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Research article

Semi-analytical and numerical study of fractal fractional nonlinear system under Caputo fractional derivative

Obaid Algahtani[1,*], Sayed Saifullah[2] and Amir Ali[2]

[1] Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[2] Department of Mathematics, University of Malakand, Chakdara, Dir(L), Khyber Pakhtunkhwa, Pakistan

* **Correspondence:** Email: obalgahtani@ksu.edu.sa.

Abstract: The article aims to investigate the fractional Drinfeld-Sokolov-Wilson system with fractal dimensions under the power-law kernel. The integral transform with the Adomian decomposition technique is applied to investigate the general series solution as well as study the applications of the considered model with fractal-fractional dimensions. For validity, a numerical case with appropriate subsidiary conditions is considered with a detailed numerical/physical interpretation. The absolute error in the considered exact and obtained series solutions is also presented. From the obtained results, it is revealed that minimizing the fractal dimension reinforces the amplitude of the solitary wave solution. Moreover, one can see that reducing the fractional order α marginally reduces the amplitude as well as alters the nature of the solitonic waves. It is also revealed that for insignificant values of time, solutions of the coupled system in the form of solitary waves are in good agreement. However, when one of the parameters (fractal/fractional) is one and time increases, the amplitude of the system also increases. From the error analysis, it is noted that the absolute error in the solutions reduces rapidly when x enlarges at small-time t, whereas, increment in iterations decreases error in the system. Finally, the results show that the considered method is a significant mathematical approach for studying linear/nonlinear FPDE's and therefore can be extensively applied to other physical models.

Keywords: integral transform; Adomian decomposition method; Drinfeld-Sokolov-Wilson equation; fractal-fractional operator; Caputo fractional operator
Mathematics Subject Classification: 35Bxx, 35Qxx, 37Mxx, 65Mxx, 41Axx

1. Introduction

Fractional calculus (FC) is widely applied to investigate many physical phenomena, including viscoelasticity, electromagnetism, damping, traffic structures, robotics, telecommunications, diffusion, wave propagation, signal processing, chaos, heat transfer, device recognition, electronics, identification, modeling, percolation and genetic algorithms, control systems, as well as irreversibility [1, 2]. In FC, the integer order differential and integral operators are extended to fractional order, because the classical order operators do not work to study many complex systems [3]. Further, the fractional operators give realistic and more accurate results when compared with classical ones [4, 5].

Many fractional operators have been defined with different types of kernels, like Riemann–Liouville (R-L), Hilfer, Caputo-Fabrizio (CF), Caputo and Atangana Baleanu in Caputo sense (ABC) [6, 7]. The R-L and Caputo operators are the power-law convolutions having the first derivative, while the CF operator is the convolution of exponential decay laws having the first-order derivative together with the Delta–Dirac property. Furthermore, the prior exponential decay type kernel has been extended to the Mittage-Leffler type, which gives better results for studying a variety of physical systems [8]. Similarly, a new type of fractional operator has been introduced to combine the ideas of Caputo and proportional derivatives [9].

There are many advantages and drawbacks to fractional operators. For example, results of time dependent FDEs with Caputo's operator usually reveal weak singularities at time ($t = 0$). Similarly, every Riemann Liouville (R-L) and Caputo derivatives of real order $\alpha > 0$ is a left-inverse operator for the RL fractional integral which represented as Volterra-like convolution integro-differential operators with kernel $k(t) = t^{m-1-\alpha}/\Gamma(m-\alpha),\ \alpha \leq m]$. If α is not an integer, then the kernel is inadequately singular at $t = 0$. Hence, locally absolutely integrable on the positive real axis. Besides the advantages, the disadvantages include the fact that, with the use of the Riemann-Liouville definition, the fractional order (FO) derivative of a constant is not zero. Further, the Riemann-Liouville and Caputo definitions have singular kernels. In most of the mathematical models, stability analysis is very important for the model. Using fractional orders of the operators used in the model, the stability analysis becomes more difficult [8, 9].

Besides the fractional-order operators, another novel idea has been proposed to extend the concept of classical differentiation to fractal ones, so that, if the fractal order becomes *one*, one can recover the classical operator [10, 11]. Similarly, when the system under consideration is differentiable, then the fractal order derivative is equal to $\beta t^{\beta-1}$. The basic idea which combines fractional and fractal differentiations and integrations is known as fractal-fractional (FF) differentiation and integration [12, 13]. The fractal differentiation and integration got a lot of interest because many physical and engineering applications such as an aquifer, turbulence, and porous media preserve fractal properties [10, 11]. In fractal derivative, the parameter is ascended in agreement with t^{α}. This new type of derivative has developed to model certain real-world problems when classical physical formulations, particularly, Darcy's law, Fick's law, and Fourier laws, are not applicable. It should be noted that these formulations cannot be applied to non-integral fractal dimensional media and are supposed to be dependent on Euclidean geometry [14].

