AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16733–16740.
DOI:10.3934/math.2022918
Received: 12 May 2022
Revised: 01 July 2022
Accepted: 08 July 2022
Published: 12 July 2022

Research article

Bifurcation analysis and classification of all single traveling wave solution in fiber Bragg gratings with Radhakrishnan-Kundu-Lakshmanan equation

Kun Zhang[1,2], **Xiaoya He**[1] **and Zhao Li**[1,2,*]

[1] College of Computer Science, Chengdu University, Chengdu 610106, China

[2] Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, Chengdu University, China

* **Correspondence:** Email: lizhao10.26@163.com.

Abstract: The current work studies the bifurcation and the classification of single traveling wave solutions of the coupled version of Radhakrishnan-Kundu-Lakshmanan equation that usually describes the dynamics of optical pulses in fiber Bragg gratings, which is also described by a family of nonlinear Schrödinger equations with cubic nonlinear terms. The solutions of the hyperbolic functions, the rational functions, the trigonometric functions and the Jacobian functions are retrieved by using the complete discrimination system of polynomial. By selecting appropriate parameters, phase portraits, two-dimension graphics and three-dimension graphics of the obtained solutions are drawn.

Keywords: Radhakrishnan-Kundu-Lakshmanan; single traveling wave solution; complete discrimination system; bifurcation; phase portraits
Mathematics Subject Classification: 35C05, 35C07, 35R11

1. Introduction

The Radhakrishnan-Kundu-Lakshmanan (RKL) equation in polarization-preserving fibers is given as [1–11]

$$iq_t + aq_{xx} + b|q|^2 q = i\lambda(|q|^2 q)_x - i\delta q_{xxx}, \tag{1.1}$$

where $q = q(t, x)$ stands for the wave profile. The parameters a, b, λ and δ are real constants.

The RKL equation are usually used to describe the pulse propagation in polarization-preserving fibers. In recent years, many experts have focused on the research of the RKL equation. Moreover, many classical methods are also used to construct the traveling wave solutions [12–20] of the RKL equation (for details, please refer to [1–10] and its references). But the coupled systems are usually used to simulate models form the fields of physics, nonlinear optics and engineering technology, see [21–23]. Therefore, the discussion of the coupled version of RKL equation has very important practical and theoretical significance.

The coupled version of RKL equation in fiber Bragg gratings is described as follows [24]

$$\begin{aligned} &iu_t + a_1 v_{xx} + (b_1|u|^2 + c_1|v|^2)u + i\alpha_1 u_x + \beta_1 v + \sigma_1 u^* v^2 = i[\lambda_1(|u|^2u)_x + \gamma_1(|v|^2u)_x - \delta_1 u_{xxx}], \\ &iv_t + a_2 u_{xx} + (b_2|v|^2 + c_2|u|^2)v + i\alpha_2 v_x + \beta_2 u + \sigma_2 v^* u^2 = i[\lambda_2(|v|^2v)_x + \gamma_1(|u|^2v)_x - \delta_2 v_{xxx}], \end{aligned} \tag{1.2}$$

where $u = u(t,x)$ and $v = v(t,x)$ denote the wave profiles, the parameters a_j, b_j, c_j, α_j, β_j, γ_j, σ_j and $\lambda_j (j = 1,2)$ are real constants. Equation (1.2) usually describe the dynamics of optical pulses in fiber Bragg gratings, which is also described by a family of nonlinear Schrödinger equations with cubic nonlinear terms. In [24], Elsayed et al. obtained the bright, dark and singular solitons solutions of Eq (1.2) by using the extended auxiliary equation method and unified Riccati equation technique. However, as far as I can, the discussion on the work of phase diagrams and single wave solutions has not been reported. This paper will focus on this issue.

The article is organized as follows. In Section 2, we draw the phase portraits and obtain the classification of single traveling wave solution of the coupled version of RKL equation. In Section 3, we present a conclusion.

