AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16632–16648.
DOI: 10.3934/math.2022911
Received: 13 March 2022
Revised: 16 June 2022
Accepted: 02 July 2022
Published: 11 July 2022

Research article

Quotient reflective subcategories of the category of bounded uniform filter spaces

Sana Khadim and Muhammad Qasim*

Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), H-12 Islamabad, Pakistan

* **Correspondence:** Email: qasim99956@gmail.com.

Abstract: Previously, several notions of T_0 and T_1 objects have been studied and examined in various topological categories. In this paper, we characterize each of T_0 and T_1 objects in the categories of several types of bounded uniform filter spaces and examine their mutual relations, and compare that with the usual ones. Moreover, it is shown that under T_0 (resp. T_1) condition, the category of preuniform (resp. semiuniform) convergence spaces and the category of bornological (resp. symmetric) bounded uniform filter spaces are isomorphic. Finally, it is proved that the category of each of T_0 (resp. T_1) bounded uniform filter space are quotient reflective subcategories of the category of bounded uniform filter spaces.

Keywords: topological category; bounded uniform filter space; T_0 objects; T_1 objects; initial and final lifts; quotient-reflective
Mathematics Subject Classification: 54A05, 54B30, 54D10, 54E70, 18B35

1. Introduction

It is well-known that general topology fails to have the concepts of uniformity (uniform convergence and uniform continuity), hereditary of quotients, Cartesian closedness, Cauchy continuity, total boundedness, and completeness. Therefore, several attempts has been made to overcome these deficiencies including Kent convergence spaces [27], quasiuniform spaces [19], generalized topological spaces [17], seminearness spaces [26] and nearness spaces [22]. But none of them have led to fulfilling all the above properties. In 1995, Preuss [35] developed the concept of semiuniform convergence spaces (a basic structure in the domain of convenient topology) that solves almost all the flaws that appeared in **Top** (category of topological spaces and continuous maps). Moreover, by exempting symmetric condition from the semi-uniform convergence spaces, Preuss obtained preuniform convergence spaces and as a special case semiuniform convergence spaces Pre. Later, in 2018 and 2019, Leseberg [31, 33] extended the idea of Preuss and defined his concept on bounded structures in terms of bounded uniform filter spaces. Interestingly, not only **PUConv** (the category of preuniform convergence spaces and uniformly continuous maps) are embedded in **b-UFIL** (the category of bounded uniform filter spaces and bounded continuous maps) but also **BORN** (category of bornological spaces and continuous maps that are embedded in bounded spaces) can easily be embedded in b-UFIL as its subcategories. Also, the category **b-UFIL** forms a strong topological universe [31].

Classical separation axioms of general topology have many applications in almost all areas of Mathematics. In algebraic topology, an alternative characterization of locally semi-simple coverings in terms of light morphisms are achieved with the help of classical T_0 [24]. Furthermore, in lambda calculus and denotational semantics of programming language, various topological models have been built by using the T_0 separation axiom [38, 39] where Hausdorff topological spaces failed to do so. Other treatments with these axioms can be found in digital topology where they are used to characterize the digital line, in computer graphs and image processing and to construct cellular complex [21,28,29]. After stating such significance of T_0 and T_1 separation properties, several mathematicians have extended this idea to arbitrary topological categories [2, 15, 20, 23, 34] and the generic point method of topos theory by Johnstone has been used [25] due to the fact that in topos theory, generally speaking, the objects may not have points, yet, they always have a generic element. One of their primarily usage is to define each of T_3, T_4, regular, completely regular, and normal objects in an abstract topological category [6].

The main concepts in general topology depend upon notion of closedness. Thus, several generalizations of the classical separation axioms at some point p (local considering) have been inspected in [2] where the primary purpose of this generalization was to interpret the notion of closed sets and strongly closed sets in arbitrary set based topological categories. Moreover, the notions of compactness, Hausdorffness and perfectness have been generalized by using these closed and strongly closed sets in any topological category over sets [2, 7]. Further, they are suitable for the formation of closure operators [16] in several well-known topological categories [10, 12, 18, 37] and used to extend several fundamental theorems of general topology including Urysohn lemma and Tietze extension theorem [13, 14].