In recent years, fractal-fractional differential equations (FFDE's) have been widely studied in electrical networks, chaotic processes, biological processes, and fluid mechanics [15–17]. To study fractal-fractional models, typically numerical methods for-instance Riccati, Chebyshev cardinal functions and Jacobi polynomials are applied that are time and memory-consuming. There are also a variety of analytical approaches that can be used, like discretization and Homotopy techniques. The discretization is considered complicated to obtain an accurate approximation, while the Homotopy analysis requires predefined parameters, where the solution of the problem is dependent on these parameters. It has been noted that the Laplace transform method (LTM) is the most consistent technique, as it does not require predefined parameters or any kind of discretization [18].

The considered systems originate from the coupled KdV equations

$$\phi_t - \frac{1}{2}(\phi_{xxx} + 6\phi\phi_x) = 2\,b\psi\psi_x, \tag{1.1}$$
$$\psi_t + \psi_{xxx} + 3\phi\phi_x = 0, \tag{1.2}$$

where $\psi(x,t)$, $\phi(x,t)$ play an important part to illustrate the interface of typical long waves with a variety of dispersion relationships. It has been proved that Eq (1.2) represents a particular example of the four-reduced Kadomtsev-Petviashvili (KP) grading [19, 20] together with affine Lie algebras [21, 22]. The derivation has converted to the most fascinating model named Drinfeld–Sokolov–Wilson (DSW) system

$$\begin{cases} \psi_t + \sigma\phi\phi_x = 0, \\ \phi_t - \gamma\phi_{xxx} + \eta\psi\phi_x + \zeta\psi_x\phi = 0. \end{cases} \tag{1.3}$$

The parameters σ, γ, η, ζ can be chosen accordingly. It should be noted that the spital and temporal variables $x \in \xi = [a, b] \subseteq \mathbb{R}$ and $t \in [0, T]$. One can see in the literature that, Eq (1.3) has extensively studied for shallow water-waves, water dispersion, fluid mechanics, traveling waves and doubly periodic wave solutions and the dispersion of nonlinear surface gravity waves through a straight/level seabed [23, 24]. The considered equation has also investigated to study a variety of natural occurrence by applying several methods [25, 26]. We will particularly study the proposed system with fractional derivative and fractal dimensions with particular the subsidiary conditions

$$\psi(x,0) = p(x), \qquad \phi(x,0) = q(x). \tag{1.4}$$

Recently, a lot of efforts have been made to develop effective techniques to investigate and examine the solutions of complex NLPDE's and systems of NLPDE's. In this connection, several direct and computational techniques have been presented, including the Lie groups [27], the Hirota method [28], the Exp-function method [29], the tanh-coth method [30] and the advanced tanh-coth method [31]. Nevertheless, it is found that, LADM is one of the most implicit and compatible computational technique to investigate the approximate solution to NLPDE's. The extended Laplace transform method (ELTM) [32, 33] is a technique for treating nonlinear differential equations that is different from the Laplace Adomian decomposition method (LADM). The nonlinear terms are handled via a theorem called the transformation of series in the ELTM, which avoids the integrations associated with the Adomian decomposition approach. Another recent publication [34] on the combined applications of Laplace transformation and the Adomian decomposition method is "A method for inverting the Laplace transforms of two classes of rational transfer functions in control engineering", which deals with Laplace inversion of ratios with polynomials having non-integer orders of the transform variable "s". The ADM was established by George Adomian is an implicit method for both numerical and analytical solution of differential equations that occur in the simulating physical problems [35–37]. The most significant of the techniques is the Adomian polynomial that offers the convergence of series solutions of the non-linear terms in the system. Since the method does not necessitate unnecessary linearization, perturbation, or other constrictive procedures and assumptions that may, occasionally significantly, alter the problem being addressed, it is particularly well suited to solving physical problems.

The Laplace transform decomposition method is a very effective analytical technique and has been successfully used to solve different problems in integer-order as well as fractional calculus to study numerous systems [38]. Motivated by its efficiency and fast convergence, we use the Laplace transform for Caputo fractal-fractional derivative. The Laplace transform with fractal-fractional dimensions and a power-law kernel is calculated in a systematic manner in this manuscript. It should be emphasized that when the fractal order equals one, the suggested technique recovers the transform. However, when both orders are equal to one, the proposed method recovers the conventional considered transform. As an application of the proposed technique, the method is applied to the governing system considered with fractal fractional dimensions under Caputo fractional derivative.

