|u\|_i^2 = \int_{B_i^{r_k}} \left(a|\nabla u|^2 + V(|x|)u^2\right) dx.$$

Hereafter, we always regard $u_i \in H_i^{\boldsymbol{r}_k}$ as an element in $H^1\left(\mathbb{R}^3\right)$ by setting $u \equiv 0$ in $\mathbb{R}^3 \backslash B_i^{\boldsymbol{r}_k}$, and denote by

$$A(u_i, u_j) = \int_{\mathbb{R}^3} |\nabla u_i|^2 |\nabla u_j|^2 dx \tag{2.1}$$

and

$$B(u_i, u_j) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_i^2(x) u_j^2(y)}{4\pi|x-y|} dxdy. \tag{2.2}$$

After that, we introduce an auxiliary functional $J_{b,\lambda} : \mathcal{H}^{\boldsymbol{r}_k} \to \mathbb{R}$ related to $I_{b,\lambda} : H \to \mathbb{R}$, defined by

$$\begin{aligned}
J_{b,\lambda}(u_1, \ldots, u_{k+1}) &:= I_{b,\lambda}\left(\sum_{i=1}^{k+1} u_i\right) \\
&= \sum_{i=1}^{k+1} \frac{1}{2} \|u_i\|_i^2 + \frac{b}{4} \sum_{i=1}^{k+1} \left(\int_{B_i^{r_k}} |\nabla u_i|^2 dx\right)^2 + \frac{\lambda}{4} \sum_{i=1}^{k+1} \int_{B_i^{r_k}} \int_{B_i^{r_k}} \frac{u_i^2(x) u_i^2(y)}{4\pi|x-y|} dxdy \\
&\quad + \sum_{i=1}^{k+1} \sum_{j \neq i} \frac{b}{4} \int_{B_i^{r_k}} |\nabla u_i|^2 dx \int_{B_j^{r_k}} |\nabla u_j|^2 dx + \sum_{i=1}^{k+1} \sum_{j \neq i} \frac{\lambda}{4} \int_{B_j^{r_k}} \int_{B_i^{r_k}} \frac{u_i^2(x) u_i^2(y)}{4\pi|x-y|} dxdy \\
&\quad - \sum_{i=1}^{k+1} \int_{B_i^{r_k}} K(|x|) F(u) dx \\
&= \sum_{i=1}^{k+1} \left(\frac{1}{2}\|u_i\|_i^2 + \frac{\lambda}{4} A(u_i, u_i) - \frac{1}{p} \int_{B_i^{r_k}} K(|x|) F(u) dx\right) \\
&\quad + \frac{b}{4} \sum_{i=1}^{k+1} \sum_{j \neq i} A(u_i, u_j) + \frac{\lambda}{4} \sum_{i=1}^{k+1} \sum_{j \neq i} B(u_i, u_j).
\end{aligned}$$

Therefore, for each $i = 1, \ldots, k+1$,

$$\begin{aligned}
\partial_{u_i} J_{b,\lambda}(u_1, \ldots, u_{k+1}) u_i = \|u_i\|_i^2 + bA(u_i, u_i) + \lambda B(u_i, u_i) - \int_{B_i^{r_k}} K(|x|) F(u) dx \\
+ b \sum_{i=1}^{k+1} \sum_{j \neq i} A(u_i, u_j) + \lambda \sum_{i=1}^{k+1} \sum_{j \neq i} B(u_i, u_j).
\end{aligned}$$

Moreover, if $(u_1,\dots,u_{k+1})\in\mathcal{H}^{\boldsymbol{r}_k}$ is a critical point of $J_{b,\lambda}$, then each component u_i satisfies

$$\begin{cases} -(a+b\sum_{j=1}^{k+1}\|\nabla u_j\|_{L^2}^2)\Delta u+V(|x|)u+\lambda\left(\int_{\mathbb{R}^3}\frac{|\sum_{j=1}^{k+1}u_j(y)|^2}{4\pi|x-y|}dy\right)u=K(|x|)f(u), & x\in B_i^{\boldsymbol{r}_k},\\ u=0, & x\notin B_i^{\boldsymbol{r}_k}. \end{cases} \tag{2.3}$$

Least energy radial solution of (1.1) which change signs exactly k times will constructed by gluing the solutions of the system (2.3). To this end, analogous to Nehari manifold, we define the set $\mathcal{N}_k^{\boldsymbol{r}_k}$ by

$$\begin{aligned} \mathcal{N}_k^{\boldsymbol{r}_k}:&=\{(u_1,\cdots,u_{k+1})\in\mathcal{H}^{\boldsymbol{r}_k}\mid u_i\neq 0,\ \partial_{u_i}J_{b,\lambda}(u_1,\cdots,u_{k+1})\,u_i=0 \text{ for } i=1,\cdots,k+1\}\\ &=\Big\{(u_1,\cdots,u_{k+1})\in\mathcal{H}^{\boldsymbol{r}_k}\mid u_i\neq 0 \text{ and for each } i=1,\cdots,k+1,\\ &\qquad \|u_i\|_i^2+b\sum_{j=1}^{k+1}A(u_j,u_i)+\lambda\sum_{j=1}^{k+1}B(u_j,u_i)-\int_{B_i^{\boldsymbol{r}_k}}K(|x|)F(u_i)dx=0\Big\}, \end{aligned} \tag{2.4}$$

and consider the least energy level

$$d(\boldsymbol{r}_k)=\inf_{(u_1,\cdots,u_{k+1})\in\mathcal{N}_k^{\boldsymbol{r}_k}}J_{b,\lambda}(u_1,\cdots,u_{k+1}). \tag{2.5}$$

We will use constraint minimizer on $\mathcal{N}_k^{\boldsymbol{r}_k}$ to seek a critical point of $J_{b,\lambda}$ with nonzero component. Until then, we would like to present some properties of $F(u)$ and results from matrix theory to ensure that $\mathcal{N}_k^{\boldsymbol{r}_k}$ is nonempty.

Lemma 2.1. *Assume that* (**F1**)–(**F4**) *holds, then for any* $t>0$ *and* $\tau\in\mathbb{R}\backslash\{0\}$, *we have*

$$K(|x|)F(\tau)-K(|x|)F\left(t^{1/4}\tau\right)+\frac{t-1}{4}K(|x|)f(\tau)\tau-\frac{\theta V_0\left(t^{1/2}-1\right)^2}{4}\tau^2\leq 0, \tag{2.6}$$

where θ *has been given in* (**F4**).

Proof. In fact, a straightforward computation shows that

$$\begin{aligned} &K(|x|)F(\tau)-K(|x|)F\left(t^{1/4}\tau\right)+\frac{t-1}{4}K(|x|)f(\tau)\tau-\frac{\theta V_0\left(t^{1/2}-1\right)^2}{4}\tau^2\\ &=\int_t^1\left(\frac{1}{4}K(|x|)f\left(s^{1/4}\tau\right)s^{-3/4}\tau-\frac{1}{4}K(|x|)f(\tau)\tau-\frac{\theta V_0}{4}\left(s^{-1/2}-1\right)\tau^2\right)ds\\ &=\frac{\tau^4}{4}\int_t^1\left(\frac{K(|x|)f\left(s^{1/4}\tau\right)}{\left(s^{1/4}\tau\right)^3}-\frac{K(|x|)f(\tau)}{\tau^3}+\theta V_0\frac{1-s^{-1/2}}{\tau^2}\right)ds\\ &=\frac{\tau^4}{4}\int_t^1\left(K(|x|)\left[\frac{f(\tau)}{\tau^3}-\frac{f\left(s^{1/4}\tau\right)}{\left(s^{1/4}\tau\right)^3}\right]\mathrm{sign}(1-s)+\theta V_0\frac{\left|1-s^{-1/2}\right|}{\tau^2}\right)\mathrm{sign}(s-1)ds, \end{aligned} \tag{2.7}$$

which combines with (**F4**) ensure (2.6). □

As a byproduct of the last lemma is the following corollary.

Corollary 2.1. *Assume that* (**F1**)–(**F4**) *hold, then for any* $\tau\in\mathbb{R}$, *we have*

$$K(|x|)F(\tau)-\frac{1}{4}K(|x|)f(\tau)\tau-\frac{\theta V_0}{4}\tau^2\leq 0.$$

Proof. In fact, the corollary follows by letting $t \to 0$ in (2.6). □

Lemma 2.2. *(Gersgorin disc Theorem [34, Therome 1.1]) For any matrix $A = \left(a_{i,j}\right) \in \mathbb{C}^{n\times n}$ and any eigenvalue $\lambda \in \sigma(A) := \{\mu \in \mathbb{C} : \det(\mu E - A) = 0\}$, there is a positive integer $l \in \{1, \ldots, n\}$ such that*

$$\left|\lambda - a_{l,l}\right| \leq \sum_{j\neq l} \left|a_{l,j}\right|.$$

By virtue of this lemma, we have the following results.

Lemma 2.3. *[12] For any $b_{ij} = b_{ji} > 0$ with $i \neq j \in \{1, \ldots, n\}$ and $s_i > 0$ with $i = 1, \ldots, n$, define the matrix $C := \left(c_{ij}\right)_{n\times n}$ by*

$$c_{ij} = \begin{cases} -\sum_{l\neq i} s_l b_{il}/s_i & \text{for } i = j, \\ b_{ij} > 0 & \text{for } i \neq j. \end{cases}$$

Then the real symmetric matrix $\left(c_{ij}\right)_{n\times n}$ is non-positive definite.

