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Research article

A posteriori mesh method for a system of singularly perturbed initial value problems

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Abstract: A system of singularly perturbed initial value problems with weak constrained conditions on the coefficients is considered. First the system of second-order singularly perturbed problems is transformed into a system of first-order singularly perturbed problems with integral terms, which facilitates the subsequent stability and a posteriori error analyses. Then a hybrid difference method with the use of interpolating quadrature rules is utilized to approximate the transformed system. Next a posteriori error analysis for the discretization scheme on an arbitrary mesh is presented. A solution-adaptive algorithm based on a posteriori error estimation is devised to generate a posteriori mesh and obtain approximation solution. Finally numerical experiments show a uniform convergence behavior of second-order for the scheme, which improves the previous results and achieves the optimal convergence order under the given discrete scheme.

Keywords: singular perturbation; hybrid difference scheme; mesh equidistribution; a posteriori error analysis; second order problems
Mathematics Subject Classification: 65L10, 65L12

1. Introduction

In this article we consider the following system of second-order singularly perturbed initial value problems

$$\begin{cases} \varepsilon_1 u_1''(x) + (a_1(x)u_1(x))' + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), & x \in \Omega, \\ \varepsilon_2 u_2''(x) + (a_2(x)u_2(x))' + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), & x \in \Omega, \\ u_1(0) = A_1, \quad u_1'(0) = B_1/\varepsilon_1, \\ u_2(0) = A_2, \quad u_2'(0) = B_2/\varepsilon_2, \end{cases} \tag{1.1}$$

where $\mathbf{u}(x) = (u_1(x), u_2(x))^T$, $\Omega = (0, 1]$, ε_1 and ε_2 are small positive parameters, A_1, A_2, B_1 and B_2 are given constants, $a_k(x) \geq \alpha_k > 0$, $f_k(x)$ and $b_{kl}(x)$ for $k, l = 1, 2$ are given sufficiently smooth functions. Under these conditions, system (1.1) has a unique solution $\mathbf{u}(x)$ with exponential boundary layers [1]. Such systems of singularly perturbed initial value problems arise in many areas of science and engineering, for example, control theory [2], optically bistable devices [3].

There are some literatures on numerical methods for singularly perturbed initial value problems. The earliest literature on solving singularly perturbed initial value problems can be traced back to the 1970s. Recently, Cakir [4] proposed a finite difference scheme on a Shishkin-type mesh to solve a second-order singularly perturbed quasilinear initial value problem. Kumar et al. [5] and Cen et al. [6] used difference schemes both on the Shishkin mesh and the Bakhvalov mesh to solve a system of first-order singularly perturbed initial value problems. Liu and Chen [7] developed a backward Euler difference scheme on an adaptive moving mesh for solving a system of first-order singularly perturbed initial value problems. In the previous works, some assumptions about $(b_{kl})_{2\times 2}$ are often needed to ensure the stability of numerical methods. But the coefficient functions $b_{kl}(x)$ with $1 \leq k, l \leq 2$ in the system (1.1) only satisfies the smoothness condition, which is weaker than the hypotheses given in the above literatures and brings out difficulties in numerical analysis.

For grasping the properties of the singularly perturbed problems, one effective method is to generate layer adapted meshes. If a priori information about the properties of exact solution is available, a priori layer-adapted meshes can be selected to fit the singularity of the exact solution [8]. In [1] a hybrid difference scheme on a priori mesh is developed for the system (1.1) and almost second-order convergence is proved. A posteriori meshes also can be used to fit the singularity of the exact solution, which don't need to know a priori information of the exact solution and are also devised to be very fine where sharp layers appear in the solution. Due to none a priori information needed, a posteriori mesh method has more extensive application. Some literatures have used a posteriori mesh methods to solve singularly perturbed problems, see [9–15] for instance. However, there is no literature on the use of a posteriori mesh method to solve second-order singularly perturbed initial value problems with weak constrained conditions on the coefficients.

The aim of this work is to construct a novel numerical scheme on a posteriori mesh for the system (1.1) with weak constrained conditions on the coefficients and improve the results given in [1]. For simplifying the problem to facilitate the subsequent stability analysis and a posteriori error estimation, the system of second-order singularly perturbed problems is transformed into a system of first-order singularly perturbed problems with integral terms. Then a hybrid difference method with the use of interpolating quadrature rules is developed to approximate the transformed system. Next a posteriori error analysis for the discretization scheme on an arbitrary mesh is presented by introducing a piecewise quadratic interpolation function. Based on a posteriori error estimation, a solution-adaptive algorithm is devised to generate a posteriori mesh for adapting the singularity of the exact solution and obtain an accurate approximation solution. Numerical experiments show a uniform convergence behavior of second-order for the scheme, which demonstrates that a posteriori mesh method improves the results given in [1] and achieves the optimal convergence order under the given discrete scheme.

Notation. Throughout the paper, C represents a positive constant independent of the mesh, which can represent different positive constants in different places. For any function g defined on $\bar{\Omega}$, we denote $g_i = g(x_i)$ for simplifying the notation and use $\|\cdot\|$ to represent the (pointwise) maximum norm on $\bar{\Omega}$.