2. Bifurcation analysis and single traveling wave solution of (1.2)

2.1. Mathematical analysis

Substituting the traveling wave transformation

$$u(t,x) = U(\xi)e^{i\theta(t,x)},\ v(t,x) = V(\xi)e^{i\theta(t,x)},\ \xi = x - ct,\ \theta(t,x) = -kx + wt + \theta_0, \tag{2.1}$$

into Eq (1.2), then integrating it again and separating real parts and imaginary parts of Eq (1.2), we have

$$\begin{aligned} &a_1V'' + 3\delta_1 kU'' - (w - \alpha_1 k + \delta_1 k^3)U - (a_1k^2 - \beta_1)V + (b_1 - \lambda k)U^3 + (c_1 + \sigma_1 - k\gamma_1)UV^2 = 0, \\ &a_2U'' + 3\delta_2 kV'' - (w - \alpha_2 k + \delta_2 k^3)V - (a_2k^2 - \beta_2)U + (b_2 - \lambda k)V^3 + (c_2 + \sigma_2 - k\gamma_2)VU^2 = 0, \\ &\delta_1 U''' - (c - \alpha_1 + 3k^2\delta_1)U' - 2a_1kV' - 3\lambda_1 U^2U' - 2\gamma_1 UVV' - \gamma_1 U'V^2 = 0, \\ &\delta_2 V''' - (c - \alpha_2 + 3k^2\delta_2)V' - 2a_2kU' - 3\lambda_2 V^2V' - 2\gamma_2 UVU' - \gamma_2 V'U^2 = 0, \end{aligned} \tag{2.2}$$

where c stands for the speed. k represents wave number. w is the phase constant. $\theta(t,x)$ represents phase component of soliton.

Making $V = AU$ $(A \neq 1)$, and substituting it into Eq (2.2) yields

$$\begin{aligned} &(a_1A + 3\delta_1 k)U'' - [w - \alpha_1 k + \delta_1 k^3 + A(a_1k^2 - \beta_1)]U + [b_1 - \lambda_1 k + A^2(c_1 + \sigma_1 - k\gamma_1)]U^3 = 0, \\ &(a_2 + 3\delta_2 kA)U'' - [a_2k^2 - \beta_2 + A(w - \alpha_2 k + \delta_2 k^3)]U + A[A^2(b_2 - \lambda_2 k) + c_2 + \sigma_2 - k\gamma_2]U^3 = 0, \\ &\delta_1 U''' - (c - \alpha_1 + 3k^2\delta_1 + 2a_1kA)U' - 3(\lambda_1 + \gamma_1 A^2)U'U^2 = 0, \\ &\delta_2 AU''' - [2a_2k + A(c - \alpha_2 + 3k^2\delta_2)]U' - 3A(\lambda_2 A^2 + \gamma_2)U'U^2 = 0. \end{aligned} \tag{2.3}$$

Integrating both sides of the third and fourth equations of Eq (2.3) at the same time, we can get

$$\begin{aligned} &\delta_1 U'' - (c - \alpha_1 + 3k^2\delta_1 + 2a_1kA)U - (\lambda_1 + \gamma_1 A^2)U^3 = 0, \\ &\delta_2 AU'' - [2a_2k + A(c - \alpha_2 + 3k^2\delta_2)]U - A(\lambda_2 A^2 + \gamma_2)U^3 = 0. \end{aligned} \tag{2.4}$$

From (2.4), we can easily obtain

$$\delta_1 = A\delta_2,\ c - \alpha_1 + 3k^2\delta_1 + 2a_1kA = 2a_2k + A(c - \alpha_2 + 3k^2\delta_2),\ \lambda_1 + \gamma_1A^2 = A(\lambda_2A^2 + \gamma_2). \quad (2.5)$$

Then, we can calculate that

$$c = \frac{\alpha_1 - A\alpha_2 - 2a_1kA + 2a_2k}{1 - A}. \quad (2.6)$$

Therefore, the first equation of Eq (2.4) can be simplified to

$$U''(\xi) - l_1U(\xi) - l_2U^3(\xi) = 0, \quad (2.7)$$

where $l_1 = \frac{c-\alpha_1+3k^2\delta_1+2a_1kA}{\delta_1}$, $l_2 = \frac{\lambda_1+\gamma_1A^2}{\delta_1}$, $\delta_1 \neq 0$.