Lemma 2.4. *Defining map $G_1 : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ and $G_2 : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ by*

$$G_1(t_1, \cdots, t_{k+1}) = \sum_{i,j=1}^{k+1} \left(t_i^{1/2} t_j^{1/2} - t_i\right) A(u_i, u_j), \tag{2.8}$$

and

$$G_2(t_1, \cdots, t_{k+1}) = \sum_{i,j=1}^{k+1} \left(t_i^{1/2} t_j^{1/2} - t_i\right) B(u_i, u_j), \tag{2.9}$$

where $A(u_i, u_j)$ and $B(u_i, u_j)$ are defined in (2.1) *and* (2.2). *Then the map G_1 and G_2 are strictly concave in $(\mathbb{R}_{>0})^{k+1}$. Moreover, for $(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{\geq 0})^{k+1}$, $G_m(t_1, \cdots, t_{k+1}) \leq G_m(1, \cdots, 1) = 0$, where $m \in \{1, 2\}$.*

Proof. We just need to check this result for G_1, and then the proof of G_2 is similar, we omit it. Indeed, direct computations give that

$$\frac{\partial G_1}{\partial t_i} = \sum_{\substack{l=1 \\ l\neq i}}^{k+1} \left(\frac{1}{2} t_i^{-\frac{1}{2}} t_l^{\frac{1}{2}} - 1\right) A(u_i, u_l),$$

$$\frac{\partial^2 G_1}{\partial t_i^2} = \sum_{\substack{l=1 \\ l\neq i}}^{k+1} \left(-\frac{1}{4} t_i^{-\frac{3}{2}} t_l^{\frac{1}{2}} - 1\right) A(u_i, u_l) \quad \text{and} \quad \frac{\partial^2 G_1}{\partial t_i t_j} = \frac{1}{4} t_i^{-\frac{1}{2}} t_j^{-\frac{1}{2}} A(u_i, u_j) \text{ for } i \neq j.$$

Defining the matrix $\left(a_{i,j}\right)_{(k+1)\times(k+1)}$ by $a_{i,j} = \frac{1}{4} t_i^{-\frac{1}{2}} t_j^{-\frac{1}{2}} A(u_i, u_j)$, then

$$\left(\frac{\partial^2 G_1}{\partial t_i t_j}\right)_{(k+1)\times(k+1)} = \begin{cases} -\sum_{l\neq i} t_l a_{il}/t_i \text{ for } i = j, \\ a_{ij} > 0 \text{ for } i \neq j, \end{cases}$$

which is a negative definite matrix in $(\mathbb{R}_{>0})^{k+1}$ by Lemma 2.3. Therefore, G_1 is a strictly concave function in $(\mathbb{R}_{>0})^{k+1}$. Moreover, notice that $\nabla G_1(1, \cdots, 1) = 0$, we deduce that $(1, \cdots, 1)$ is a unique global maximum point of G_1 on $(\mathbb{R}_{>0})^{k+1}$ and

$$\max_{(\mathbb{R}_{>0})^{k+1}} G_1(t_1, \cdots, t_{k+1}) = G_1(1, \cdots, 1) = 0.$$

Consequently, by the continuity of G_1,

$$\max_{(\mathbb{R}_{\geq 0})^{k+1}} G_1(t_1, \cdots, t_{k+1}) = \max_{(\mathbb{R}_{>0})^{k+1}} G_1(t_1, \cdots, t_{k+1}) = g(1, \cdots, 1) = 0.$$

As a result, we complete the proof of this lemma. □

3. Critical points of the auxiliary functional

In this section, we seek a critical point of the auxiliary functional $J_{b,\lambda}$. Firstly, we prove the following lemma.

Lemma 3.1. *Assume that* (**V**), (**K**), (**F1**) − (**F4**) *hold and* $\boldsymbol{r}_k \in \Gamma_k$ *is fixed. Then for any* $(u_1, \cdots, u_{k+1}) \in \mathcal{H}^{\boldsymbol{r}_k} \backslash \{\mathbf{0}\}$ *and* $(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{\geq 0})^{k+1} := [0, \infty)^{k+1}$*, there holds*

$$\begin{aligned} &J_{b,\lambda}\left(t_1^{\frac{1}{4}} u_1, \cdots, t_{k+1}^{\frac{1}{4}} u_{k+1}\right) - J_{b,\lambda}(u_1, \cdots, u_{k+1}) \\ &\leq \sum_{i=1}^{k+1} \left[\frac{t_i - 1}{4} \partial_{u_i} J_{b,\lambda}(u_1, \cdots, u_{k+1}) u_i - (1 - \theta) \frac{\left(t_i^{1/2} - 1\right)^2}{4} \|u_i\|_i^2\right], \end{aligned} \tag{3.1}$$

where $\theta \in (0, 1)$ *is defined in* (**F4**).

Proof. For each $(u_1, \cdots, u_{k+1}) \in \mathcal{H}^{\boldsymbol{r}_k}$ with $u_i \neq 0$ for $i = 1, \cdots, k + 1$, we have

$$\begin{aligned} &J_{b,\lambda}\left(t_1^{\frac{1}{4}} u_1, \cdots, t_{k+1}^{\frac{1}{4}} u_{k+1}\right) - J_{b,\lambda}(u_1, \cdots, u_{k+1}) \\ &= \sum_{i=1}^{k+1} \left(\frac{1}{2} t_i^{\frac{1}{2}} \|u_i\|_i^2 + \frac{b}{4} t_i^{\frac{1}{2}} \sum_{j=1}^{k+1} t_j^{\frac{1}{2}} A(u_j, u_i) + \frac{\lambda}{4} t_i^{\frac{1}{2}} \sum_{j=1}^{k+1} t_j^{\frac{1}{2}} B(u_j, u_i) - \int_{B_i^{r_k}} K(|x|) F(t_i^{\frac{1}{4}} u_i)\right) \\ &\quad - \sum_{i=1}^{k+1} \left(\frac{1}{2} \|u_i\|_i^2 + \frac{b}{4} \sum_{j=1}^{k+1} A(u_j, u_i) + \frac{\lambda}{4} \sum_{j=1}^{k+1} B(u_j, u_i) - \int_{B_i^{r_k}} K(|x|) F(u_i)\right) \\ &= \sum_{i=1}^{k+1} \left[\frac{1}{2}(t_i^{\frac{1}{2}} - 1) \|u_i\|_i^2 + \frac{b}{4} \sum_{j=1}^{k+1} \left(t_j^{\frac{1}{2}} t_i^{\frac{1}{2}} - 1\right) A(u_j, u_i) + \frac{\lambda}{4} \sum_{j=1}^{k+1} \left(t_j^{\frac{1}{2}} t_i^{\frac{1}{2}} - 1\right) B(u_j, u_i) \right. \\ &\qquad \left. + \int_{B_i^{r_k}} k(|x|) \left[F(t_i^{\frac{1}{4}} u_i) - F(u_i)\right] dx\right] \\ &= \sum_{i=1}^{k+1} \frac{t_i - 1}{4} \partial_{u_i} J_{b,\lambda}(u_1, \cdots, u_{k+1}) u_i - \sum_{i=1}^{k+1} \frac{(t_i^{\frac{1}{2}} - 1)^2}{4} \|u_i\|_i^2 \\ &\quad + \frac{b}{4} \sum_{i,j=1}^{k+1} (t_j^{\frac{1}{2}} t_i^{\frac{1}{2}} - t_i) A(u_j, u_i) + \frac{\lambda}{4} \sum_{i,j=1}^{k+1} (t_j^{\frac{1}{2}} t_i^{\frac{1}{2}} - t_i) B(u_j, u_i) \\ &\quad + \sum_{i=1}^{k+1} \int_{B_i^{r_k}} \left(K(|x|) F(u_i) - K(|x|) F\left(t_i^{1/4} u_i\right) + \frac{t_i - 1}{4} K(|x|) f(u_i) u_i\right) dx \\ &\leq \sum_{i=1}^{k+1} \left[\frac{t_i - 1}{4} \partial_{u_i} J_{b,\lambda}(u_1, \cdots, u_{k+1}) u_i - (1 - \theta) \frac{(t_i^{\frac{1}{2}} - 1)^2}{4} \|u_i\|_i^2\right] \\ &\quad + \frac{b}{4} G_1(t_1, \cdots, t_{k+1}) + \frac{\lambda}{4} G_2(t_1, \cdots, t_{k+1}) \\ &\quad + \sum_{i=1}^{k+1} \int_{B_i^{r_k}} \left(K(|x|) \left(F(u_i) - F\left(t_i^{1/4} u_i\right)\right) + \frac{t_i - 1}{4} K(|x|) f(u_i) u_i - \theta V_0 \frac{(t_i^{\frac{1}{2}} - 1)^2}{4} |u_i|^2\right) dx. \end{aligned} \tag{3.2}$$

Now, using Lemmas 2.1 and 2.4, we can easily get the conclusion of Lemma 3.1. □

From now on, we always assume that (**V**), (**K**), (**F1**)–(**F4**) hold. We are devoted to finding the critical points of $J_{b,\lambda}$. Firstly, for $(u_1, \cdots, u_{k+1}) \in \mathcal{H}^{\mathbf{r}_k}$, we define a map $\Psi^{\mathbf{u}} : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ by

$$\Psi^{u}(t_1, \cdots, t_{k+1}) := J_{b,\lambda}\left(t_1^{\frac{1}{4}} u_1, \cdots, t_{k+1}^{\frac{1}{4}} u_{k+1}\right). \tag{3.3}$$

With the help of Lemma 3.1, we are devoted to prove that the set $\mathcal{N}_k^{\mathbf{r}_k}$, defined in (2.4), is nonempty in the following Lemma.

Lemma 3.2. *Assume $\mathbf{r}_k \in \Gamma_k$ is fixed. Then for any $(u_1, \cdots, u_{k+1}) \in \mathcal{H}^{\mathbf{r}_k} \backslash \{\mathbf{0}\}$, there exists a unique $(k+1) - tuple$ $(\bar{t}_1, \cdots, \bar{t}_{k+1})$ of positive numbers such that*

$$\Psi^{\mathbf{u}}(\bar{t}_1, \cdots, \bar{t}_{k+1}) = \max_{(\mathbb{R}_{\geq 0})^{k+1}} \Psi^{\mathbf{u}}(t_1, \cdots, t_{k+1}), \tag{3.4}$$

and

$$\left((\bar{t}_1)^{\frac{1}{4}} u_1, \cdots, (\bar{t}_{k+1})^{\frac{1}{4}} u_{k+1}\right) \in \mathcal{N}_k^{\mathbf{r}_k}. \tag{3.5}$$

Proof. By the definition of $\Psi^{\mathbf{u}}$ and the assumption (**V**), (**K**), (**F1**)–(**F4**), we deduce that $\Psi^{\mathbf{u}}(t_1, \ldots, t_{k+1}) \to 0$ as $|(t_1, \ldots, t_{k+1})| \to 0$ and $\Psi^{\mathbf{u}}(t_1, \ldots, t_{k+1}) \to -\infty$ as $|(t_1, \ldots, t_{k+1})| \to \infty$, which yields $\Psi^{\mathbf{u}}$ possess at least one global maximal point $(\bar{t}_1, \ldots, \bar{t}_{k+1})$.