2. The continuous problem

By integrating both sides of equations, the system (1.1) can be transformed into

$$\begin{cases} L_1\mathbf{u}(x) \equiv \varepsilon_1 u_1'(x) + a_1(x)u_1(x) + \int_0^x b_{11}(s)u_1(s)\mathrm{d}s + \int_0^x b_{12}(s)u_2(s)\mathrm{d}s = \bar{f}_1(x), & x \in \Omega, \\ L_2\mathbf{u}(x) \equiv \varepsilon_2 u_2'(x) + a_2(x)u_2(x) + \int_0^x b_{21}(s)u_1(s)\mathrm{d}s + \int_0^x b_{22}(s)u_2(s)\mathrm{d}s = \bar{f}_2(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where

$$\bar{f}_k(x) = B_k + a_k(0)A_k + \int_0^x f_k(s)\mathrm{d}s, \qquad k = 1, 2.$$

The following lemmas give the stability results for the exact solution $\mathbf{u}(x)$ of system (2.1), which will be used in a posteriori error analysis.

Lemma 2.1. If the function $v(x)$ is the solution of the following initial value problem

$$\begin{cases} \varepsilon v'(x) + p(x)v(x) = q(x), & x \in \Omega, \\ v(0) = A, \end{cases} \tag{2.2}$$

with $\varepsilon > 0$, $p(x) \geq \alpha > 0$, $|q(x)| \leq Q(x)$ and $Q(x) > 0$ is a nondecreasing continuous function, then we have

$$|v(x)| \leq |A| + \alpha^{-1}Q(x), \qquad x \in \bar{\Omega}.$$

Proof. The solution $v(x)$ of problem (2.2) can be explicitly written as

$$v(x) = A \exp\left(-\varepsilon^{-1}\int_0^x p(t)\mathrm{d}t\right) + \varepsilon^{-1}\int_0^x q(t)\exp\left(-\varepsilon^{-1}\int_t^x p(s)\mathrm{d}s\right)\mathrm{d}t.$$

Then we have

$$\begin{aligned} |v(x)| &\leq |A|\, e^{-\alpha x/\varepsilon} + Q(x)\int_0^x \varepsilon^{-1}e^{-\alpha(x-t)/\varepsilon}\mathrm{d}t \\ &\leq |A| + \alpha^{-1}Q(x), \end{aligned}$$

where we have used $Q(x)$ is a positive and nondecreasing continuous function.

Lemma 2.2. The solution $\mathbf{u}(x)$ of system (2.1) has the following bound

$$|u_1(x)| + |u_2(x)| \leq C\left(|A_1| + |A_2| + \left\|\bar{f}_1\right\| + \left\|\bar{f}_2\right\|\right), \qquad x \in \bar{\Omega}.$$

Proof. It is easy to get the following estimates

$$\begin{aligned} &\left|\bar{f}_k(x) - \int_0^x b_{kk}(s)u_k(s)\mathrm{d}s - \int_0^x b_{k,3-k}(s)u_{3-k}(s)\mathrm{d}s\right| \\ &\leq \left|\bar{f}_k\right| + \|b_{kk}\|\int_0^x |u_k(s)|\,\mathrm{d}s + \left\|b_{k,3-k}\right\|\int_0^x |u_{3-k}(s)|\,\mathrm{d}s \end{aligned}$$

for $k = 1, 2$. Then, by applying Lemma 2.1 to (2.1) we have

$$|u_k(x)| \le |A_k| + \alpha^{-1}\left[\left\|\bar{f}_k\right\| + \|b_{kk}\| \int_0^x |u_k(s)|\,\mathrm{d}s + \left\|b_{k,3-k}\right\| \int_0^x |u_{3-k}(s)|\,\mathrm{d}s\right]$$

for $k = 1, 2$. Thus, from the above inequalities we can obtain the following estimate

$$|u_1(x)| + |u_2(x)| \le C\left[|A_1| + |A_1| + \left\|\bar{f}_1\right\| + \left\|\bar{f}_2\right\| + \alpha^{-1}\int_0^x (|u_1(s)| + |u_2(s)|)\,\mathrm{d}s\right].$$

Hence, by using the Gronwall's inequality we have the following result

$$|u_1(x)| + |u_2(x)| \le C\left(|A_1| + |A_1| + \left\|\bar{f}_1\right\| + \left\|\bar{f}_2\right\|\right).$$

So far, we conclude that the lemma holds true.

3. Discretization scheme

In this section a hybrid difference scheme with the use of interpolating quadrature rules on a posteriori mesh will be developed for system (2.1). First, an arbitrary mesh $\Omega^N = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ with mesh sizes $h_i = x_i - x_{i-1}$ for $1 \le i \le N$ is introduced. Then, the discretization scheme on Ω^N can be constructed as follows

$$\begin{cases} L_1^N \mathbf{U}_i = \tilde{f}_{1,i}, \;\; L_2^N \mathbf{U}_i = \tilde{f}_{2,i}, & i = 1, 2, \cdots, N, \\ U_{1,0} = A_1, \quad U_{2,0} = A_2, \end{cases} \tag{3.1}$$

where $\mathbf{U} = (U_1, U_2)^T$ is the discrete approximation of the exact solution $\mathbf{u} = (u_1, u_2)^T$,

$$L_k^N \mathbf{U}_i = \begin{cases} \varepsilon_k D^- U_{k,i} + \frac{1}{2}\left(g_{k,i-1} + g_{k,i}\right), & \frac{\varepsilon_k}{h_i} \ge \frac{a_{k,i-1}}{2}, \\ \varepsilon_k D^- U_{k,i} + g_{k,i}, & \text{otherwise}, \end{cases} \tag{3.2}$$

$$\tilde{f}_{k,i} = \begin{cases} \frac{1}{2}\left(F_{k,i-1} + F_{k,i}\right), & \frac{\varepsilon_k}{h_i} \ge \frac{a_{k,i-1}}{2}, \\ F_{k,i}, & \text{otherwise}, \end{cases} \tag{3.3}$$