2.2. Phase portraits

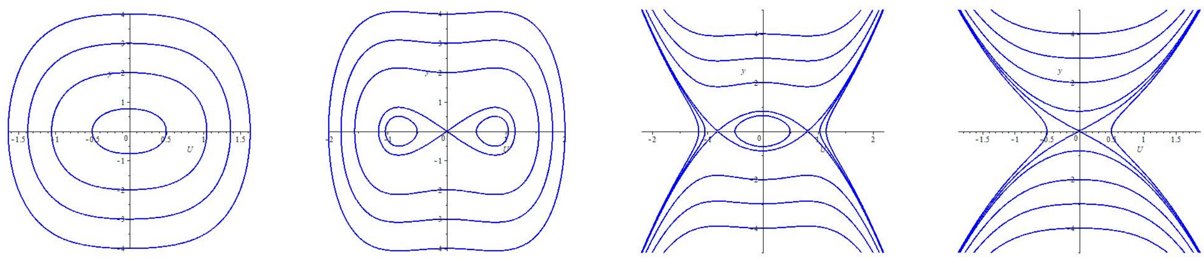
Here, we denote $\frac{dU}{d\xi} = y$, then system (2.7) becomes the following two-dimensional system

$$\begin{cases} \frac{dU(\xi)}{d\xi} = y, \\ \frac{dy}{d\xi} = l_1U(\xi) + l_2U^3(\xi), \end{cases} \quad (2.8)$$

with Hamiltonian system

$$H(U, y) = \frac{1}{2}y^2 - \frac{l_1}{2}U^2(\xi) - \frac{l_2}{4}U^4(\xi) = h. \quad (2.9)$$

The phase portraits of system (2.8) are shown in Figure 1.



(a) $l_1 < 0, l_2 < 0$ (b) $l_1 > 0, l_2 < 0$ (c) $l_1 < 0, l_2 > 0$ (d) $l_1 > 0, l_2 > 0$

Figure 1. Phase portraits of system (2.8).

2.3. Single traveling wave solution

Multiplying U' both sides of Eq (2.7) and integrating once with respect to ξ, we can get

$$(U')^2 = \frac{l_2}{2}U^4 + l_1U^2 + 2l_0, \quad (2.10)$$

where l_0 is the constant. Then, we take the following transformation

$$U = \pm\sqrt{(2l_2)^{-\frac{1}{3}}\Phi},\ p = 4l_1(2l_2)^{-\frac{2}{3}},\ q = 8l_0(2l_2)^{-\frac{1}{3}},\ \xi_1 = (2l_2)^{\frac{1}{3}}\xi. \quad (2.11)$$

Inserting (2.11) into (2.10), we have

$$(\Phi_{\xi_1})^2 = \Phi(\Phi^2 + p\Phi + q). \tag{2.12}$$

Next, we can get the integral expression of Eq (2.12)

$$\pm(\xi_1 - \xi_0) = \int \frac{d\Phi}{\sqrt{\Phi(\Phi^2 + p\Phi + q)}}. \tag{2.13}$$

Here, we set $F(\Phi) = \Phi^2 + p\Phi + q$ and $\Delta = p^2 - 4q$. According to the root of $F(\Phi) = 0$, the solution of Eq (2.13) has the following four cases.

Case 1. $\Delta = 0$ and $\Phi > 0$.