Firstly, we prove that $\bar{t}_i > 0$ for all $i = 1, \ldots, k+1$. Otherwise, there exists $i_0 \in \{1, \ldots, k+1\}$ such that $\bar{t}_i = 0$. Without loss of generality, we assume $\bar{t}_1 = 0$. Notice that

$$\begin{aligned}
\Psi^{\mathbf{u}}(\tau, \bar{t}_2, \ldots, \bar{t}_{k+1}) = &\frac{1}{2}\tau^{\frac{1}{2}}\|u_1\|_1^2 + \frac{b}{4}\tau A(u_1, u_1) + \frac{\lambda}{4}\tau B(u_1, u_1) - \int_{B_1^{r_k}} K(|x|)F(\tau^{\frac{1}{4}} u_1)dx \\
&+ \frac{b}{2}\sum_{j=2}^{k+1} \tau^{\frac{1}{2}}(\bar{t}_j)^{\frac{1}{2}} A(u_1, u_j) + \frac{\lambda}{2}\sum_{j=2}^{k+1} \tau^{\frac{1}{2}}(\bar{t}_j)^{\frac{1}{2}} B(u_1, u_j) \\
&+ \sum_{i=2}^{k+1}\left[\frac{(\bar{t}_i)^{\frac{1}{2}}}{2}\|u_i\|_i^2 + \frac{b}{4}\bar{t}_i A(u_i, u_i) + \frac{\lambda}{4}\bar{t}_i B(u_i, u_i) - \int_{B_i^{r_k}} K(|x|)F((\bar{t}_i)^{\frac{1}{4}} u_i)dx\right] \\
&+ \frac{b}{4}\sum_{i=2}^{k+1}\sum_{j\neq i}(\bar{t}_i)^{\frac{1}{2}}(\bar{t}_j)^{\frac{1}{2}} A(u_i, u_j) + \frac{\lambda}{4}\sum_{i=2}^{k+1}\sum_{j\neq i}(\bar{t}_i)^{\frac{1}{2}}(\bar{t}_j)^{\frac{1}{2}} B(u_i, u_j)
\end{aligned} \tag{3.6}$$

is increasing with respect to $\tau > 0$ for τ small enough, which leads to a contradiction. Therefore, $\bar{t}_i > 0$ for all $i = 1, \ldots, k+1$.

Since $(\bar{t}_1, \ldots, \bar{t}_{k+1})$ is a global maximum point of Ψ^u, namely,

$$\frac{\partial \Psi^{\mathbf{u}}}{\partial t_i}(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{k+1}) = \frac{1}{4}\partial_{u_i} J_{b,\lambda}\left((\bar{t}_1)^{\frac{1}{4}} u_1, \cdots, (\bar{t}_{k+1})^{\frac{1}{4}} u_{k+1}\right)(\bar{t}_i)^{-\frac{3}{4}} u_i = 0,$$

which implies (3.5).

Thus, our results will be proved if we show the global maximum point of Ψ is unique. Indeed, by Lemma 3.1, if $(u_1, \cdots, u_{k+1}) \in \mathcal{N}_k^{\mathbf{r}_k}$, one has

$$J_{b,\lambda}(u_1, \cdots, u_{k+1}) \geq J_{b,\lambda}(t_1^{\frac{1}{4}} u_1, \cdots, t_{k+1}^{\frac{1}{4}} u_{k+1}) + (1-\theta)\sum_{i=1}^{k+1} \frac{\left(t_i^{1/2} - 1\right)^2}{4}\|u_i\|_i^2. \tag{3.7}$$

Suppose on the contrary that there exists another maximum point $(s_1, \cdots, s_{k+1})$ of $\Psi^{\mathbf{u}}$ and $(s_1, \cdots, s_{k+1}) \neq (1, \cdots, 1)$, Choosing $t_i = \frac{1}{s_i}$ for all $1 \leq i \leq k+1$ and by virtue of (3.7), we have

$$\begin{aligned} J_{b,\lambda}(u_1, \cdots, u_{k+1}) &> J_{b,\lambda}(s_1^{\frac{1}{4}} u_1, \cdots, s_{k+1}^{\frac{1}{4}} u_{k+1}) \\ &> J_{b,\lambda}(t_1^{\frac{1}{4}} s_1^{\frac{1}{4}} u_1, \cdots, t_{k+1}^{\frac{1}{4}} s_{k+1}^{\frac{1}{4}} u_{k+1}) = J_{b,\lambda}(u_1, \cdots, \quad u_{k+1}), \end{aligned}$$

which is impossible. This completes the proof of Lemma 3.2. □

Recall the definition of $d(\boldsymbol{r}_k)$ in (2.5), and then we have the following result.

Lemma 3.3. *For each $\boldsymbol{r}_k \in \Gamma_k$, there is a minimizer $\left(u_1^{\boldsymbol{r}_k}, \cdots, u_{k+1}^{\boldsymbol{r}_k}\right) \in \mathcal{N}_k^{\boldsymbol{r}_k}$ with $(-1)^{i+1} u_i^{\boldsymbol{r}_k} > 0$ in $B_i^{\boldsymbol{r}_k}$ for $i = 1, \cdots, k+1$ satisfying*

$$J_{b,\lambda}\left(u_1^{\boldsymbol{r}_k}, \cdots, u_{k+1}^{\boldsymbol{r}_k}\right) = d\left(\boldsymbol{r}_k\right).$$

Proof. For $(u_1, \cdots, u_{k+1}) \in \mathcal{N}_k^{\boldsymbol{r}_k}$, denoted by $u = \sum_{i=1}^{k+1} u_i$. From Corollary 2.1 , we deduce that

$$\begin{aligned} J_{b,\lambda}\left(u_1, \cdots, u_{k+1}\right) &= I_{b,\lambda}\left(\sum_{i=1}^{k+1} u_i\right) = I_{b,\lambda}(u) - \frac{1}{4} I'_\lambda(u) u \\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} K(|x| f(u) u - K(|x|) F(u)\right) dx \\ &\geq \frac{1}{4}\|u\|^2 - \frac{\theta V_0}{4} \int_{\mathbb{R}^3} u^2 dx \geq \frac{1-\theta}{4}\|u\|^2. \end{aligned} \tag{3.8}$$

Suppose that $\left\{\left(u_1^n, \cdots, u_{k+1}^n\right)\right\}_{n=1}^{\infty} \subset \mathcal{N}_k^{\boldsymbol{r}_k}$ is a minimizing sequence of $J_{b,\lambda}\big|_{\mathcal{N}_k}$, whose existence and boundedness are guaranteed by (3.8). Then, up to a subsequence, which we still denote by $\left(u_1^n, \cdots, u_{k+1}^n\right)$, it converges to an element $\left(u_1^0, \cdots, u_{k+1}^0\right)$ weakly in $\mathcal{H}^{\boldsymbol{r}_k}$.

From now on the proof will be divided into several steps.

Step 1. We proof that $u_i^0 \neq 0$ for all i. In fact, we have two cases: Either u_i^n convergence to u_i^0 strongly in $\mathcal{H}_i^{\boldsymbol{r}_k}$ or it converges to u_i^0 weakly but not strongly in $\mathcal{H}_i^{\boldsymbol{r}_k}$. In the former case, since $\left(u_1^n, \cdots, u_{k+1}^n\right) \in \mathcal{N}_k^{\boldsymbol{r}_k}$, then by (**K**), (**F1**), (**F2**) and the Sobolev embedding theorem, it follows that

$$\begin{aligned} \left\|u_i^n\right\|_i^2 + b\sum_{j=1}^{k+1} A(u_j^n, u_i^n) + \lambda \sum_{j=1}^{k+1} B(u_j^n, u_i^n) &= \int_{\mathbb{R}^3} K(|x|) f\left(u_i^n\right) u_i^n dx \\ &\leq \left(\varepsilon \int_{\mathbb{R}^3} \left|u_i^n\right|^2 dx + C(\varepsilon) \int_{\mathbb{R}^{N3}} \left|u_i^n\right|^p dx\right) \\ &\leq \frac{1}{2}\left\|u_i^n\right\|_i^2 + C\left\|u_i^n\right\|_i^p, \end{aligned} \tag{3.9}$$

which implies $\|u_i^n\|_i \geq C_0$. Consequently, u_i^0 is not zero. In the latter case, we have $\|u_i^0\|_i < \liminf_{n\to\infty} \|u_i^n\|_i$. Besides, due to the compactly embedding $\mathcal{H}_i^{\boldsymbol{r}_k} \hookrightarrow L^q$ for $2 < q < 6$, it follows by Strauss's compactness Lemma [30] that for all $i = 1, \cdots, k+1$,

$$\int_{\mathbb{R}^3} K(|x|) F\left(u_i^n\right) dx \to \int_{\mathbb{R}^3} K(|x|) F\left(u_i^0\right) dx \tag{3.10}$$

and

$$\int_{\mathbb{R}^3} K(|x|)f\left(u_i^n\right)u_i^n dx \to \int_{\mathbb{R}^3} K(|x|)f\left(u_i^0\right)u_i^0 dx. \tag{3.11}$$

Therefore, we deduce that

$$\left\|u_i^0\right\|_i^2 < \int_{\mathbb{R}^3} K(|x|)f(u_i^0)u_i^0 dx \le \frac{1}{2}\left\|u_i^0\right\|_i^2 + C\left\|u_i^0\right\|_i^p, \tag{3.12}$$

and we also have u_i^0 is not zero.