$$g_{k,i} = a_{k,i} U_{k,i} + \int_0^{x_i} \left[b_{kk}^I(s) U_k^I(s) + b_{k,3-k}^I(s) U_{3-k}^I(s)\right] \mathrm{d}s, \tag{3.4}$$

$$F_{k,i} = B_k + a_k(0) A_k + \int_0^{x_i} f_k^I(s)\mathrm{d}s, \tag{3.5}$$

and

$$D^- U_{k,i} = \frac{U_{k,i} - U_{k,i-1}}{h_i}, \qquad U_k^I(x) = U_{k,i} + (x - x_i)\, D^- U_{k,i}, ,$$
$$b_{kl}^I(x) = b_{kl,i} + (x - x_i)\, D^- b_{kl,i}, \qquad f_k^I(x) = f_{k,i} + (x - x_i)\, D^- f_{k,i}$$

for $x \in [x_{i-1}, x_i]$ and $k, l = 1, 2$.

The stability results of the discretization scheme (3.1) are given in the following lemmas.

Lemma 3.1. If the mesh function V on Ω^N is the solution of the following initial value problem

$$\begin{cases} \varepsilon D^- V_i + \frac{1}{2}\left(p_{i-1}V_{i-1} + p_i V_i\right) = \frac{1}{2}\left(q_{i-1} + q_i\right), & \frac{\varepsilon}{h_i} \geq \frac{p_{i-1}}{2}, \\ \varepsilon D^- V_i + p_i V_i = q_i, & \text{otherwise}, \\ V_0 = A \end{cases}$$

with $\varepsilon > 0$, $p_i \geq \alpha > 0$, $|q_i| \leq Q_i$, and $Q > 0$ is a nondecreasing mesh function, then we have

$$|V_i| \leq |A| + \alpha^{-1} Q_i, \qquad 0 \leq i \leq N.$$

Proof. This result can be proved by using the discrete maximum principle. See the literature [16, Lemma 3] for details.

Lemma 3.2. The solution $\mathbf{U}$ of the discretization scheme (3.1) has the following bound

$$\left|U_{1,i}\right| + \left|U_{2,i}\right| \leq C\left(|A_1| + |A_2| + \left\|\tilde{f}_1\right\| + \left\|\tilde{f}_2\right\|\right), \qquad 0 \leq i \leq N.$$

Proof. By applying Lemma 3.1 and the discrete Gronwall's inequality to the discretization scheme (3.1) we can obtain the desired result.

Now we demonstrate a posteriori error estimation for the scheme (3.1) on an arbitrary mesh Ω^N. Let $\bar{\mathbf{U}}(x) = \left(\bar{U}_1(x), \bar{U}_2(x)\right)^T$ be a piecewise quadratic interpolation function vector of the computed solution $\mathbf{U}$ as that in [12, 17], which is defined by

$$\bar{U}_k(x) = U_{k,i} + (x - x_i)\, D^- U_{k,i} + \frac{1}{2}\,(x - x_{i-1})\,(x - x_i)\, D^- D^- U_{k,i}, \qquad x \in [x_{i-1}, x_i] \tag{3.6}$$

for $1 \leq i \leq N$ and $k = 1, 2$, where

$$D^- D^- U_{k,i} = \frac{D^- U_{k,i} - D^- U_{k,i-1}}{h_i}, \qquad D^- U_{k,0} = 0.$$

Then we can easily get the following results

$$\bar{U}_k(x_i) = U_{k,i}, \quad \left[\bar{U}_k(x)\right]' = D^- U_{k,i} + \left(x - x_{i-1/2}\right) D^- D^- U_{k,i}, \qquad x \in [x_{i-1}, x_i] \tag{3.7}$$

for $1 \leq i \leq N$ and $k = 1, 2$, where $x_{i-1/2} = \left(x_{i-1} + x_i\right)/2$.

Theorem 3.3. Let $\mathbf{u}(x)$ be the exact solution vector of system (2.1), $\mathbf{U}$ be the numerical solution vector of the discretization scheme (3.1) on an arbitrary mesh Ω^N and $\bar{\mathbf{U}}(x)$ be its piecewise quadratic interpolation function vector defined in (3.6). Then, under the assumptions $\left|a_k''(x)\right| \leq C$ and $\left|b_{kl}''(x)\right| \leq C$ for $k, l = 1, 2$, the following estimate

$$\begin{aligned} \left\|\bar{\mathbf{U}}(x) - \mathbf{u}(x)\right\|_{\bar{\Omega}} \leq\ & C \max_{1\leq i\leq N} h_i^2 \Big\{1 + \left|D^- U_{1,i}\right| + \left|D^- U_{2,i}\right| + \left|D^- f_{1,i}\right| + \left|D^- f_{2,i}\right| \\ & + \left|D^- D^- U_{1,i}\right| + \left|D^- D^- U_{2,i}\right|\Big\} \end{aligned}$$

holds true.

Proof. We derive a posteriori error estimation for the discretization scheme (3.1) based on the stability results given in Lemma 2.2. According to the characteristics of the discretization scheme, we estimate the error $\left|L_k\left(\bar{\mathbf{U}}(x)-\mathbf{u}(x)\right)\right|$ in the case $\frac{\varepsilon_k}{h_i} > \frac{a_{k,i-1}}{2}$ and the case $\frac{\varepsilon_k}{h_i} \leq \frac{a_{k,i-1}}{2}$ respectively.