When $p < 0$ and $l_1 < 0$, the solution of Eq (1.2) is

$$u_1(t,x) = \pm\sqrt{-\frac{l_1}{l_2}\tanh^2(\frac{(-2l_1)^{\frac{1}{2}}}{2}(x - ct - \xi_0))}e^{i(-kx+wt+\theta_0)}, \tag{2.14}$$

$$u_2(t,x) = \pm\sqrt{-\frac{l_1}{l_2}\coth^2(\frac{(-2l_1)^{\frac{1}{2}}}{2}(x - ct - \xi_0))}e^{i(-kx+wt+\theta_0)}. \tag{2.15}$$

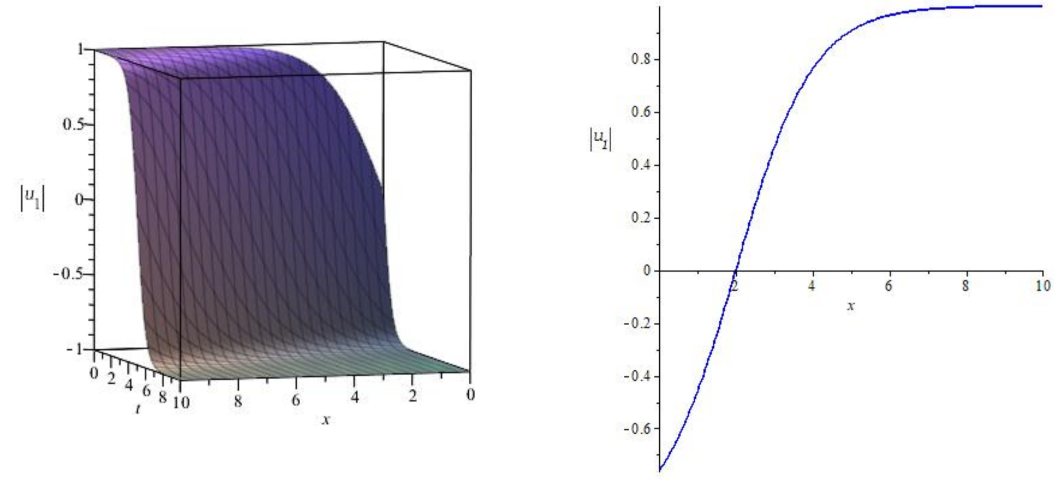
When $p > 0$ and $l_1 > 0$, the solution of Eq (1.2) is

$$u_3(t,x) = \sqrt{\frac{l_1}{l_2}\tan^2(\frac{(2l_1)^{\frac{1}{2}}}{2}(x - ct - \xi_0))}e^{i(-kx+wt+\theta_0)}. \tag{2.16}$$

When $p = 0$ and $l_2 > 0$, the solution of Eq (1.2) is

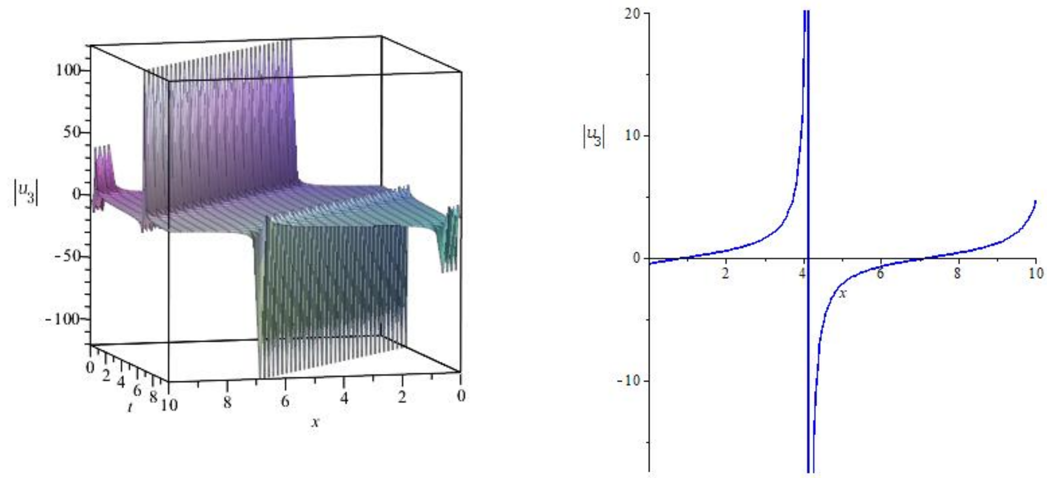
$$u_4(t,x) = \pm\sqrt{\frac{2}{l_2(x - ct - \xi_0)^2}}e^{i(-kx+wt+\theta_0)}. \tag{2.17}$$

By selecting appropriate parameters, we draw the solution $|u_1(t,x)|$ and $|u_3(t,x)|$ of two-dimensional and three-dimensional graphics as shown in Figures 2 and 3, respectively.