Step 2. We claim that $\left(u_1^n,\ldots,u_{k+1}^n\right) \to \left(u_1^0,\ldots,u_{k+1}^0\right)$ strongly in $\mathcal{H}^{\boldsymbol{r}_k}$. By contradiction, if this is not the case then, there exists $i \in \{1,\cdots,k+1\}$ such that $\|u_i^0\|_i < \liminf_{n\to\infty}\|u_i^n\|_i$. Namely, $\left(u_1^0,\ldots,u_{k+1}^0\right) \notin \mathcal{N}_k^{\boldsymbol{r}_k}$. Since $u_i \neq 0$ for all $i = 1,\cdots,k+1$, by virtue of Lemma 3.2, we deduce that there exists $\left(t_1^0,\ldots,t_{k+1}^0\right) \neq (1,\ldots,1)$ satisfying $\left((t_1^0)^{\frac{1}{4}}u_1^0,\ldots,(t_{k+1}^0)^{\frac{1}{4}}u_{k+1}^0\right) \in \mathcal{N}_k^{\boldsymbol{r}_k}$ and

$$\begin{aligned}
d(\boldsymbol{r}_k) &= \inf_{(u_1,\cdots,u_{k+1})\in\mathcal{N}_k^{\boldsymbol{r}_k}} J_{b,\lambda}\left(u_1,\cdots,u_{k+1}\right)\\
&\le J_{b,\lambda}\left((t_1^0)^{\frac{1}{4}}u_1^0,\ldots,(t_{k+1}^0)^{\frac{1}{4}}u_{k+1}^0\right)\\
&= \frac{1}{2}\sum_{i=1}^{k+1}\left(t_i^0\right)^{\frac{1}{2}}\left\|u_i^0\right\|_i^2 + \frac{b}{4}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}(t_i^0)^{\frac{1}{2}}(t_j^0)^{\frac{1}{2}}A(u_i,u_j) + \frac{\lambda}{4}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}(t_i^0)^{\frac{1}{2}}(t_j^0)^{\frac{1}{2}}B(u_i,u_j)\\
&\quad - \sum_{i=1}^{k+1}\int_{B_i^{r_k}} K(|x|)F((t_i^0)^{\frac{1}{4}}u_i^0)dx\\
&< \frac{1}{2}\sum_{i=1}^{k+1}\left(\left(t_i^0\right)^{\frac{1}{2}}\liminf_{n\to\infty}\left\|u_i^n\right\|_i^2\right) + \frac{b}{4}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}(t_i^0)^{\frac{1}{2}}(t_j^0)^{\frac{1}{2}}A(u_i^n,u_j^n)\\
&\quad + \frac{\lambda}{4}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}(t_i^0)^{\frac{1}{2}}(t_j^0)^{\frac{1}{2}}B(u_i^n,u_j^n) - \sum_{i=1}^{k+1}\int_{B_i^{r_k}} K(|x|)F((t_i^0)^{\frac{1}{4}}u_i^n)dx\\
&\le \liminf_{n\to\infty} J_{b,\lambda}\left(u_1^n,\cdots,u_{k+1}^n\right)\\
&= d(\boldsymbol{r}_k),
\end{aligned}$$

which is impossible. Thus, $\left(u_1^0,\cdots,u_{k+1}^0\right)$ is contained in $\mathcal{N}_k^{\boldsymbol{r}_k}$ and a minimizer of $J_{b,\lambda}\big|_{\mathcal{N}_k^{r_k}}$.

Step 3. Letting

$$\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right) := \left(\left|u_1^0\right|, -\left|u_2^0\right|,\cdots,(-1)^{k+2}\left|u_{k+1}^0\right|\right),$$

we can easily check that $\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right)$ is also a minimizer of $J_{b,\lambda}\big|_{\mathcal{N}_k^{r_k}}$. Using the strong maximum principle, we get $(-1)^{i+1}u_i^{\boldsymbol{r}_k} > 0$ in $B_i^{\boldsymbol{r}_k}$ for all $i = 1,\ldots,k+1$. The proof is completed. □

In the following, we will prove the minimizer of $J_{b,\lambda}\big|_{\mathcal{N}_k^{r_k}}$, which obtained by the previous Lemma, is actually a critical point of Functional $J_{b,\lambda}$. However, we cannot use the Nehari method in [19] directly. More precisely, since the assumption (**F1**), it seems difficult to prove $\mathcal{N}_k^{\boldsymbol{r}_k}$ is a manifold as in [19]. Thus, we use the deformation lemma and Brouwer degree theory to achieve this. This idea comes from [12,33].

In what follows, for convenience, we denote by

$$\mathbf{u}=(u_1,\cdots,u_{k+1})\in\mathcal{H}^{\boldsymbol{r}_k},\ \mathbf{t}=(t_1,\cdots,t_{k+1})\in\mathbb{R}^{k+1},\ \mathbf{t}^{1/4}\bullet\mathbf{u}=\left(t_1^{1/4}u_1,\cdots,t_{k+1}^{1/4}u_{k+1}\right).$$

Lemma 3.4. *If $\boldsymbol{r}_k\in\Gamma$ and $\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right)\in\mathcal{N}_k^{\boldsymbol{r}_k}$ is a minimizer of $J_{b,\lambda}|_{\mathcal{N}_k}$ such that*

$$J_{b,\lambda}\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right)=d\left(\boldsymbol{r}_k\right),$$

then $\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right)$ is a critical point of $J_{b,\lambda}$ in $\mathcal{H}_k^{\boldsymbol{r}_k}$.

Proof. Suppose on contrary, if $\mathbf{u}^{\boldsymbol{r}_k}=\left(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}\right)$ is not a critical point of $J_{b,\lambda}$, then there exist $\delta>0$ and $\varrho>0$ such that

$$\mathbf{u}\in\mathcal{H}^{\boldsymbol{r}_k},\|\mathbf{u}-\mathbf{u}^{\boldsymbol{r}_k}\|\le 3\delta\Rightarrow\left\|(\partial_{u_1}J_{b,\lambda}(\mathbf{u}),\cdots,\partial_{u_{k+1}}J_{b,\lambda}(\mathbf{u}))\right\|\ge\varrho.$$

Let $D=\left\{\mathbf{t}\in\mathbb{R}_{\ge0}^{k+1}:\left|t_i^{1/4}-1\right|<(\frac{1}{2})^{1/4},\ \forall i=1,\cdots,k+1\right\}$. Then D is an open neighborhood of $\mathbf{1}:=(1,\cdots,1)\in\mathbb{R}^{k+1}$. It follows from Lemma 3.2 that

$$\kappa:=\max_{\partial D}J_{b,\lambda}\left(\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)<d\left(\boldsymbol{r}_k\right). \tag{3.13}$$

For $\varepsilon:=\min\left\{(m_b-\kappa)/3,1,\varrho\delta/8\right\},S:=B\left(\mathbf{u}^{\boldsymbol{r}_k},\delta\right)$, [36, Lemma 2.3] yields a deformation $\eta=(\eta_1,\cdots,\eta_{k+1})\in C\left([0,1]\times\mathcal{H}^{\boldsymbol{r}_k},\mathcal{H}^{\boldsymbol{r}_k}\right)$ such that

(i) $\eta(1,\mathbf{u})=\mathbf{u}$ if $\mathbf{u}\notin J_{b,\lambda}^{-1}\left([d(\boldsymbol{r}_k)-2\varepsilon,d(\boldsymbol{r}_k)+2\varepsilon]\right)\cap S_{2\delta}$;
(ii) $\eta\left(1,J_{b,\lambda}^{d(\boldsymbol{r}_k)+\varepsilon}\cap S\right)\subset J_{b,\lambda}^{d(\boldsymbol{r}_k)-\varepsilon}$;
(iii) $J_{b,\lambda}(\eta(1,\mathbf{u}))\le J_{b,\lambda}(\mathbf{u}),\forall\mathbf{u}\in\mathcal{H}^{\boldsymbol{r}_k}$.

By Lemma 3.2,

$$J_{b,\lambda}\left(\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)\le J_{b,\lambda}\left(\mathbf{u}^{\boldsymbol{r}_k}\right)=d(\boldsymbol{r}_k).$$

Then it follows from (ii) that

$$J_{b,\lambda}\left(\eta\left(1,\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)\right)<d(\boldsymbol{r}_k)-\varepsilon,\quad\forall\mathbf{t}\in(\mathbb{R}_{\ge0})^{k+1},\ \sum_{i=1}^{k+1}|t_i^{\frac{1}{4}}-1|<\delta/\|\mathbf{u}^{\boldsymbol{r}_k}\|. \tag{3.14}$$

On the other hand, by (iii) and (3.7), one has

$$J_{b,\lambda}\left(\eta\left(1,\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)\right)\le J_{b,\lambda}\left(\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)<d(\boldsymbol{r}_k),\quad\forall\mathbf{t}\in(\mathbb{R}_{\ge0})^{k+1},\ \sum_{i=1}^{k+1}|t_i^{\frac{1}{4}}-1|\ge\delta/\|\mathbf{u}^{\boldsymbol{r}_k}\|. \tag{3.15}$$

Combining (3.14) with (3.15), we deduce that

$$\max_{\mathbf{t}\in D}J_{b,\lambda}\left(\eta\left(1,\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right)\right)<d(\boldsymbol{r}_k). \tag{3.16}$$

Now we claim that

$$\left\{\eta\left(1,\mathbf{t}^{1/4}\bullet\mathbf{u}^{\boldsymbol{r}_k}\right):\mathbf{t}\in D\right\}\cap\mathcal{N}_k^{\boldsymbol{r}_k}\ne\emptyset. \tag{3.17}$$

In fact, we denote $\mathbf{\Phi} = (\Phi_1, \cdots, \Phi_{k+1}) : D \to \mathbb{R}^{k+1}$ by

$$\Phi_i(\mathbf{t}) := \partial_{u_i} E\left(\eta\left(1, \mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}\right)\right)\eta_i\left(1, \mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}\right), \quad \text{for } i = 1, \cdots, k+1.$$