For $x \in [x_{i-1}, x_i]$ we set

$$\begin{aligned}
w_k(x) &= a_k(x)\bar{U}_k(x) + \int_0^x \left[b_{kk}(s)\bar{U}_k(s) + b_{k,3-k}(s)\bar{U}_{3-k}(s)\right]\mathrm{d}s, && (3.8)\\
\bar{w}_k(x) &= a_k(x)\bar{U}_k(x) + \int_0^x \left[b^I_{kk}(s)\bar{U}_k(s) + b^I_{k,3-k}(s)\bar{U}_{3-k}(s)\right]\mathrm{d}s, && (3.9)\\
F_k(x) &= B_k + a_k(0)A_k + \int_0^x f^I_k(s)\mathrm{d}s, && (3.10)\\
\bar{w}^I_k(x) &= \bar{w}_{k,i} + (x-x_i)\,D^-\bar{w}_{k,i}, && (3.11)\\
F^I_k(x) &= F_{k,i} + (x-x_i)\,D^-F_{k,i}, && (3.12)
\end{aligned}$$

with $1 \leq i \leq N$ and $k = 1, 2$. Combining (3.4), (3.6) with (3.9) we have

$$\bar{w}_{k,i} = g_{k,i} + \frac{1}{2}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\left(s-x_{j-1}\right)\left(s-x_j\right)\left[b^I_{kk}(s)D^-D^-U_{k,j} + b^I_{k,3-k}(s)D^-D^-U_{3-k,j}\right]\mathrm{d}s. \tag{3.13}$$

Then, for $x \in (x_{i-1}, x_i)$ we can derive the following equality

$$\begin{aligned}
L_k\bar{\mathbf{U}}(x) - L_k\mathbf{u}(x) &= \varepsilon_k\left[\bar{U}_k(x)\right]' + w_k(x) - \bar{f}_k(x)\\
&= \varepsilon_k D^-U_{k,i} + (x-x_{i-1/2})\,\varepsilon_k D^-D^-U_{k,i} + \bar{w}^I_k(x) + \left(\bar{w}_k(x) - \bar{w}^I_k(x)\right)\\
&\quad + (w_k(x) - \bar{w}_k(x)) - F^I_k(x) + \left(F^I_k(x) - F_k(x)\right) + \left(F_k(x) - \bar{f}_k(x)\right), && (3.14)
\end{aligned}$$

where we have used the equalities (2.1) and (3.7). For simplifying the notation we set

$$R_{k,i} = \bar{w}_k(x) - \bar{w}^I_k(x) + w_k(x) - \bar{w}_k(x) + F^I_k(x) - F_k(x) + F_k(x) - \bar{f}_k(x), \qquad k = 1, 2. \tag{3.15}$$

Thus, for the case $\frac{\varepsilon_k}{h_i} > \frac{a_{k,i-1}}{2}$ we derive the following estimate

$$\begin{aligned}
&\left|L_k\bar{\mathbf{U}}(x) - L_k\mathbf{u}(x)\right|\\
&= \left|\frac{F_{k,i-1}+F_{k,i}}{2} - \frac{g_{k,i-1}+g_{k,i}}{2} + (x-x_{i-1/2})\,\varepsilon_k D^-D^-U_{k,i} + \bar{w}^I_{k,i} - F^I_{k,i} + R_{k,i}\right|\\
&= \left|\frac{F_{k,i-1}+F_{k,i}}{2} - \frac{\bar{w}_{k,i-1}+\bar{w}_{k,i}}{2} + (x-x_{i-1/2})\,\varepsilon_k D^-D^-U_{k,i}\right.\\
&\quad + \bar{w}_{k,i} + (x-x_i)\,D^-\bar{w}_{k,i} - F_{k,i} - (x-x_i)\,D^-F_{k,i} + R_{k,i}\Big|\\
&\quad \left.+\frac{1}{2}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\left(s-x_{j-1}\right)\left(s-x_j\right)\left[b^I_{kk}(s)D^-D^-U_{k,j} + b^I_{k,3-k}(s)D^-D^-U_{3-k,j}\right]\mathrm{d}s\right|\\
&\leq \left|(x-x_{i-1/2})\,D^-\left(\varepsilon_k D^-U_{k,i} + \bar{w}_{k,i} - F_{k,i}\right)\right| + \left|R_{k,i}\right| + C\max_{1\leq j\leq i} h_j^2\left|D^-D^-U_{k,j}\right|\\
&= \left|(x-x_{i-1/2})\,D^-\left(\frac{F_{k,i-1}+F_{k,i}}{2} - \frac{g_{k,i-1}+g_{k,i}}{2} - F_{k,i}\right)\right| + \left|R_{k,i}\right| + C\max_{1\leq j\leq i} h_j^2\left|D^-D^-U_{k,j}\right|
\end{aligned}$$

$$\le \left|\frac{1}{2}\left(x - x_{i-1/2}\right) D^-\left(\bar{w}_{k,i} - \bar{w}_{k,i-1} + F_{k,i} - F_{k,i-1}\right)\right| + \left|R_{k,i}\right| + C \max_{1\le j\le i} h_j^2 \left|D^-D^-U_{k,j}\right|$$
$$\le Ch_i^2\left(\left|D^-D^-\bar{w}_{k,i}\right| + \left|D^-D^-F_{k,i}\right|\right) + \left|R_{k,i}\right| + C \max_{1\le j\le i} h_j^2 \left|D^-D^-U_{k,j}\right|$$
$$\le C \max_{1\le j\le i} h_j^2 \left(\left|D^-D^-U_{k,j}\right| + \left|D^- f_{k,j}\right|\right) + \left|R_{k,i}\right|, \tag{3.16}$$