The rest of the article is organized as follows: Section 2 contains basic definitions associated with the fractal-fractional calculus. Section 3 presents the general solution of the considered coupled equations with the FF operator with the power-law kernel by using the LADM. Section 4 presents a numerical example of the considered model with suitable initial conditions to validate the proposed method. Section 5 concludes the article with a summary.

2. Preliminaries

Here, we define some basic definitions related to fractal-fractional calculus.

Definition 1. *[14, 39] Let $u \in C[a, c]$, then the Caputo fractional operators for $\alpha \in (0, 1]$ is defined by*

$$
{}_a^C Du(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \acute{u}(s) ds & \forall\, \alpha \in (m-1, m], \\ u^m(t) & \alpha = m. \end{cases} \tag{2.1}
$$

It should be noted that for $\alpha = 1$, the above derivative converges to classical derivative. Let $\psi(t)$ is differentiable in interval $(b,\ c)$. Let $\psi(t)$ is FF differentiable in $(b,\ c)$ with fractal order β, then FF operator with power law kernel is given by

$$
{}_a^{FFP} D_t^{\alpha,\,\beta} \psi(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \frac{d}{dt^\beta} \psi(s) ds, \quad 0 < m-1 < \alpha,\, \beta \le m,
$$

where

$$
\frac{d\psi(t)}{dt^\beta} = \lim_{t \to s} \frac{\psi(t) - \psi(s)}{t^\beta - s^\beta}.
$$

In more general form the above operator can be expressed as

$$
{}_a^{FFP} D_t^{\alpha,\,\beta,\,\gamma} \psi(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \frac{d^\gamma}{dt^\beta} \psi(s) ds, \quad 0 < m-1 < \alpha,\, \beta,\, \gamma \le m,
$$

where

$$\frac{d^{\gamma}\psi(t)}{dt^{\beta}} = \lim_{t\to s}\frac{\psi^{\gamma}(t)-\psi^{\gamma}(s)}{t^{\beta}-s^{\beta}}.$$

Definition 2. *[14] The FF integral with power law kernel is*

$${}_{0}^{F}\mathcal{I}_{t}^{\alpha} = \frac{\beta}{\Gamma(\alpha)}\int_{0}^{t} s^{\alpha-1}\psi(s)(t-s)^{\alpha-1}ds.$$

Definition 3. *[40] The Laplace transform $\mathcal{L}$ of a function $\psi(t)$ for $t > 0$ is defined by the integral*

$$\mathcal{L}[\psi(t)] = F(s) = \int_{0}^{\infty} e^{-st}\psi(t)dt. \tag{2.2}$$

Definition 4. *[40] The inverse Laplace transform of the function $F(s)$ is denoted by $\mathcal{L}^{-1}$ and is defined by*

$$\psi(t) = \mathcal{L}^{-1}\left(F(s)\right)(t) = \frac{1}{2\pi i}\lim_{t\to\infty}\int_{r-it}^{r+it} e^{st}F(s)ds,$$

where the integration is done along the vertical line $Re(s) = r$ in the complex plane such that r is greater than the real part of all singularities of $F(s)$ and $F(s)$ is bounded on the line.

Definition 5. *[40] The Laplace transform of Caputo fractional operator is defined as*

$$\mathcal{L}_{a}^{C}D_{t}^{\alpha}\psi(x,t) = s^{\alpha}\mathcal{L}\psi(x,t) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\psi_{kt}(x,0), \quad n = [\alpha]+1.$$

Remark 1. *There are some cases in the transformable functions where poles of some particular orders occur. These functions cannot be invertible analytically. For example, consider a transformed function in the form*

$$\bar{f(s)} = \frac{4\Omega}{[\pi - 2\tan^{-1}(\frac{2\Omega}{s})][s^2+4\Omega^2]},$$

using the Bromwich contour, we can find that a pole of first order at $s = 0$. The double branch points $s = \pm 2i\Omega$, two poles also at $\pm 2i\Omega$. Using inverse Laplace transform this function is not invertible.

3. The proposed method

Here, we calculate the Laplace transform and the governing model in FF sense with power law kernel. We also calculate the series solution by using the proposed method (LADM).