(a) Three-dimension graphic. (b) Two-dimension graphic.

Figure 2. The solution of Eq (1.2) when $\delta_1 = 2$, $A = 2$, $a_1 = -1$, $\alpha_1 = 2$, $k = 1$, $c = 1$, $\gamma_1 = \frac{1}{2}$, $\lambda_1 = \frac{1}{2}$, $\xi_0 = 0$.



(a) Three-dimension graphic. (b) Two-dimension graphic.

Figure 3. The solution of Eq (1.2) when $\delta_1 = 2$, $A = 2$, $a_1 = -1$, $\alpha_1 = 2$, $k = 1$, $c = 1$, $\gamma_1 = 1$, $\lambda_1 = -3$, $\xi_0 = 0$.

Case 2. $\Delta > 0$ and $q = 0$.

When $\Phi > -p$ and $p < 0$, the solution of Eq (1.2) is

$$u_5(t,x) = \pm\sqrt{\frac{l_1}{l_2}\tanh^2\left(\frac{(2l_1)^{\frac{1}{2}}}{2}(x - ct - \xi_0)\right) - \frac{2l_1}{l_2}}\, e^{i(-kx+wt+\theta_0)}, \tag{2.18}$$

$$u_6(t,x) = \sqrt{\frac{l_1}{l_2}\coth^2(\frac{(2l_1)^{\frac{1}{2}}}{2}(x-ct-\xi_0)) - \frac{2l_1}{l_2}} e^{i(-kx+wt+\theta_0)}. \tag{2.19}$$

When $\Phi > -p$ and $p > 0$, the solution of Eq (1.2) is

$$u_7(t,x) = \pm\sqrt{-\frac{l_1}{l_2}\tan^2(\frac{(-2l_1)^{\frac{1}{2}}}{2}(x-ct-\xi_0)) + \frac{2l_1}{l_2}} e^{i(-kx+wt+\theta_0)}. \tag{2.20}$$

Case 3. $\Delta > 0$ and $p \neq 0$.

Assume that there are constants α, β and γ satisfying $\alpha < \beta < \gamma$, here one of them is zero and two other constants are the roots of $F(\Phi) = 0$. Thus, when $\alpha < \Phi < \beta$, we have

$$u_8(t,x) = \pm\sqrt{(2l_2)^{-\frac{1}{3}}[\alpha + (\beta-\alpha)\mathrm{sn}^2(\frac{\sqrt{\gamma-\alpha}}{2})(2l_2)^{\frac{1}{3}}(\xi_1 - \xi_0), m]} e^{i(-kx+wt+\theta_0)}, \tag{2.21}$$

$$u_9(t,x) = \pm\sqrt{(2l_2)^{-\frac{1}{3}}[\frac{-\beta\mathrm{sn}^2(\frac{1}{2}\sqrt{\gamma-\alpha}(2l_2)^{\frac{1}{3}}(\xi-\xi_0), m) + \gamma}{\mathrm{cn}^2(\frac{1}{2}\sqrt{\gamma-\alpha}(2l_2)^{\frac{1}{3}}(\xi-\xi_0), m)}]} e^{i(-kx+wt+\theta_0)}, \tag{2.22}$$

where $m^2 = \frac{\beta-\alpha}{\gamma-\alpha}$.

Case 4. $\Delta < 0$.

When $\Phi > \beta$, we have

$$u_{10}(t,x) = \pm\sqrt{2(\frac{l_0}{l_2})^{\frac{1}{2}}[\frac{2}{1+\mathrm{cn}(2(l_0l_2)^{\frac{1}{4}}(\xi-\xi_0), m)} - 1]} e^{i(-kx+wt+\theta_0)}, \tag{2.23}$$

where $m^2 = \frac{(2l_0)^{\frac{1}{2}} - l_1}{2(2l_0)^{\frac{1}{2}}}$.