where we have used the equalities (2.1), (3.1)–(3.14). For the case $\frac{\varepsilon_k}{h_i} \le \frac{a_{k,i-1}}{2}$, we derive the following estimate

$$\left|L_k\bar{\mathbf{U}}(x) - L_k\mathbf{u}(x)\right|$$
$$= \left|F_{k,i} - g_{k,i} + \left(x - x_{i-1/2}\right)\varepsilon_k D^-D^-U_{k,i} + \bar{w}^I_{k,i} - F^I_{k,i} + R_{k,i}\right|$$
$$= \Big|F_{k,i} - \bar{w}_{k,i} + \left(x - x_{i-1/2}\right)\varepsilon_k D^-D^-U_{k,i} + \bar{w}_{k,i} + \left(x - x_i\right) D^-\bar{w}_{k,i} - F_{k,i} - \left(x - x_i\right) D^-F_{k,i}$$
$$+R_{k,i} + \frac{1}{2}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\left(s - x_{j-1}\right)\left(s - x_j\right)\left[b^I_{kk}(s)D^-D^-U_{k,j} + b^I_{k,3-k}(s)D^-D^-U_{3-k,j}\right]\mathrm{d}s\Big|$$
$$\le \left|\left(x_i - x_{i-1/2}\right)\varepsilon_k D^-D^-U_{k,i}\right| + \left|R_{k,i}\right| + C \max_{1\le j\le i} h_j^2 \left|D^-D^-U_{k,j}\right|$$
$$\le C \max_{1\le j\le i} h_j^2 \left|D^-D^-U_{k,j}\right| + \left|R_{k,i}\right|, \tag{3.17}$$

where we also have used the equalities (2.1), (3.1)–(3.14). Furthermore, from the equality (3.15) we derive the following estimate

$$\begin{aligned}\left|R_{k,i}\right| &\le \left|\bar{w}_k(x) - \bar{w}^I_k(x)\right| + \left|w_k(x) - \bar{w}_k(x)\right| + \left|F^I_k(x) - F_k(x)\right| + \left|F_k(x) - \bar{f}_k(x)\right| \\ &\le C \max_{1\le i\le N} h_i^2 \Big\{1 + \left|D^-U_{1,i}\right| + \left|D^-U_{2,i}\right| + \left|D^-f_{1,i}\right| + \left|D^-f_{2,i}\right| \\ &\quad + \left|D^-D^-U_{1,i}\right| + \left|D^-D^-U_{2,i}\right|\Big\},\end{aligned} \tag{3.18}$$

where we have used the following linear interpolation remainder estimate

$$\left\|g(x) - g^I(x)\right\|_{\bar{\Omega}} \le \max_{1\le i\le N}\left\{\frac{h_i^2}{2}\sup_{(x_{i-1},x_i)}\left|g''(x)\right|\right\}$$

and the assumptions $\left|a''_k(x)\right| \le C$ and $\left|b''_{kl}(x)\right| \le C$ for $k, l = 1, 2$. Then we can derive the following estimate from the inequalities (3.16)–(3.18),

$$\begin{aligned}\|L_k\mathbf{U}(x) - L_k\mathbf{u}(x)\|_{\bar{\Omega}} &\le C \max_{1\le i\le N} h_i^2 \Big\{1 + \left|D^-U_{1,i}\right| + \left|D^-U_{2,i}\right| + \left|D^-f_{1,i}\right| + \left|D^-f_{2,i}\right| \\ &\quad + \left|D^-D^-U_{1,i}\right| + \left|D^-D^-U_{2,i}\right|\Big\}.\end{aligned} \tag{3.19}$$

Thus we can conclude that the lemma holds true by applying Lemma 2.2 with (3.19).

Next, we devise a solution-adapted algorithm to generate a posteriori adapted mesh and obtain the approximation solution. Based on the idea of the adaptive algorithm given by Kopteva and Stynes [18], we solve the following equidistribution problem to generate a posteriori adapted mesh

$$M_i h_i = \frac{1}{N}\sum_{j=1}^{N} M_j h_j, \qquad i = 1, \ldots, N,$$

where M_i is selected based on a posteriori error estimation as follows

$$M_i = 1 + \left|D^- U_{1,i}\right|^{1/2} + \left|D^- U_{2,i}\right|^{1/2} + \left|D^- f_{1,i}\right|^{1/2} + \left|D^- f_{2,i}\right|^{1/2} + \left|D^- D^- U_{1,i}\right|^{1/2} + \left|D^- D^- U_{2,i}\right|^{1/2}.$$

Algorithm:

Step 1. Define $\Omega^{N,(0)} = \left\{x_i^{(0)} \mid x_i^{(0)} = i/N, 0 \le i \le N\right\}$ as the initial iteration mesh with $k = 0$.

Step 2. Solve the discretization system (3.1) on $\Omega^{N,(k)} = \left\{x_i^{(k)} \mid 0 \le i \le N\right\}$ with the mesh size $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$ to get the approximation solution $\left\{U_i^{(k)}\right\}_{i=0}^{N}$.

Step 3. Let $l_i^{(k)} = h_i^{(k)} M_i^{(k)}$ and $I_i^{(k)} = \sum_{j=1}^{i} l_j^{(k)}$. Go to Step 5 if

$$\max_{1\le i\le N}\left\{l_i^{(k)}\right\} \le C_0 I_N^{(k)}/N$$

for a constant $C_0 > 1$ selected by the user; else continue with the next step.