3.1. Fractal-fractional Laplace transform with power law

Let a continuous function $\psi(t) \in \mathcal{H}^1$ for $0 \leq t \leq T$. Further, consider

$$^{FFP}D_t^{\alpha,\beta}\psi(t) = B, \tag{3.1}$$

where B is an exterior function to chosen accordingly and $0 < \alpha, \beta \leq 1$. We can simplify Eq (3.1) in the form [17]

$$^{C}D_t^{\alpha}\psi(t) = \beta t^{\beta-1}B. \tag{3.2}$$

Applying Laplace transform to Eq (3.2), we obtain

$$s^{\alpha}\mathcal{L}(\psi(t)) - s^{\alpha-1}\psi(0) = \mathcal{L}(\beta t^{\beta-1}B), \quad \mathcal{L}(\psi(t)) = \frac{\psi(0)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}(\beta t^{\beta-1}B).$$

Similarly, applying inverse $\mathcal{L}^{-1}$, we obtain

$$\psi(t) = \mathcal{L}^{-1}\left[\frac{\psi(0)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}(\beta t^{\beta-1}B)\right].$$

When B is a function of x, then $\mathcal{L}(\beta t^{\beta-1}B) = \mathcal{L}(\beta t^{\beta-1}B(x)) = \Gamma(\beta+1)B(x)/s^{\beta}$, where $\mathcal{L}(t^{\beta}) = \Gamma(\beta)/s^{\beta}$ and $\beta\Gamma(\beta) = \Gamma(\beta+1)$.

$$\psi(t) = \psi(0) + \mathcal{L}^{-1}\left[\frac{\Gamma(\beta+1)}{s^{\alpha+\beta}}\right]B(x), \quad \phi(t) = \phi(0) + \frac{\Gamma(\beta+1)t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}B(x). \tag{3.3}$$

The above relation is Laplace transform of fractal fractional operator with power law kernel.

3.2. The governing model in fractal-fractional sense with power law kernel

Let us suppose model (1.3) in fractal fractional sense with power law kernel as

$$\begin{cases} ^{FFP}D_t^{\alpha,\beta}\psi + \sigma\phi\phi_x = 0, \\ ^{FFP}D_t^{\alpha,\beta}\phi + \gamma\phi_{xxx} + \eta\psi\phi_x + \zeta\psi_x\phi = 0, \end{cases} \tag{3.4}$$

with $0 < \alpha,\beta \leq 1$ and subsidiary conditions (SCs)

$$\psi(x,0) = p(x) \quad and \quad \phi(x,0) = q(x), \tag{3.5}$$

Regrouping Eq (3.4) gives

$$\begin{aligned} ^{C}D_t^{\alpha}\psi &= \beta t^{\beta-1}\{-\sigma\phi\phi_x\}, \\ ^{C}D_t^{\alpha}\phi &= \beta t^{\beta-1}\{-\gamma\phi_{xxx} - \eta\psi\phi_x - \zeta\psi_x\phi\}. \end{aligned} \tag{3.6}$$

Using Laplace transform to both sides

$$\begin{aligned} \mathcal{L}[^{C}D_t^{\alpha}]\psi &= \mathcal{L}[\beta t^{\beta-1}\{-\sigma\phi\phi_x\}], \\ \mathcal{L}[^{C}D_t^{\alpha}]\phi &= \mathcal{L}[\beta t^{\beta-1}\{-\gamma\phi_{xxx} - \eta\psi\phi_x - \zeta\psi_x\phi\}]. \end{aligned}$$

Applying the definition discussed in the subsection 3.1 for power law kernel gives

$$\begin{aligned}\mathcal{L}[\psi_t] &= \frac{p(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[\beta t^{\beta-1}(-\sigma\phi\phi_x)\right],\\ \mathcal{L}[\phi_t] &= \frac{q(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[\beta t^{\beta-1}(-\gamma\phi_{xxx} - \eta\psi\phi_x - \zeta\psi_x\phi)\right].\end{aligned} \tag{3.7}$$

Consider ψ and ϕ in the series form

$$\psi = \sum_{n=0}^{\infty}\psi_n, \quad \phi = \sum_{n=0}^{\infty}\phi_n, \tag{3.8}$$

the non-linear terms are decomposed as

$$\phi\phi_x = \sum_{n=0}^{\infty}A_n, \quad \psi\phi_x = \sum_{n=0}^{\infty}B_n \quad and \quad \psi_x\phi = \sum_{n=0}^{\infty}C_n, \tag{3.9}$$

where A_n, B_n and C_n represents the Adomian polynomials [41] described for the above terms as

$$\begin{aligned}A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{n}\lambda^k\phi_k\right)\left(\sum_{k=0}^{n}\lambda^k\phi_{kx}\right)\right]_{\lambda=0}, \quad B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{n}\lambda^k\psi_k\right)\left(\sum_{k=0}^{n}\lambda^k\phi_{kx}\right)\right]_{\lambda=0},\\ C_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{n}\lambda^k\psi_{kx}\right)\left(\sum_{k=0}^{n}\lambda^k\phi_k\right)\right]_{\lambda=0}.\end{aligned}$$