Remark 2.1. From the linear transformation $V = AU$, traveling wave transformation (2.1) and the obtained solution $u(t,x)$, we can easily get the solution $v(t,x)$.

3. Conclusions

In this paper, the phase diagram is drawn with the help of Maple software and planar dynamic system theory. Moreover, the complete discrimination system of polynomial method has been applied to construct the single traveling wave solutions of the coupled version of RKL equation in fiber Bragg gratings. The solution obtained in the paper is also very effective in physics, which can help physicists understand the propagation of traveling wave in coupled RKL equation. Moreover, we have also depicted two-dimensional and three-dimensional graphs of Eq (1.2).

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Biswas, Optical soliton perturbation with Radhakrishnan-Kundu-Lakshmanan equation by traveling wave hypothesis, *Optik*, **171** (2018), 217–220. http://dx.doi.org/10.1016/j.ijleo.2018.06.043

2. M. Annamalai, N. Veerakumar, S. Narasimhan, A. Selvaraj, Q. Zhou, A. Biswas, et al., Algorithm for dark solitons with Radhakrishnan-Kundu-Lakshmanan model in an optical fiber, *Results Phys.*, **30** (2021), 104806. http://dx.doi.org/10.1016/j.rinp.2021.104806

3. A. Biswas, M. Ekici, A. Sonmezoglu, A. Alshomrani, Optical solitons with Radhakrishnan-Kundu-Lakshmanan equation by extended trial function scheme, *Optik*, **160** (2018), 415–427. http://dx.doi.org/10.1016/j.ijleo.2018.02.017

4. S. ur Rehman, J. Ahmad, Modulation instability analysis and optical solitons in birefringent fibers to RKL equation without four wave mixing, *Alex. Eng. J.*, **60** (2021), 1339–1354. http://dx.doi.org/10.1016/j.aej.2020.10.055

5. A. Biswas, Y. Yıldırım, E. Yasar, M. Mahmood, A. Alshorani, Q. Zhou, et al., Optical soliton perturbation for Radhakrishnan-Kundu-Lakshmanan equation with a couple of integration schemes, *Optik*, **163** (2018), 126–136. http://dx.doi.org/10.1016/j.ijleo.2018.02.109

6. Y. Yıldırım, A. Biswas, Q. Zhou, A. Alzahrani, M. Belic, Optical solitons in birefringent fibers with Radhakrishnan-Kundu-Lakshmanan equation by a couple of strategically sound integration architectures, *Chinese J. Phys.*, **65** (2020), 341–354. http://dx.doi.org/10.1016/j.cjph.2020.02.029

7. D. Lu, A. Seadawy, M. Khater, Dispersive optical soliton of the generalized Radhakrishnan-Kundu-Lakshmanan dynamical equation with power law nonlinearity and its applications, *Optik*, **164** (2018), 54–64. http://dx.doi.org/10.1016/j.ijleo.2018.02.082

8. N. Raza, A. Javid, Dynamics of optical solitons with Radhakrishnan-Kundu-Lakshmanan model via two reliable integration schemes, *Optik*, **178** (2019), 557–566. http://dx.doi.org/10.1016/j.ijleo.2018.09.133

9. A. Ghose-Choudhury, S. Garai, Solutions of the variabel coefficient Radhakrishnan-Kundu-Lakshmanan equation using the method of similarity reduction, *Optik*, **241** (2021), 167254. http://dx.doi.org/10.1016/j.ijleo.2021.167254

10. S. Garai, A. Ghose-Choudhury, On the solution of the generalized Radhakrishnan-Kundu-Lakshmanan equation, *Optik*, **243** (2021), 167374. http://dx.doi.org/10.1016/j.ijleo.2021.167374

11. G. Akram, M. Sadaf, M. Dawood, Abundant soliton solutions for Radhakrishnan-Kundu-Lakshmanan equation with Kerr law non-linearity by improved $\tan(\frac{\Phi(\xi)}{2})$-expansion technique, *Optik*, **247** (2021), 167787. http://dx.doi.org/10.1016/j.ijleo.2021.167787