Step 4. Let $Y_i^{(k)} = i I_N^{(k)}/N$. Generate a new mesh $\Omega^{N,(k+1)} = \left\{x_i^{(k+1)} | 0 \le i \le N\right\}$ by interpolating $\left(Y_i^{(k)}, x_i^{(k+1)}\right)$ to $\left(I_i^{(k)}, x_i^{(k)}\right)$ linearly. Return to Step 2 with setting $k = k + 1$.

Step 5. Take $\Omega^{N,*} = \Omega^{N,(k)}$ as the final adapted mesh and $\left\{U_i^*\right\}_{i=0}^{N} = \left\{U_i^{(k)}\right\}_{i=0}^{N}$ as the approximation solution.

Remark. In order to facilitate the analysis of the method, this paper only considers the system with two equations. This method can be easily extended to the systems with $M > 2$ equations.

4. Numerical experiments

In this section we give some numerical experiments on two test examples which have been considered in [1].

Example 4.1. Consider the following system of singularly perturbed initial value problems

$$\begin{cases} \varepsilon_1 u_1''(x) + u_1'(x) - (4 + x)u_1(x) + (1 + 2x)\, u_2(x) = f_1(x), & x \in (0, 1], \\ \varepsilon_2 u_2''(x) + u_2'(x) + (1 + x)u_1(x) + \left(2 + x^2\right) u_2(x) = f_2(x), & x \in (0, 1], \\ u_1(0) = 1, \quad u_1'(0) = 1/\varepsilon_1, \\ u_2(0) = 1, \quad u_2'(0) = 2/\varepsilon_2, \end{cases}$$

where $\mathbf{f}(x) = (f_1(x), f_2(x))^T$ is chosen such that $\mathbf{u}(x) = (u_1(x), u_2(x))^T$ with $u_1(x) = 1 - e^{-x/\varepsilon_1} + \varepsilon_2 e^{-x/\varepsilon_2} + x^2 + e^x - \varepsilon_2$ and $u_2(x) = 3 - 2e^{-x/\varepsilon_2} + x(1 + x) - \sin x$ is the exact solution vector. The singular perturbation parameters are chosen from the set $S_\varepsilon = \left\{(\varepsilon_1, \varepsilon_2) \,\middle|\, \varepsilon_1 = 2^{-0}, 2^{-8}, 2^{-16}, 2^{-24}, 2^{-32}, 2^{-40},\ \ \varepsilon_2 = 2^4, 2^{-4}, 2^{-12}, 2^{-20}, 2^{-28}, 2^{-36}\right\}$.

Following the idea of Kopteva and Stynes [18] describing the mesh movement, we use Figure 1, which needs to read from the bottom to the top, to indicate the generation process of a posteriori adapted mesh for Example 4.1 with $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$ and $N = 128$. From Figure 1 we know that the mesh refinement is carried out in the boundary layer region by the solution-adaptive algorithm based on the singular behavior of the exact solution. The maximum error is defined by

$$e^N = \max_{1\le i\le N}\left\{\left|u_{1,i} - U_{1,i}\right|, \left|u_{2,i} - U_{2,i}\right|\right\}$$

and the convergence rate is calculated by

$$r^N = \log_2\left(e^N/e^{2N}\right).$$

Error estimates and convergence rates of the hybrid difference scheme (3.1) on a posteriori adapted mesh for Example 4.1 with different combinations of ε_1 and ε_2 are presented in Table 1.

Table 1. Error estimates and convergence rates of the hybrid difference scheme on a posteriori adapted mesh and on a Shishkin mesh for Example 4.2.

		Number of mesh points N					
$(\varepsilon_1, \varepsilon_2)$		128	256	512	1024	2048	4096
$\left(2^0, 2^4\right)$	a posteriori	2.0667e-5	5.1643e-6	1.2891e-6	3.2192e-7	8.0418e-8	2.0096e-8
		2.001	2.002	2.002	2.001	2.001	-
	Shishkin	2.9745e-5	5.1425e-6	1.2856e-6	3.2140e-7	8.0349e-8	2.0087e-8
		2.000	2.000	2.000	2.000	1.999	-
$\left(2^{-8}, 2^{-4}\right)$	a posteriori	6.0787e-3	3.9669e-4	1.0200e-4	2.5678e-5	6.4437e-6	1.6140e-6
		3.938	1.959	1.990	1.995	1.997	-
	Shishkin	1.5063e-0	5.7009e-1	1.9378e-1	6.2442e-2	1.9370e-2	5.8467e-3
		1.402	1.557	1.634	1.689	1.728	-
$\left(2^{-16}, 2^{-12}\right)$	a posteriori	1.2770e-3	3.2798e-4	8.3452e-5	2.2417e-5	7.4204e-6	1.3353e-6
		1.961	1.975	1.896	1.595	2.474	-
	Shishkin	1.8161e-0	6.5715e-1	2.2198e-1	7.1324e-2	2.2085e-2	6.6632e-3
		1.467	1.566	1.638	1.691	1.729	-
$\left(2^{-24}, 2^{-20}\right)$	a posteriori	1.4432e-3	3.6782e-4	9.3220e-5	2.3617e-5	5.9523e-6	1.4972e-6
		1.972	1.980	1.981	1.988	1.991	-
	Shishkin	1.8178e-0	6.5774e-1	2.2213e-1	7.1372e-2	2.2101e-2	6.6631e-3
		1.467	1.566	1.638	1.691	1.730	-
$\left(2^{-32}, 2^{-28}\right)$	a posteriori	1.4585e-3	3.7898e-4	9.4827e-5	2.3833e-5	6.0004e-6	1.5091e-6
		1.944	1.999	1.992	1.990	1.991	-
	Shishkin	1.8178e-0	6.5774e-1	2.2213e-1	7.1372e-2	2.2101e-2	6.6631e-3
		1.467	1.566	1.638	1.691	1.730	-
$\left(2^{-40}, 2^{-36}\right)$	a posteriori	1.5068e-3	3.8255e-4	9.5562e-5	2.4170e-5	6.0252e-6	1.5118e-6
		1.978	2.001	1.983	2.004	1.995	-
	Shishkin	1.8178e-0	5.1425e-6	1.2856e-6	3.2140e-7	8.0349e-8	2.0087e-8
		1.467	1.566	1.638	1.691	1.730	-