Applying $\mathcal{L}^{-1}$ to Eq (3.7), together with Eqs (3.8) and (3.9) and Eq (3.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= p(x) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\sigma\beta t^{\beta-1}\left(-\sum_{n=0}^{\infty}A_n\right)\right\}\right],\\ \sum_{n=0}^{\infty}\phi_n(x,t) &= q(x)) + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}\left(-\gamma\sum_{n=0}^{\infty}\phi_{nxxx} - \eta\sum_{n=0}^{\infty}B_n - \zeta\sum_{n=0}^{\infty}C_n\right)\right\}\right].\end{aligned} \tag{3.10}$$

Comparing terms on both sides in Eq (3.10), we obtain the series solution

$$\begin{aligned}\psi_0 &= p(x), & \phi_0 &= q(x),\\ \psi_1 &= \mathcal{L}^{-1}\left[\tfrac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\sigma A_0)\right\}\right], & \phi_1 &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\gamma\phi_{0xxx} - \eta B_0 - \zeta C_0)\right\}\right],\\ \psi_2 &= \mathcal{L}^{-1}\left[\tfrac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\sigma A_1)\right\}\right], & \phi_2 &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\gamma\phi_{1xxx} - \eta B_1 - \zeta C_1)\right\}\right],\\ \psi_3 &= \mathcal{L}^{-1}\left[\tfrac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\sigma A_2)\right\}\right], & \phi_3 &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\beta t^{\beta-1}(-\gamma\phi_{2xxx} - \eta B_2 - \zeta C_2)\right\}\right].\end{aligned}$$

The general series solution can be obtained in the form

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n, \quad \phi(x,t) = \sum_{n=0}^{\infty}\phi_n. \tag{3.11}$$

4. Applications of the method

4.1. Example

For validation of the proposed technique, consider the following numerical example

$$\begin{cases} {}^{FFP}D_t^{\alpha,\beta}\psi + 3\phi\phi_x = 0, \\ {}^{FFP}D_t^{\alpha,\beta}\phi + 2\phi_{xxx} + 2\psi\phi_x + \psi_x\phi = 0, \end{cases} \tag{4.1}$$

with

$$\psi(x,0) = 3\,\mathrm{sech}^2(x), \qquad \phi(x,0) = 2\,\mathrm{sech}(x). \tag{4.2}$$

The exact solution of Eq (4.1) can obtained in the form [42]

$$\psi = \frac{3\gamma}{2} sech^2\left(\sqrt{\frac{\gamma}{2}}(x-\gamma t)\right), \qquad \phi = \pm\gamma sech\left(\sqrt{\frac{\gamma}{2}}(x-\gamma t)\right). \tag{4.3}$$

Following the procedure presented in Section 3 together with IC's Eq (4.2), we get the approximate series solution to Eq (4.1):

$$\begin{aligned}
\psi_0 &= 3sech^2(x),\\
\phi_0 &= 2sech(x),\\
\psi_1 &= \frac{12\,\Gamma(\beta+1)t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\ \mathrm{sech}^2(x)\tanh(x),\\
\phi_1 &= \frac{4\,\Gamma(\beta+1)t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,\mathrm{sech}(x),\\
\psi_2 &= 24\frac{\beta\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}\frac{\Gamma(\alpha+2\beta-1)t^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}\left[\tanh^3(x) - \mathrm{sech}^4(x) - 6\,\mathrm{sech}(x)\tanh^2(x)\right]\mathrm{sech}^2(x),\\
\phi_2 &= -4\frac{\beta\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}\frac{\Gamma(\alpha+2\beta-1)t^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}[1 + 48\,\mathrm{sech}^4(x) + 6\,\mathrm{sech}^5(x) - 38\,\mathrm{sech}^2(x) - 6\,\mathrm{sech}^3(x)\\
&\quad +6\,\mathrm{sech}^2(x)\tanh(x)]\,\mathrm{sech}(x).
\end{aligned}$$

The final approximate solutions can be expressed as:

$$\psi = \sum_{n=0}^{\infty}\psi_n, \qquad \phi = \sum_{n=0}^{\infty}\phi_n. \tag{4.4}$$

4.2. Absolute error estimate

The absolute error analysis between Eqs (4.3) and (4.4) is shown in the following table (Table 1).