Then for all $\mathbf{t} \in \partial D$, the properties of deformation lemma implies $\eta\left(1, \mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}\right) = \mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}$ and $\Phi_i(\mathbf{t}) := \partial_{u_i} E\left(\mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}\right) t_i^{1/4} u_i^{\boldsymbol{r}_k}$ for all $i = 1, \cdots, k+1$. On the other hand, by virtue of Lemma 3.2 and Brouwer degree theory, it follows that

$$\deg\left(\left(\frac{\partial \Psi^{\mathbf{u}^{\boldsymbol{r}_k}}}{\partial t_1}, \cdots, \frac{\partial \Psi^{\mathbf{u}^{\boldsymbol{r}_k}}}{\partial t_{k+1}}\right), D, \mathbf{0}\right) = 1,$$

where $\Psi^{\mathbf{u}^{\boldsymbol{r}_k}}$ has been defined in (3.3). Therefore, for all $\mathbf{t} \in \partial D$ we have

$$\begin{aligned} 1 &= \deg\left(\left(\frac{\partial \Psi^{\mathbf{u}^{\boldsymbol{r}_k}}}{\partial t_1}, \cdots, \frac{\partial \Psi^{\mathbf{u}^{\boldsymbol{r}_k}}}{\partial t_{k+1}}\right), D, \mathbf{0}\right) \\ &= \deg\left(\left(\partial_{u_1} J_{b,\lambda}\left(u^{\boldsymbol{r}_k}\right) \frac{1}{4} t_1^{-3/4} u_1, \cdots, \partial_{u_{k+1}} J_{b,\lambda}\left(u^{\boldsymbol{r}_k}\right) \frac{1}{4} t_{k+1}^{-3/4} u_{k+1}\right), D, \mathbf{0}\right) \\ &= \deg\left(\left(\frac{1}{4t_1}\Phi_i(\mathbf{t}), \cdots, \frac{1}{4t_{k+1}}\Phi_{k+1}(\mathbf{t})\right), D, 0\right), \end{aligned}$$

which shows the correctness of the (3.17). Consequently, we deduce that

$$\sup_{\mathbf{t} \in D} J_{b,\lambda}\left(\eta\left(1, \mathbf{t}^{1/4} \bullet \mathbf{u}^{\boldsymbol{r}_k}\right)\right) \geq d(\boldsymbol{r}_k), \tag{3.18}$$

which leads to a contradiction with (3.16). Hence, $\mathbf{u}^{\boldsymbol{r}_k} = \left(u_1^{\boldsymbol{r}_k}, \cdots, u_{k+1}^{\boldsymbol{r}_k}\right)$ is a critical point of $J_{b,\lambda}$ and the proof is finished. □

4. Existence of the radial sign-changing solutions

Recall the infimum level $d(\boldsymbol{r}_k)$ defined in (2.5), then we have the following results.

Lemma 4.1. *For any $p \in (4, 6)$ and $\boldsymbol{r}_k = (r_1, \ldots, r_k) \in \Gamma_k$*

(i) if $r_i - r_{i-1} \to 0$ for some $i \in \{1, \ldots, k\}$, then $d(\boldsymbol{r}_k) \to +\infty$;
(ii) if $r_k \to \infty$, then $d(\boldsymbol{r}_k) \to +\infty$;
(iii) d is continuous in Γ_k. As a consequence, there exists a $\overline{\boldsymbol{r}}_k \in \Gamma_k$ such that

$$d(\overline{\boldsymbol{r}}_k) = \inf_{\boldsymbol{r}_k \in \Gamma_k} d(\boldsymbol{r}_k).$$

Proof. (i) In view of Lemma 3.3, combining the Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} \left\|u_i^{\boldsymbol{r}_k}\right\|_i^2 &= \int_{\mathbb{R}^3} K(|x|) f\left(u_i^{\boldsymbol{r}_k}\right) u_i^{\boldsymbol{r}_k} dx - \lambda \sum_{i,j=1}^{k+1} \int_{\mathbb{R}^3} \int_{B_i^{\boldsymbol{r}_k}} \frac{|u_i^{\boldsymbol{r}_k}(y)|^2 |u_j^{\boldsymbol{r}_k}(x)|^2}{4\pi|x-y|} dx dy \\ &\quad - b \int_{\mathbb{R}^3} \sum_{j=1}^{k+1} \left|\nabla u_j^{\boldsymbol{r}_k}(x)\right|^2 dx \int_{B_i^{\boldsymbol{r}_k}} \left|\nabla u_i^{\boldsymbol{r}_k}\right|^2 dx \\ &\leq \varepsilon \int_{\mathbb{R}^3} \left|u_i^{\boldsymbol{r}_k}\right|^2 dx + C(\varepsilon) \int_{\mathbb{R}^3} \left|u_i^{\boldsymbol{r}_k}\right|^p dx \\ &\leq C\varepsilon \left\|u_i^{\boldsymbol{r}_k}\right\|_i^2 + CC(\varepsilon) \left\|u_i^{\boldsymbol{r}_k}\right\|_i^p \left|B_i^{\boldsymbol{r}_k}\right|^{\frac{6-p}{6}}. \end{aligned}$$

If $r_i - r_{i-1} \to 0$, then $\left\|\sum_{j=1}^{k+1} u_j^{\boldsymbol{r}_k}(x)\right\| \to +\infty$ due to $p \in (4,6)$. Thus,

$$d(\boldsymbol{r}_k) = J_{b,\lambda}\left(u_1^{\boldsymbol{r}_k}, \ldots, u_{k+1}^{\boldsymbol{r}_k}\right) = I_{b,\lambda}\left(\sum_{j=1}^{k+1} u_j^{\boldsymbol{r}_k}\right) \geq \frac{1-\theta}{4}\left\|\sum_{i=1}^{k+1} u_i^{\boldsymbol{r}_k}\right\|^2 \to +\infty, \tag{4.1}$$

then, (i) holds.

(ii) Recalling the Strauss inequality [26], we can find $a_0 > 0$ such that, for all radial function $u \in H_V$,

$$u(x) \leq \frac{a_0\|u\|}{|x|},$$

for a.e. $|x| > 1$. so we have

$$\begin{aligned}
\left\|u_{k+1}^{\boldsymbol{r}_k}\right\|_{k+1}^2 &\leqslant \int_{\mathbb{R}^3} K(|x|) f\left(u_i^{\boldsymbol{r}_k}\right) u_i^{\boldsymbol{r}_k} dx \\
&\leq \varepsilon \int_{\mathbb{R}^3} \left|u_i^{\boldsymbol{r}_k}\right|^2 dx + C(\varepsilon) \int_{\mathbb{R}^3} \left|u_i^{\boldsymbol{r}_k}\right|^p dx \\
&\leq C\varepsilon \left\|u_{k+1}^{\boldsymbol{r}_k}\right\|_{k+1}^2 + CC(\varepsilon)\|u_{k+1}^{\boldsymbol{r}_k}\|_{k+1}^{p-2} \int_{B_{k+1}^{r_k}} |x|^{2-p}|u_{k+1}^{\boldsymbol{r}_k}|^2 dx \\
&\leq C\varepsilon \left\|u_{k+1}^{\boldsymbol{r}_k}\right\|_{k+1}^2 + CC(\varepsilon)|r_k|^{2-p}\|u_{k+1}^{\boldsymbol{r}_k}\|_{k+1}^{p-2} \int_{B_{k+1}^{r_k}} |u_{k+1}^{\boldsymbol{r}_k}|^2 dx \\
&\leq C\varepsilon \left\|u_{k+1}^{\boldsymbol{r}_k}\right\|_{k+1}^2 + CC(\varepsilon)|r_k|^{2-p}\|u_{k+1}^{\boldsymbol{r}_k}\|_{k+1}^{p}
\end{aligned} \tag{4.2}$$

which yields that $\left\|\sum_{j=1}^{k+1} u_j^{\boldsymbol{r}_k}(x)\right\| \to +\infty$ as $r_k \to \infty$ due to $p \in (4,6)$. Therefore,

$$d(\boldsymbol{r}_k) \geq \frac{1-\theta}{4}\left\|\sum_{i=1}^{k+1} u_i^{\boldsymbol{r}_k}\right\|^2 \to +\infty \quad \text{as } r_k \to \infty,$$

which implies (ii).

(iii) Take a sequence $\left\{\boldsymbol{r}_k^n\right\}_{n=1}^{\infty}$ satisfying $\boldsymbol{r}_k^n \to \bar{\boldsymbol{r}}_k \in \Gamma_k$. We will prove the conclusion by showing $d(\bar{\boldsymbol{r}}_k) \geq \limsup_{n\to\infty} d\left(\boldsymbol{r}_k^n\right), d(\bar{\boldsymbol{r}}_k) \leq \liminf_{n\to\infty} d\left(\boldsymbol{r}_k^n\right)$.

First, we prove that $d(\bar{\boldsymbol{r}}_k) \geq \limsup_{n\to\infty} d\left(\boldsymbol{r}_k^n\right)$. In order to emphasize that $v_i^{\boldsymbol{r}_k^n}$ is radial in $B_i^{\boldsymbol{r}_k^n}$, we will rewrite $v_i^{\boldsymbol{r}_k^n}(|x|) = v_i^{\boldsymbol{r}_k^n}(r)$. Define $v_i^{\boldsymbol{r}_k^n} : \left[r_{i-1}^n, r_i^n\right] \to \mathbb{R}$ by

$$v_i^{\boldsymbol{r}_k^n}(r) = \begin{cases} t_i^n u_i^{\bar{\boldsymbol{r}}_k}\left(\bar{r}_{i-1} + \frac{\bar{r}_i - \bar{r}_{i-1}^n}{r_i^n - r_{i-1}^n}\left(r - r_{i-1}^n\right)\right), & i = 1, \ldots, k, \\ t_{k+1}^n u_{k+1}^{\bar{\boldsymbol{r}}_k}\left(\frac{\bar{r}_k}{r_k^n} r\right), & i = k+1, \end{cases}$$

where $\left(u_1^{\boldsymbol{r}_k^n}, \ldots, u_{k+1}^{\boldsymbol{r}_k^n}\right)$ and $\left(u_1^{\bar{\boldsymbol{r}}_k}, \ldots, u_{k+1}^{\bar{\boldsymbol{r}}_k}\right)$ are minimizers of $J_{b,\lambda}\big|_{\mathcal{N}_k^{\boldsymbol{r}_k^n}}$ and $J_{b,\lambda}\big|_{\mathcal{N}_k^{\bar{\boldsymbol{r}}_k}}$ respectively. $\left(t_1^n, \ldots, t_{k+1}^n\right)$ is the unique $(k+1)$ tuple of positive numbers such that $\left(v_1^{\boldsymbol{r}_k^n}, \ldots, v_{k+1}^{\boldsymbol{r}_k^n}\right) \in \mathcal{N}_k^{\boldsymbol{r}_k^n}$. By the definition of $\left(u_1^{\boldsymbol{r}_k^n}, \ldots, u_{k+1}^{\boldsymbol{r}_k^n}\right)$, we know that

$$J_{b,\lambda}(v_1^{\boldsymbol{r}_k^n}, \ldots, v_{k+1}^{\boldsymbol{r}_k^n}) \geq J_{b,\lambda}(u_1^{\boldsymbol{r}_k^n}, \ldots, u_{k+1}^{\boldsymbol{r}_k^n}) = d(\boldsymbol{r}_k^n). \tag{4.3}$$