12. W. Rabie, A. Seadawy, H. Ahmed, Highly dispersive optical solitons to the generalized third-order nonlinear Schrödinger dynamical equation with applications, *Optik*, **241** (2021), 167109. http://dx.doi.org/10.1016/j.ijleo.2021.167109

13. M. El-Sheikh, H. Ahmed, A. Arnous, W. Rabie, A. Biswas, A. Alshomrani, et al., Optical solitons in birefringent fibers with Lakshmanan-Porsezian-Daniel model by modified simple equation, *Optik*, **192** (2019), 162899. http://dx.doi.org/10.1016/j.ijleo.2019.05.105

14. H. Eldidamony, H. Ahmed, A. Zaghrout, Y. Ali, A. Arnous, Optical solitons with Kudryashov's

quintuple power law nonlinearity having nonlinear chromatic dispersion using modified extended direct algebraic method, *Optik*, **262** (2022), 169235. http://dx.doi.org/10.1016/j.ijleo.2022.169235

15. I. Samir, N. Badra, A. Seadawy, H. Ahmed, A. Arnous, Exact wave solutions of the fourth order nonlienar partial differential equation of optical fiber pulses by using different methods, *Optik*, **230** (2021), 166313. http://dx.doi.org/10.1016/j.ijleo.2021.166313
16. A. Seadawy, H. Ahmed, W. Rabie, A. Biswas, Chirp-free optical solitons in fiber bragg gratings with dispersive reflectivity having polynomial law of nonlinearity, *Optik*, **225** (2021), 165681. http://dx.doi.org/10.1016/j.ijleo.2020.165681
17. K. Nisar, M. Inc, A. Jhangeer, M. Muddasar, B. Infal, New soliton solutions of Heisenberg ferromagnetic spin chain model, *Pramana-J. Phys.*, **96** (2022), 28. http://dx.doi.org/10.1007/s12043-021-02266-y
18. M. Khater, A. Jhangeer, H. Rezazadeh, L. Akinyemi, M. Akbar, M. Inc, Propagation of new dynamics of longitudinal bud equation among a magneto-electro-elastic round rod, *Mod. Phys. Lett. B*, **35** (2021), 2150381. http://dx.doi.org/10.1142/S0217984921503814
19. Z. Li, Bifurcation and traveling wave solution to fractional Biswas-Arshed equation with the beta time derivative, *Chaos Soliton. Fract.*, **160** (2022), 112249. http://dx.doi.org/10.1016/j.chaos.2022.112249
20. A. Jhangeer, M. Muddassar, J. Awrejcewicz, Z. Naz, M. Riaz, Phase portrait,multi-stability,sensitivity and chaotic analysis of Gardner's equation with their wave turbulence and solitons solutions, *Results Phys.*, **32** (2022), 104981. http://dx.doi.org/10.1016/j.rinp.2021.104981
21. Z. Li, Z. Lian, Optical solitons and single traveling wave solutions for the Triki-Biswas equation describing monomode optical fibers, *Optik*, **258** (2022), 168835. http://dx.doi.org/10.1016/j.ijleo.2022.168835
22. T. Han, Z. Li, Classification of all single traveling wave solutions of fractional coupled Boussinesq equations via the complete discrimination system method, *Adv. Math. Phys.*, **2021** (2021), 3668063. http://dx.doi.org/10.1155/2021/3668063
23. T. Han, Z. Li, X. Zhang, Bifurcation and new exact traveling wave solutions to time-space coupled fractional nonlinear Schrödinger equation, *Phys. Lett. A*, **395** (2021), 127217. http://dx.doi.org/10.1016/j.physleta.2021.127217
24. E. Zayed, R. Shohib, M. Alngar, Y. Yıldırım, Optical solitons in fiber Bragg gratings with Radhakrishnan-Kundu-Lakshmanan equation using two integration schemes, *Optik*, **245** (2021), 167635. http://dx.doi.org/10.1016/j.ijleo.2021.167635