Table 1. $\alpha = \beta = 1$, $\gamma = 2$ are considered for error estimate.

(x,t)	Exact	ψ	$\lvert$ Exact$-\psi\rvert$	Exact	ϕ	$\lvert$ Exact$-\phi\rvert$
(-4,0.1)	0.0027	0.0026	1.2192×10^{-4}	0.0370	0.5790	2.09×10^{-2}
(-2,0.1)	0.1438	0.1375	6.3×10^{-3}	0.4790	0.4395	3.9500×10^{-2}
(0,0.1)	2.8831	2.8	3.1×10^{-3}	2	1.78	2.2×10^{-1}
(2,0.1)	0.3107	0.3010	9.1×10^{-3}	0.4790	0.6402	1.6120×10^{-1}
(4,0.1)	0.0060	0.0058	2.0801×10^{-4}	0.0370	0.0872	5.02×10^{-2}
(-4,0.05)	0.0033	0.0033	3.5091×10^{-5}	0.0370	0.0657	2.87×10^{-2}
(-2,0.05)	0.1747	0.1729	1.8×10^{-3}	0.4790	0.4830	4×10^{-3}
(0,0.05)	2.9702	2.9700	1.9887×10^{-4}	2	1.9450	5.55×10^{-2}
(2,0.05)	0.2568	0.2546	2.2×10^{-3}	0.4790	0.5844	1.0540×10^{-1}
(4,0.05)	0.0049	0.0049	4.5790×10^{-5}	0.0370	0.0804	4.34×10^{-2}

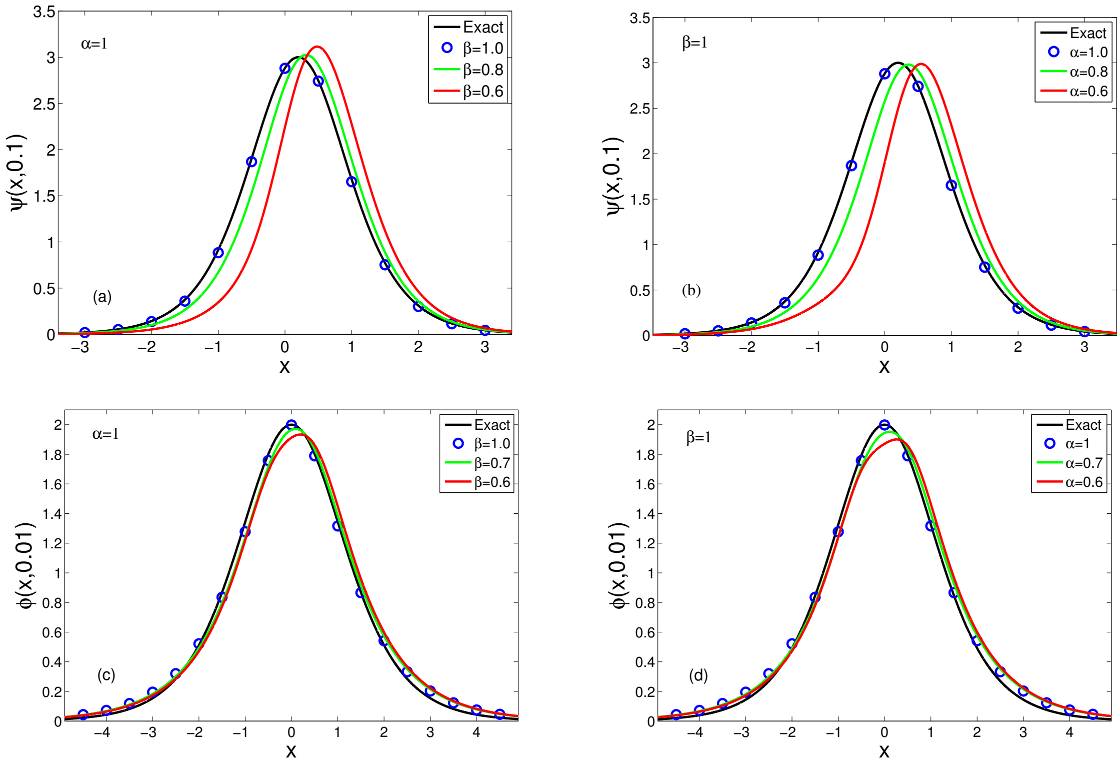

Figure 1. Comparison of exact and approximate solutions of (ψ, ϕ) given in Eqs (4.3) and (4.4) for different values of α and β respectively.