Since $\boldsymbol{r}_k^n \to \bar{\boldsymbol{r}}_k \in \Gamma_k$, by a straightforward computation, one can easily get the following equations,

$$
\begin{aligned}
&\int_{B_i^{\boldsymbol{r}_k^n}} \left|v_i^{\boldsymbol{r}_k^n}\right|^2 dx = (t_i^n)^2 \int_{B_i^{\bar{\boldsymbol{r}}_k}} \left|u_i^{\bar{\boldsymbol{r}}_k}\right|^2 dx + o_n(1);\\
&\left\|v_i^{\boldsymbol{r}_k^n}\right\|_i^2 = (t_i^n)^2 \left\|u_i^{\bar{\boldsymbol{r}}_k}\right\|_i^2 + o_n(1);\\
&\int_{B_i^{\boldsymbol{r}_k^n}} f(v_i^{\boldsymbol{r}_k^n}) v_i^{\boldsymbol{r}_k^n} dx = \int_{B_i^{\bar{\boldsymbol{r}}_k}} f(t_i^n u_i^{\bar{\boldsymbol{r}}_k}) t_i^n u_i^{\bar{\boldsymbol{r}}_k} dx + o_n(1);\\
&\int_{B_i^{\boldsymbol{r}_k^n}} \int_{B_i^{\boldsymbol{r}_k^n}} \frac{v_i^{\boldsymbol{r}_k^n}(x) v_i^{\boldsymbol{r}_k^n}(y)}{4\pi|x-y|} dxdy = (t_i^n)^4 \int_{B_i^{\bar{\boldsymbol{r}}_k}} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \frac{u_i^{\bar{\boldsymbol{r}}_k}(x) u_i^{\bar{\boldsymbol{r}}_k}(y)}{4\pi|x-y|} dxdy + o_n(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{B_i^{\boldsymbol{r}_k^n}} \left|\nabla v_i^{\boldsymbol{r}_k^n}(x)\right|^2 dx \int_{B_j^{\boldsymbol{r}_k^n}} \left|\nabla v_j^{\boldsymbol{r}_k^n}(x)\right|^2 dx\\
&\qquad = (t_i^n)^2 \left(t_j^n\right)^2 \int_{B_i^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_i^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx \int_{B_i^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_j^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx + o_n(1).
\end{aligned}
$$

According to $\left(v_1^{\boldsymbol{r}_k^n}, \ldots, v_{k+1}^{\boldsymbol{r}_k^n}\right) \in \mathcal{N}_k^{\boldsymbol{r}_k^n}$ and $\left(u_1^{\bar{\boldsymbol{r}}_k}, \ldots, u_{k+1}^{\bar{\boldsymbol{r}}_k}\right) \in \mathcal{N}_k^{\bar{\boldsymbol{r}}_k^n}$, there holds that

$$
\begin{aligned}
&\left\|u_i^{\bar{\boldsymbol{r}}_k}\right\|_i^2 + b \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_i^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx \int_{B_j^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_j^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx\\
&\qquad + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \frac{u_i^{\bar{\boldsymbol{r}}_k}(x) u_i^{\bar{\boldsymbol{r}}_k}(y)}{4\pi|x-y|} dxdy - \int_{B_i^{\bar{\boldsymbol{r}}_k}} K(|x|) f\left(u_i^{\bar{\boldsymbol{r}}_k}\right) u_i^{\bar{\boldsymbol{r}}_k} dx = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(t_i^n\right)^2 \|u_i^{\bar{\boldsymbol{r}}_k}\|_i^2 + b(t_i^n)^2 (t_j^n)^2 \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_i^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx \int_{B_j^{\bar{\boldsymbol{r}}_k}} \left|\nabla u_j^{\bar{\boldsymbol{r}}_k}(x)\right|^2 dx\\
&\quad + \lambda (t_i^n)^4 \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \int_{B_i^{\bar{\boldsymbol{r}}_k}} \frac{u_i^{\bar{\boldsymbol{r}}_k}(x) u_i^{\bar{\boldsymbol{r}}_k}(y)}{4\pi|x-y|} dxdy - \int_{B_i^{\bar{\boldsymbol{r}}_k}} K(|x|) f\left(t_i^n u_i^{\bar{\boldsymbol{r}}_k}\right) t_i^n u_i^{\bar{\boldsymbol{r}}_k} dx = o_n(1).
\end{aligned}
$$

This combined with Lemma 3.2 we have $\lim_{n\to\infty} t_i^n = 1$ for all i. Hence, from (4.3) we can see that

$$
d\left(\bar{\boldsymbol{r}}_k\right) = J_{b,\lambda}\left(u_1^{\bar{\boldsymbol{r}}_k}, \ldots, u_{k+1}^{\bar{\boldsymbol{r}}_k}\right) = \limsup_{n\to\infty} J_{b,\lambda}\left(v_1^{\boldsymbol{r}_k^n}, \ldots, v_{k+1}^{\boldsymbol{r}_k^n}\right) \geq \limsup_{n\to\infty} d\left(\boldsymbol{r}_k^n\right). \tag{4.4}
$$

Next, we prove that $d\left(\bar{\boldsymbol{r}}_k\right) \leq \liminf_{n\to\infty} d\left(\boldsymbol{r}_k^n\right)$. By the same argument as former case, let $w_i^{\boldsymbol{r}_k^n} = [\bar{r}_{i-1}, \bar{r}_i] \to \mathbb{R}$ be defined by

$$
w_i^{\boldsymbol{r}_k^n}(r) = \begin{cases} s_i^n u_i^{\boldsymbol{r}_k^n}\left(r_{i-1}^n + \frac{r_i^n - r_{i-1}^n}{\bar{r}_i - \bar{r}_{i-1}}(r - \bar{r}_{i-1})\right), & \text{if} \quad i = 1, \ldots, k,\\ s_{k+1}^n u_{k+1}^{\boldsymbol{r}_k^n}\left(\frac{r_k^n}{\bar{r}_k} r\right), & \text{if } i = k+1, \end{cases} \tag{4.5}
$$

where $\left(s_1^n,\dots,s_{k+1}^n\right)\in(\mathbb{R}_+)^{k+1}$ such that $\left(w_1^{\boldsymbol{r}_k^n},\dots,w_{k+1}^{\boldsymbol{r}_k^n}\right)\in\mathcal{N}_k^{\bar{\boldsymbol{r}}_k}$. By the same arguments, we can deduce $s_i^n\to 1$ as $n\to\infty$ for all $i=1,\dots,k+1$. Hence,

$$\begin{aligned} d\left(\bar{\boldsymbol{r}}_k\right) &= J_{b,\lambda}\left(u_1^{\bar{\boldsymbol{r}}_k},\dots,u_{k+1}^{\bar{\boldsymbol{r}}_k}\right)\\ &\le \liminf_{n\to\infty} J_{b,\lambda}\left(w_1^{\boldsymbol{r}_k^n},\dots,w_{k+1}^{\boldsymbol{r}_k^n}\right)=\liminf_{n\to\infty} J_{b,\lambda}\left(u_1^{\boldsymbol{r}_k^n},\dots,u_{k+1}^{\boldsymbol{r}_k^n}\right)=\liminf_{n\to\infty} d\left(\boldsymbol{r}_k^n\right). \end{aligned}$$

This combined with (4.4) yields that d is continuous in Γ_k. Furthermore, taking account into $(i),(ii)$, we know that there is a $\bar{\boldsymbol{r}}_k\in\Gamma_k$ such that $d\left(\bar{\boldsymbol{r}}_k\right)=\inf_{\boldsymbol{r}_k\in\Gamma_k} d\left(\boldsymbol{r}_k\right)$. Hence, (iii) holds. □

Proof of Theorem 1.1. By Lemmas 3.3 and 4.1, we deduce that there exists $(\bar{\boldsymbol{r}}_k)\in\Gamma_k$ and $\mathbf{u}^{\bar{\boldsymbol{r}}_k}=\left(u_1^{\bar{\boldsymbol{r}}_k},\cdots,u_{k+1}^{\bar{\boldsymbol{r}}_k}\right)\in\mathcal{N}_k^{\bar{\boldsymbol{r}}_k}$ with $(-1)^{i+1}u_i^{\bar{\boldsymbol{r}}_k}>0$ in $B_i^{\bar{\boldsymbol{r}}_k}$ such that

$$J_{b,\lambda}\left(u_1^{\bar{\boldsymbol{r}}_k},\dots,u_{k+1}^{\bar{\boldsymbol{r}}_k}\right)=d\left(\bar{\boldsymbol{r}}_k\right)=\inf_{\boldsymbol{r}_k\in\Gamma_k} d\left(\boldsymbol{r}_k\right),$$

which implies

$$c_k=d\left(\bar{\boldsymbol{r}}_k\right)=I_{b,\lambda}\left(\sum_{i=1}^{k+1}u_i^{\bar{\boldsymbol{r}}_k}\right),$$

where c_k has been defined in (1.8). Now, we claim that $u_k=\sum_{i=1}^{k+1}u_i^{\bar{\boldsymbol{r}}_k}$ is solution of (1.1). Suppose to the contrary that the claim does not hold, then by density argument, there exists a radial function $\psi\in C_0^\infty(\mathbb{R}^3)$ such that

$$I'_{b,\lambda}(u_k)\psi=-2.$$

We denote a function $g\in C\left(\mathbb{R}^{k+1}\times\mathbb{R};H\right)$ by

$$g(\mathbf{t},\varepsilon):=\sum_{i=1}^{k+1}t_iu_i^{\bar{\boldsymbol{r}}_k}+\varepsilon\psi.$$

Since $\sum_{i=1}^{k+1}u_i^{\bar{\boldsymbol{r}}_k}$ is continuous and has exactly k nodes, take into account that g is also continuous, we deduce that there exists $\tau>0$ small enough, such that $g(\mathbf{t},\varepsilon)$ also changes sign k times and

$$I'_{b,\lambda}(g(\mathbf{t},\varepsilon))\psi<-1,\quad \forall(\mathbf{t},\varepsilon))\in D_\tau\times[0,\tau], \tag{4.6}$$

where $D_\tau:=\left\{\mathbf{t}=(t_1,\cdots,t_{k+1})\in\mathbb{R}^{k+1}:|t_i-1|<\tau \text{ for all } 1\le i\le k+1\right\}$.