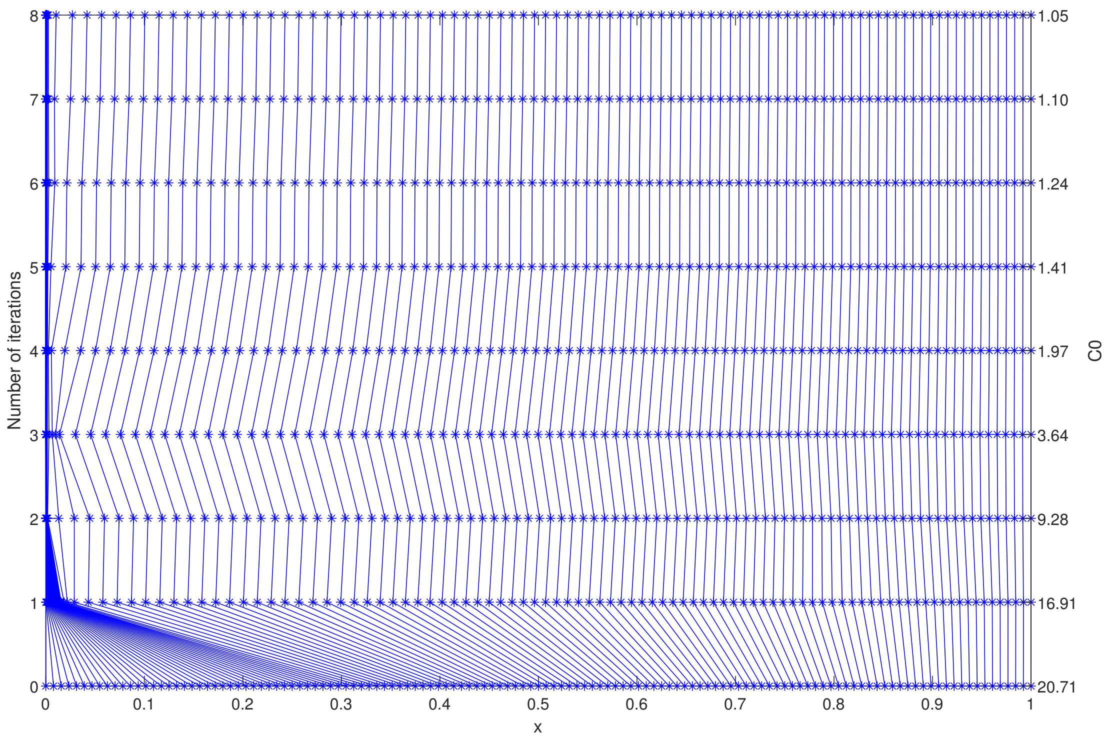

Figure 1. Generation process of the mesh for Example 4.1 with $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$ and $N = 128$.

Example 4.2 Consider the following system of singularly perturbed initial value problems

$$\begin{cases} \varepsilon_1 u_1''(x) + ((2+x)u_1(x))' + (1+x)u_1(x) + (2+\cos x)\,u_2(x) = 1 + x + e^{-x}, & x \in (0,1], \\ \varepsilon_2 u_2''(x) + \left((1+x^2)u_2(x)\right)' - u_1(x) + (2+x)\,u_2(x) = 1 + x^2, & x \in (0,1], \\ u_1(0) = 1, \quad u_1'(0) = 1/\varepsilon_1, \\ u_2(0) = 1, \quad u_2'(0) = 1/\varepsilon_2. \end{cases}$$

The singular perturbation parameters are also chosen from the set $S_\varepsilon = \left\{(\varepsilon_1, \varepsilon_2) \,\middle|\, \varepsilon_1 = 2^{-0}, 2^{-8}, 2^{-16}, 2^{-24}, 2^{-32}, 2^{-40}, \varepsilon_2 = 2^{4}, 2^{-4}, 2^{-12}, 2^{-20}, 2^{-28}, 2^{-36}\right\}$.

Figure 2, which also needs to read from the bottom to the top, indicates the generation process of a posteriori adapted mesh for Example 4.2 with $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$ and $N = 128$. From Figure 2 we know that the mesh refinement is also carried out in the boundary layer region by the solution-adaptive algorithm based on the singular behavior of the exact solution. Since the exact solution of Example 4.2 is not available, the double mesh principle is utilized to measure the errors and the convergence rates as follows

$$e^N = \max_{1 \le i \le N} \left\{ \left| U_{1,i}^N - \bar{U}_1^{2N}(x_i) \right|, \left| U_{2,i}^N - \bar{U}_2^{2N}(x_i) \right| \right\}, \quad r^N = \log_2 \left(e^N / e^{2N} \right),$$

where $\bar{\mathbf{U}}^{2N}(x)$ is its piecewise quadratic interpolation function vector defined in (3.6) for the computed solution $\mathbf{U}^{2N}$. Error estimates and convergence rates of the hybrid difference scheme (3.1) on a posteriori adapted mesh for Example 4.2 are presented in Table 2.