4.3. Discussion

For the numerical demonstration, the parameters $\sigma = 3,\ \gamma = \eta = 2$ *and* $\zeta = 1$ are used. The effect of fractal order variable β and stable fractional order α with time ($t = 0.1$) for approximate solution ψ are displayed in Figure 1 (a), while, Figure 1 (b) displays the effect of fractional order variable α with stable fractal order β of the approximate solution ψ. One can see that a good agreement is obtained.

The bottom panel of Figure 1 depicts the behaviour of ϕ with a variety values of β by keeping fixed α fixed and then changing α with fix value of β. As a conclusion, it is observed that the amplitude increases by decreasing the fractal dimension β. Similarly, decreasing α, to some extent decreases the amplitude as well as alters the shape of the solitonic solution.

The absolute error between Eqs (4.3) and (4.4) for $\alpha = \beta = 1$, $\gamma = 2$ is calculated in Table 1 and plotted in Figure 2. It is observed that the error in the system decreases when x increases for small value of time (t). It is noted that, aggregating in iterations diminishes the absolute error. It is interesting to note that the higher order correction for dispersion may be added using the new mathematical parameter (time fractional order α) in the modulation of such systems for different waves phenomenons.

The physical conduct of the obtained approximate solutions ψ versus ϕ is depicted in Figure 3 (a) and (b). The behaviour of ψ with differing β and α respectively with particular values of spatial variable x versus time (t) is illustrated in the top panel of Figure 4. Similarly, the behaviour of ϕ with different values of β and α spatial variables $x = 0.6$ versus time (t) is illustrated in Figure 5 (a) and (b). It is observed that, when time (t) is small enough, the solitary waves are in very good agreement. It is also observed that intensifying time (t) rapidly enhances the wave propagation when one of the fractal or fractional variables (α and β) is not equal to one.

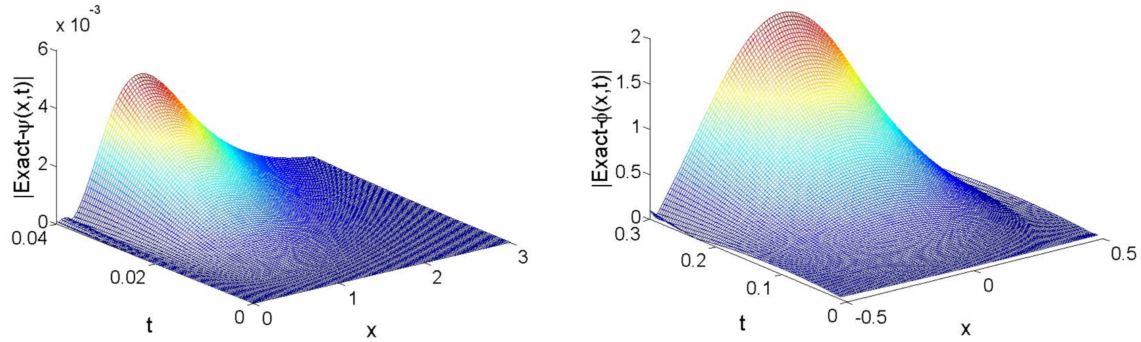

Figure 2. The surface plots of the absolute error estimate obtained for $\psi(x, t)$ [Eq (4.3)] and $\phi(x, t)$ [Eq (4.4)] presented in Table 1.

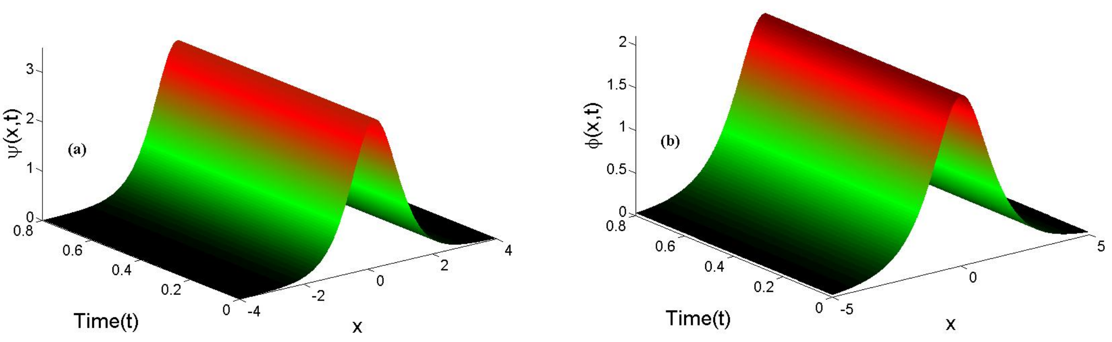

Figure 3. The surface plots of approximate solutions depicted in Figure 1 (a) and (c).