Now we choose $\eta\in C^\infty(\mathbb{R}^3)$, $0\le\eta\le 1$ with $\eta(\mathbf{t})=1$ if $\mathbf{t}\in\overline{D_{\frac{\tau}{4}}}$ and $\eta(\mathbf{t})=0$ if $\mathbf{t}\notin D_{\frac{\tau}{2}}$. Furthermore, we define another continuous function $\tilde{g}:\mathbb{R}^{k+1}\to H$ by

$$\tilde{g}(\mathbf{t}):=\sum_{i=1}^{k+1}t_iu_i^{\bar{\boldsymbol{r}}_k}+\tau\eta(\mathbf{t})\psi.$$

Similarly, for all $\mathbf{t}\in D_\tau$, $\tilde{g}(\mathbf{t})$ also changes sign k times and has k nodes $0<r_1(\mathbf{t})<\cdots<r_k(\mathbf{t})<+\infty$.

We assert that

$$\exists\,\bar{\mathbf{t}}\in D_\tau \text{ such that } \tilde{g}(\bar{\mathbf{t}})\in\mathcal{N}_k. \tag{4.7}$$

If the assertion holds, then

$$I_{b,\lambda}(\tilde{g}(\bar{\mathbf{t}})) \geq c_k. \tag{4.8}$$

However, if $\bar{\mathbf{t}} \in \overline{D_{\frac{\tau}{2}}}$, then $\eta(\bar{\mathbf{t}}) > 0$. This combines with (4.6) implies that

$$\begin{aligned} I_{b,\lambda}(\tilde{g}(\bar{\mathbf{t}})) &= I_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{t}_i u_i^{\bar{r}_k}\right) + \int_0^1 \left\langle I'_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{t}_i u_i^{\bar{r}_k} + \mu\tau\eta(\bar{\mathbf{t}})\psi\right), \tau\eta(\bar{\mathbf{t}})\psi\right\rangle d\mu \\ &\leq I_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{t}_i u_i^{\bar{r}_k}\right) - \tau\eta(\bar{\mathbf{t}}) \\ &< c_k. \end{aligned} \tag{4.9}$$

On the other hand, if $\bar{\mathbf{t}} \notin \overline{D_{\frac{\tau}{2}}}$, then $\eta(\bar{\mathbf{t}}) = 0$, it follows from Lemma 3.2 that

$$I_{b,\lambda}(\tilde{g}(\bar{\mathbf{t}})) = I_{b,\lambda}\left(\sum_{i=1}^{k+1} (\bar{t}_i) u_i^{\bar{r}_k}\right) < I_{b,\lambda}\left(\sum_{i=1}^{k+1} u_i^{\bar{r}_k}\right) = c_k. \tag{4.10}$$

Consequently, (4.9) and (4.10) lead to a contradiction with (4.8). Therefore, it is enough to prove (4.7). Indeed, we denote by $\Omega(\sigma_i(\mathbf{t}), \sigma_{i+1}(\mathbf{t})) = \left\{x \in \mathbb{R}^N : |x| \in (\sigma_i(\mathbf{t}), \sigma_{i+1}(\mathbf{t}))\right\}$ and define a map $H : \overline{D_\tau} \to \mathbb{R}^{k+1}$ by

$$\mathbf{H}(\mathbf{t}) = (H_1(\mathbf{t}), \cdots, H_{k+1}(\mathbf{t})) \quad \text{with components} \quad H_i(\mathbf{s}) := \left\langle J'_{b,\lambda}(\bar{g}(\mathbf{t})), \bar{g}(\mathbf{t})|_{\Omega(\sigma_i(\mathbf{t}),\sigma_{i+1}(\mathbf{t}))}\right\rangle.$$

Clearly $H \in C\left(\overline{D_\tau}, \mathbb{R}^{k+1}\right)$ and for $\mathbf{t} \in \partial D_\tau$, we have

$$\begin{aligned} H_i(\mathbf{t}) = J'_{b,\lambda}\left(\sum_{j=1}^{k+1} t_j u_i^{\bar{r}_k}\right) t_i u_i^{\bar{r}_k} &= t_i^2 \left\|u_i^{\bar{r}_k}\right\|_j^2 + b t_i^2 \sum_{j=1}^{k+1} t_j^2 A(u_i^{\bar{r}_k}, u_j^{\bar{r}_k}) + \lambda t_i^2 \sum_{j=1}^{k+1} t_j^2 B(u_i^{\bar{r}_k}, u_j^{\bar{r}_k}) \\ &\quad - \int_{\mathbb{R}^N} K(|x|) f\left(t_i u_i^{\bar{r}_k}\right) t_i u_i^{\bar{r}_k}. \end{aligned} \tag{4.11}$$

Since for small $\tau > 0$,

$$H_i(1-\tau, \cdots, 1-\tau) > 0 \text{ and } H_i(1+\tau, \cdots, 1+\tau) < 0.$$

Then by a straightforward computations, we deduce that

$$\begin{aligned} &H_i(t_1, \cdots, t_{i-1}, 1-\tau, t_{i+1}, \cdots, t_{k+1}) > 0 \quad \text{for all } t_j \in (1-\tau, 1+\tau),\ j \neq i, \\ &H_i(t_1, \cdots, t_{i-1}, 1+\tau, t_{i+1}, \cdots, t_{k+1}) < 0 \quad \text{for all } t_j \in (1-\tau, 1+\tau),\ j \neq i. \end{aligned}$$

Therefore, by the Miranda theorem [25], there exists $\bar{\mathbf{t}} \in D_\tau$ such that $\mathbf{H}(\bar{\mathbf{t}}) = 0$, which implies that $\tilde{g}(\bar{\mathbf{t}}) \in \mathcal{N}_k$. Thus, the assertion (4.7) is confirmed and the proof of Theorem 1.1 is completed. □

5. Energy comparison and the asymptotic behaviors of the nodal solutions

For any integer $k \geq 0$, by Theorem 1.1, we know that the problem (1.1) has a radial solution u_k which changes sign exactly k-times. Now, we are ready to prove the energy comparison property.

Proof of Theorem 1.2. By Theorem 1.1, there exists $\bar{\boldsymbol{r}}_{k+1} \in \Gamma_{k+1}$ and $u_{k+1} = \sum_{i=1}^{k+1} u_i^{\bar{\boldsymbol{r}}_{k+1}}$ such that u_{k+1} is the solution of (1.1) with $k+1$ nodes. Moreover, $(u_1^{\bar{\boldsymbol{r}}_{k+1}}, \cdots, u_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}, u_{k+2}^{\bar{\boldsymbol{r}}_{k+1}}) \in \mathcal{N}_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}$ and $I_\lambda(u_{k+1}) = d(\bar{\boldsymbol{r}}_{k+1}) = \inf_{\boldsymbol{r}_{k+1} \in \Gamma_{k+1}} d(\boldsymbol{r}_{k+1})$. Meanwhile, Lemma 3.2 implies that there exists $(a_1, \cdots, a_{k+1})$ such that

$$(a_1 u_1^{\bar{\boldsymbol{r}}_{k+1}}, \cdots, a_{k+1} u_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}) \in \mathcal{N}_k^{\bar{\boldsymbol{r}}_k}. \tag{5.1}$$

Thus, using (5.1) and Lemma 3.2 we see that

$$\begin{aligned} I_{b,\lambda}(u_k) &\leq I_{b,\lambda}\left(\sum_{i=1}^{k+1} a_i u_i^{\bar{\boldsymbol{r}}_{k+1}}\right) = J_{b,\lambda}\left(a_1 u_1^{\bar{\boldsymbol{r}}_{k+1}}, \cdots, a_{k+1} u_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}\right) \\ &= J_{b,\lambda}\left(a_1 u_1^{\bar{\boldsymbol{r}}_{k+1}}, \cdots, a_{k+1} u_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}, 0\right) < J_{b,\lambda}\left(u_1^{\bar{\boldsymbol{r}}_{k+1}}, \cdots, u_{k+1}^{\bar{\boldsymbol{r}}_{k+1}}, u_{k+2}^{\bar{\boldsymbol{r}}_{k+2}}\right) = I_{b,\lambda}(u_{k+1}). \end{aligned}$$

On the other hand, by using the Non-Nehari manifold method in [6], we deduce that there exist a ground state solution u_0 of (1.1), such that $u_0 \in \mathcal{N}$ and $I_{b,\lambda}(u_0) = c_0$. Besides, for $u_k = \sum_{i=1}^{k+1} u_i^{\bar{\boldsymbol{r}}_k}$ being a solution of (1.1) with k nodes, by Lemma 3.3 we have known $(-1)^{i+1} u_i^{\bar{\boldsymbol{r}}_k}$ is positive for each $i = 1, \cdots, k+1$. Thus, it follows from Lemma 3.2 and Corollary 4.3 in [6] that

$$\begin{aligned} c_k = I_{b,\lambda}(u_k) &= \max_{\mathbf{t} \in \mathbb{R}_{\geq 0}^{k+1}} J_{b,\lambda}(\mathbf{t} \bullet u_k) \\ &= \max_{\mathbf{t} \in \mathbb{R}_{\geq 0}^{k+1}} \left(\sum_{i=1}^{k+1} I_{b,\lambda}(t_i u_i^{\bar{\boldsymbol{r}}_k}) + \sum_{i=1}^{k+1} \sum_{j \neq i} \frac{b}{4} t_i^2 t_j^2 A(u_i^{\bar{\boldsymbol{r}}_k}, u_j^{\bar{\boldsymbol{r}}_k}) + \sum_{i=1}^{k+1} \sum_{j \neq i} \frac{\lambda}{4} t_i^2 t_j^2 B(u_i^{\bar{\boldsymbol{r}}_k}, u_j^{\bar{\boldsymbol{r}}_k}) \right) \\ &> \max_{\mathbf{t} \in \mathbb{R}_{\geq 0}^{k+1}} \left(\sum_{i=1}^{k+1} I_{b,\lambda}(t_i u_i^{\bar{\boldsymbol{r}}_k}) \right) \geq (k+1) c_0. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. For $b, \lambda > 0$, we denote by $u_k^{b,\lambda}$ instead of u_k to emphasize the dependence of b and λ, where u_k has been given by Theorem 1.1. For $b = \lambda = 0$, by using the similar argument as the proof of Theorem 1.1, we can obtain a solution v_k^0 of equation (1.4) with precisely k nodes. Moreover, $I_0'(v_k^0) = 0$ and $I_0(v_k^0) = c_k^0$.