Table 2. Error estimates and convergence rates of the hybrid difference scheme on a posteriori adapted mesh and on a Shishkin mesh for Example 4.2.

		Number of mesh points N					
$(\varepsilon_1, \varepsilon_2)$		128	256	512	1024	2048	4096
$\left(2^{0}, 2^{4}\right)$	a posteriori	8.4785e-6	2.0766e-6	5.2200e-7	1.2864e-7	3.2183e-8	8.0762e-9
		2.030	1.992	2.021	1.999	1.995	-
	Shishkin	1.8591e-5	4.6187e-6	1.1510e-6	2.8729e-7	7.1762e-8	1.7956e-8
		2.009	2.005	2.002	2.001	1.999	-
$\left(2^{-8}, 2^{-4}\right)$	a posteriori	3.1241e-4	7.4682e-5	1.7305e-5	4.2817e-6	1.1237e-6	2.9106e-7
		2.065	2.110	2.015	1.930	1.949	-
	Shishkin	1.8967e-2	7.5138e-3	2.7208e-3	9.1998e-4	2.9473e-4	9.0771e-5
		1.336	1.466	1.564	1.642	1.699	-
$\left(2^{-16}, 2^{-12}\right)$	a posteriori	4.7987e-4	1.4184e-4	3.5888e-5	9.2055e-6	2.3151e-6	5.8591e-7
		1.758	1.983	1.963	1.991	1.982	-
	Shishkin	1.8916e-2	7.5083e-3	2.7206e-3	9.2094e-4	2.9516e-4	9.0943e-5
		1.333	1.465	1.563	1.642	1.698	-
$\left(2^{-24}, 2^{-20}\right)$	a posteriori	5.4219e-4	1.5305e-4	3.8907e-5	9.6076e-6	2.4150e-6	5.8763e-7
		1.825	1.976	2.018	1.992	2.039	-
	Shishkin	1.8916e-2	7.5083e-3	2.7206e-3	9.2095e-4	2.9516e-4	9.0943e-5
		1.333	1.465	1.563	1.642	1.698	-
$\left(2^{-32}, 2^{-28}\right)$	a posteriori	6.4189e-4	1.5624e-4	3.7680e-5	9.8985e-6	2.3267e-6	6.2017e-7
		2.039	2.052	1.928	2.089	1.908	-
	Shishkin	1.8916e-2	7.5083e-3	2.7206e-3	9.2095e-4	2.9516e-4	9.0943e-5
		1.333	1.465	1.563	1.642	1.698	-
$\left(2^{-40}, 2^{-36}\right)$	a posteriori	6.0685e-4	1.4982e-4	3.8654e-5	9.8938e-6	2.4869e-6	6.1415e-7
		2.018	1.955	1.966	1.992	2.018	-
	Shishkin	1.8916e-2	7.5083e-3	2.7206e-3	9.2095e-4	2.9516e-4	9.0943e-5
		1.333	1.465	1.563	1.642	1.698	-

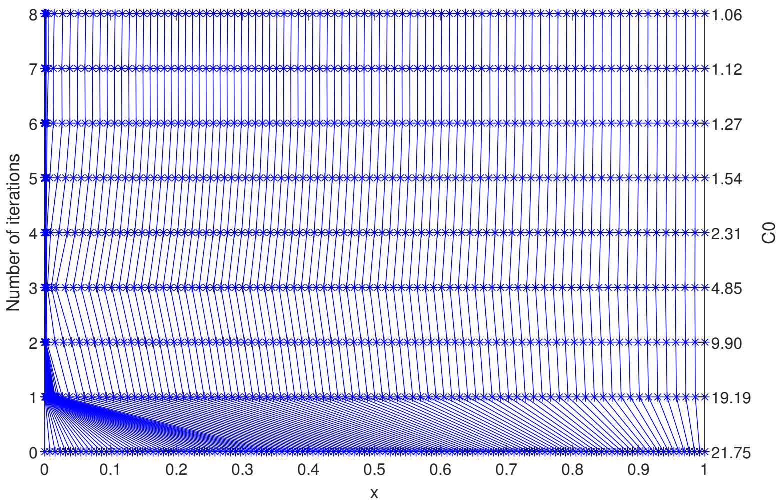

Figure 2. Generation process of the mesh for Example 4.2 with $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$ and $N = 128$.

The results in Tables 1 and 2 show a uniform convergence behavior of second-order for the scheme (3.1) on the adaptive algorithm described in Section 3. Moreover, the numerical results of the hybrid difference scheme on a Shishkin mesh given in [1] are also presented in Tables 1 and 2. The comparison of numerical results shows that the scheme presented in this paper is more accurate than the method given in [1].

5. Conclusions and discussion

In this paper, we first transformed the system of second-order singularly perturbed initial value problems with weak constrained conditions on the coefficients into a system of first-order singularly perturbed problems with integral terms. This transformation is beneficial to the subsequent stability analysis and a posteriori error estimation. And then we construct a hybrid difference scheme with the use of interpolating quadrature rules to approximate the transformed first-order singularly perturbed system. Next we derive a posteriori error analysis for the discretization scheme on an arbitrary mesh. Based on a posteriori error estimation we devise a solution-adaptive algorithm to generate a posteriori mesh and obtain approximation solution. Finally we present some numerical experiments to show a uniform convergence behavior of second-order for the scheme, which verifies that this scheme is more accurate than the method given in [1] and achieves the optimal convergence order under the given discretization scheme. In future we will extend this method to the case that B_{kl} are not sufficiently smooth functions.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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