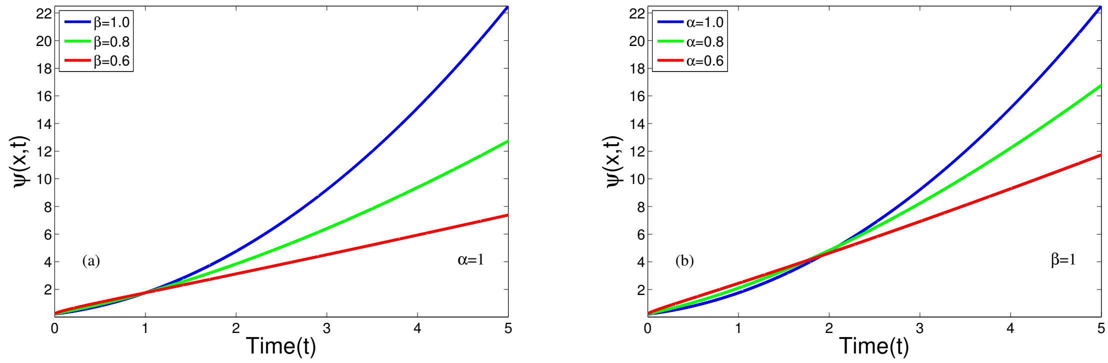

Figure 4. The behaviour of ψ for different values of α and β versus time (t).

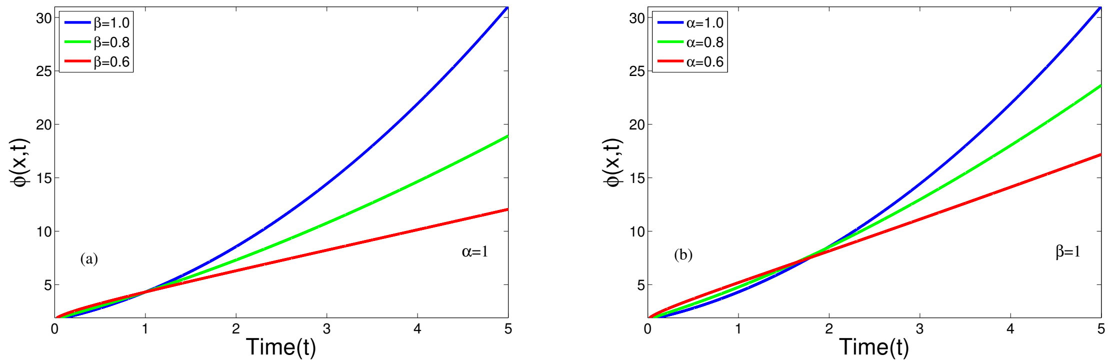

Figure 5. The behaviour of ψ for different values of α and β versus time (t).

5. Conclusions

We have studied coupled nonlinear system with fractal-fractional sense together with a power law kernel using LADM. It is observed that the proposed technique is very effective for studying such types of nonlinear coupled systems. The main advantage of the suggested method is that it can analyse systematic solutions of the considered coupled system without any perturbation, estimate the long-lasting and complex polynomials, or any discretization. It is worth mentioning that, the suggested approach gives us greater freedom to take into account different kinds of initial presumptions and equation type complexity as well as nonlinearity. Hence, as a result, the complex NDEs (NPDE's/NODE's) can be addressed immediately. The innovative aspect of the proposed method is that it uses a simple algorithm to evaluate the solution and is homotopy and axiomatically natured, allowing for a quick convergence of the obtained solution for the nonlinear section of the provided issue. The results from numerous algorithms, including q-HAM, HPM, ADM, and some other conventional procedures, are conceivably contained in it, giving it a tremendous degree of generality. When compared to existing methods, the proposed method may maintain high accuracy while requiring less effort and computing time. From the numerical analysis, it is observed that fractal dimensions play a very effective role as they enhance the system amplitude. It is also discovered that, for sufficiently small time (t) the error is minimised between the exact and approximate solutions. It will be fascinating to investigate such nonlinear systems in a time fractal-fractional context in the future, as time has a significant impact on the results.

As a future work, it will also be interesting to investigate the solutions of Boussinesq-type equations using MDLDM. Further, the sine-Gordon expansion method and the hyperbolic function method studied reported in [43] can be applied to the DWS equation with fractal fractional dimensions to study the novel type of solitary wave solutions.

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Conflict of interest

It is declared that all the authors have no conflict of interest regarding this manuscript.

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