From now on, we divide the proof into several steps.

Step 1. We claim that, for any sequence $\{\lambda_n\}$ and $\{b_n\}$ with $b_n \searrow 0$ and $\lambda_n \searrow 0$ as $n \to \infty$, $\left\{u_k^{b_n,\lambda_n}\right\}$ is bounded in H. In fact, choosing nonzero functions $\zeta_i \in C_0^\infty(B_i^{\bar{\boldsymbol{r}}_k})$ for $i = 1, \cdots, k+1$ and define a map $h_i : \mathcal{H}^{\boldsymbol{r}_k} \to \mathbb{R}$ by

$$\begin{aligned} h_i(\mathbf{t}) &= \partial_{u_i} J_{b,\lambda}(t_1^{\frac{1}{4}} \zeta_1, \cdots, t_{k+1}^{\frac{1}{4}} \zeta_{k+1}) t_i^{\frac{1}{4}} \zeta_i \\ &= t_i^{\frac{1}{2}} \|\zeta_i\|_i^2 + b t_i^{\frac{1}{2}} \sum_{j=1}^{k+1} t_j^{\frac{1}{2}} A(\zeta_i, \zeta_j) + \lambda t_i^{\frac{1}{2}} \sum_{j=1}^{k+1} t_j^{\frac{1}{2}} B(\zeta_i, \zeta_j) - \int_{B_i^{\bar{\boldsymbol{r}}_k}} K(|x|) f(t_i^{\frac{1}{4}} \zeta_i) t_i^{\frac{1}{4}} \zeta_i dx. \end{aligned}$$

Then, (**F4**) implies that, for any $b, \lambda \in [0,1]$, there exists a $(k+1)$-tuple $\mu := (\mu_1, \cdots, \mu_{k+1})$ of positive numbers, which does not depend on b and λ, such that

$$h_i(\mu) < 0, \text{ for } i = 1, \cdots, k+1.$$

Thus, in view of Lemma 3.2, there exists a unique $(k+1)$ tuple $(t_1(b,\lambda), \cdots, t_{k+1}(b,\lambda)) \in (0,+\infty)$ with $t_i(b,\lambda) < \mu_i$ such that

$$\left(\bar{\zeta}_1, \cdots, \bar{\zeta}_{k+1}\right) := \left(t_1^{\frac{1}{4}}(b,\lambda)\zeta_1, \cdots, t_{k+1}^{\frac{1}{4}}(b,\lambda)\zeta_{k+1}\right) \in \mathcal{N}_k^{\bar{r}_k}.$$

Besides, by a straightforward computation, there holds

$$\begin{aligned}
I_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{\zeta}_i\right) &= I_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{\zeta}_i\right) - \frac{1}{4} I'_{b,\lambda}\left(\sum_{i=1}^{k+1} \bar{\zeta}_i\right)\left(\sum_{i=1}^{k+1} \bar{\zeta}_i\right) \\
&= \frac{1}{4} \sum_{i=1}^{k+1} \|\bar{\zeta}_i\|_i^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(|x|)\left(f(\zeta_i)\zeta_i - F(\zeta_i)\right) dx \\
&\le \frac{1}{4} \sum_{i=1}^{k+1} \|\bar{\zeta}_i\|_i^2 + \frac{1}{4}\|K\|_{L^\infty} \int_{\mathbb{R}^3} \left(C_1 \bar{\zeta}_i^2 + C_2 \bar{\zeta}_i^p\right) dx \\
&\le \frac{1}{4} \sum_{i=1}^{k+1} \|\mu_i^{\frac{1}{4}} \zeta_i\|_i^2 + \frac{1}{4}\|K\|_{L^\infty} \int_{\mathbb{R}^3} \left(C_1 \mu_i^{\frac{1}{2}} \zeta_i^2 + C_2 \mu_i^{\frac{p}{4}} \zeta_i^p\right) dx := C_0,
\end{aligned}$$

where we used (**K**), (**F1**)–(**F4**). Moreover, by Corollary 2.1, we deduce that

$$\begin{aligned}
C_0 = I_{b,\lambda}(\sum_{i=1}^{k+1} \bar{\zeta}_i) \ge I_{b,\lambda}\left(u_k^{b,\lambda}\right) &= I_{b,\lambda}\left(u_k^{b,\lambda}\right) - \frac{1}{4} I'_{b,\lambda}\left(u_k^{b,\lambda}\right) u_k^{b,\lambda} \\
&= \frac{1}{4}\left\|u_k^{b,\lambda}\right\|^2 + \frac{1}{4} \int_{\mathbb{R}^N} K(|x|)\left(f(u_k^{b,\lambda}) u_k^{b,\lambda} - F(u_k^{b,\lambda})\right) dx \qquad (5.2) \\
&\ge \frac{1}{4}(1-\theta) \sum_{i=1}^{k+1} \left\|u_k^{b,\lambda}\right\|^2,
\end{aligned}$$

which guarantees the boundedness of $\|u_k^{b_n,\lambda_n}\|$.

Step 2. There exists a subsequence of $\{b_n\}$ and $\{\lambda_n\}$, still denoted by $\{b_n\}$ and $\{\lambda_n\}$, such that $u_k^{b_n,\lambda_n} \to w_k$ weakly in H. Then, w_k is a weak solution of (1.4). Since $u_k^{b_n,\lambda_n}$ is a sign-changing solution of (1.1) with $b = b_n$ and $\lambda = \lambda_n$, then by the Hardy-Littlewood-Sobolev inequality and the compactness of the embedding $H \hookrightarrow L^q\left(\mathbb{R}^3\right)$ for $2 < q < 6$, we deduce that $u_k^{b_n,\lambda_n} \to w_k$ strongly in H. Indeed,

$$\begin{aligned}
&\|u_k^{b_n,\lambda_n} - w_k\|^2 \\
&= \left\langle I'_{b_n,\lambda_n}\left(u_k^{b_n,\lambda_n}\right) - I'_0(w_k), u_k^{b_n,\lambda_n} - w_k\right\rangle - b_n \int_{\mathbb{R}^3} |\nabla u_k^{b_n,\lambda_n}|^2 dx \int_{\mathbb{R}^3} \nabla u_k^{b_n,\lambda_n}\left(\nabla u_k^{b_n,\lambda_n} - \nabla w_k\right) dx \\
&- \lambda_n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_k^{b_n,\lambda_n}(x)|^2 u_k^{b_n,\lambda_n}(y)\left(u_k^{b_n,\lambda_n}(y) - w_k(y)\right)}{|x-y|} dxdy + \int_{\mathbb{R}^3} K(|x|) f\left(u_k^{b_n,\lambda_n}\right)\left(u_k^{b_n,\lambda_n} - w_k\right) dx \\
&- \int_{\mathbb{R}^3} K(|x|) f(w_k)\left(u_k^{b_n,\lambda_n} - w_k\right) dx \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

which implies $w_k \neq 0$.

Step 3. Since v_k^0 is a least energy radial solution of (1.4), and we write $v_k^0 = v_{k,1} + \cdots + v_{k,k+1}$, each $v_{k,i}$ is supported on only one annulus B_i and vanishes at the complement of it. Thanks to Lemma 3.2, for each $\lambda_n > 0$, there is a unique $(k+1)$-tuple $(a_1(b_n, \lambda_n), \cdots, a_{k+1}(b_n, \lambda_n))$ of positive numbers such that

$$(a_1(b_n, \lambda_n) v_{k,1}, \cdots, a_{k+1}(b_n, \lambda_n) v_{k,k+1}) \in \mathcal{N}_k.$$

Then, for $i = 1, \cdots, k+1$, we have

$$\begin{aligned}(a_i(b_n, \lambda_n))^2 \left\|v_{k,i}\right\|_i^2 &+ b_n (a_i(b_n, \lambda_n))^2 \sum_{j=1}^{k+1} \left(a_j(b_n, \lambda_n)\right)^2 A(v_{k,i}, v_{k,j}) \\ &+ \lambda_n (a_i(b_n, \lambda_n))^2 \sum_{j=1}^{k+1} \left(a_j(b_n, \lambda_n)\right)^2 B(v_{k,i}, v_{k,j}) = \int_{B_i} K(|x|) f(a_i(\lambda_n) v_{k,i}) a_i(\lambda_n) v_{k,i} dx.\end{aligned}$$

Notice that $v_{k,i}$ satisfies

$$\left\|v_{k,i}\right\|_i^2 = \int_{B_i} K(|x|) f(v_{k,i}) v_{k,i} dx.$$

Then by (**F1**)–(**F4**) and Lemma 3.2, one can easily check that

$$(a_1(b_n, \lambda_n), \cdots, a_{k+1}(b_n, \lambda_n)) \to (1, \cdots, 1), \quad \text{as } n \to \infty.$$

Thus,

$$\begin{aligned}I_0(v_k^0) \le I_0(w_k) = \lim_{n\to\infty} I_{b_n,\lambda_n}(u_k^{b_n,\lambda_n}) &\le \lim_{n\to\infty} I_{b_n,\lambda_n}\left(\sum_{i=1}^{k+1} a_i(b_n, \lambda_n) v_{k,i}\right) \\ &= I_0\left(\sum_{i=1}^{k+1} v_{k,i}\right) = I_0(v_k^0).\end{aligned}$$

Therefore, w_k is a least energy radial solution of (1.1) which changes sign k times.

Consequently, we complete the proof of Theorem 1.3. □

6. Conclusions

This manuscript has employed the variational method to study the Kirchhoff-Schrödinger-Poisson system. By using the Gersgorin disc's theorem , Miranda theorem and Brouwer degree theory, we show the existence of infinitely many nodal solutions $\{u_k^{b,\lambda}\}$ with a prescribed number of nodes k for the system. Moreover, we prove that the energy behavior and convergence property of $\{u_k^{b,\lambda}\}$.

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Conflict of interest

The authors declare that there is no conflict of interest.

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