The salient objectives of this study are stated as under:

(i) To characterize local $\overline{T_0}$, local T_0' and local T_1 objects in the category of bounded uniform filter spaces, and examine their mutual relationship;

(ii) To give the characterization of $\overline{T_0}$, T_0' andT_1 objects in the category of bounded uniform filter spaces, and examine their mutual relationship;

(iii) To examine that under conditions of T_0 and T_1, preuniform (respectively semiuniform) spaces are isomorphic to bornological (respectively symmetric) bounded uniform filter spaces;

(iv) To examine the quotient-reflective properties of several bounded uniform filter spaces.

2. Preliminaries

For arbitrary topological categories $\mathcal{G}$ and $\mathcal{H}$, the functor $\mathfrak{F} : \mathcal{G} \to \mathcal{H}$ is said to be a topological functor, or the category $\mathcal{G}$ is said to be a topological category over $\mathcal{H}$ if $\mathfrak{F}$ is concrete (amnestic and faithful), $\mathfrak{F}$ consists of small fibers, and there exists an initial lift (or equivalently, a final lift) corresponding to every $\mathfrak{F}$-source [1, 36].

A filter σ on a set A is a non-empty collection such that finite intersection of elements of σ is in σ, and every superset of a set in σ is in σ. If $\emptyset \in \sigma$ then σ is an improper filter otherwise it is a proper filter. We write $\mathcal{F}(A)$ for the set of all filters on A. Let $v \in A$, then $[v] = \dot{v} = [\{v\}] = \{W \subset A : v \in W\}$ is a filter on A. Similarly, $[U] = \{W \subset A : W \supset U\}$ is a filter on A.

Definition 2.1. *(cf. [31]) Let X be a non-empty set, Θ^X be a non-empty subset of $P(X)$ and $\psi \subset \mathcal{F}(X \times X)$ be a non-empty set of uniform filters on the cartesian product of X with itself. A pair (Θ^X, ψ) is said to be a bounded uniform filter structure (or b-UFIL structure) on X and the corresponding triplet (X, Θ^X, ψ) is known as bounded uniform filter space (or b-UFIL space) on X if the following axioms hold:*
(b-UFIL1) $E' \subset E \in \Theta^X$ implies $E' \in \Theta^X$;
(b-UFIL2) $x \in X$ implies $\{x\} \in \Theta^X$;
(b-UFIL3) $E \in \Theta^X \backslash \emptyset$ implies $[E] \times [E] \in \psi$;
(b-UFIL4) $\sigma \in \psi$ and $\sigma \subset \sigma' \in \mathcal{F}(X \times X)$ implies $\sigma' \in \psi$.

A b-UFIL space (X, Θ^X, ψ) is a symmetric b-UFIL space provided that the following axiom holds:
(b-UFIL5) $\sigma \in \psi$ implies $\sigma^{-1} \in \psi$.

A symmetric b-UFIL space (X, Θ^X, ψ) is a symmetric bounded uniform limit space provided that the following axiom holds:
(b-UFIL6) $\sigma \in \psi$ and $\sigma' \in \psi$ implying $\sigma \cap \sigma' \in \psi$.

A b-UFIL space (X, Θ^X, ψ) is a crossbounded uniform filter space provided it satisfies the following condition:
(crb) $\sigma \in \psi$ implies $E \times E \in \sigma$ for some $E \in \Theta^X$.

If we denote by **CROSSb-UFIL** the corresponding defined full subcategory of **b-UFIL**, then it is clear that **BOUND** and **CROSSb-UFIL** are isomorphic. Hence we can introduce the following as:

A cross bounded uniform filter space (X, Θ^X, ψ) is a bornological b-UFIL space provided that the following axiom holds:
(b-UFIL7) $E, E' \in \Theta^X$ implying $E \cup E' \in \Theta^X$.

Let (X, Θ^X, ψ_X) and (Y, Θ^Y, ψ_Y) be two b-UFIL spaces and $h : X \to Y$ be a map. Then h is called bounded uniformly continuous (or buc) map if $E \in \Theta^X$ implies $h(E) \in \Theta^Y$; and $\sigma \in \psi_X$ implies $(h \times h)(\sigma) \in \psi_Y$; where $(h \times h)(\sigma) := \{V \subset Y \times Y : \exists U \in \sigma \mid (h \times h)[U] \subset V\}$ with $(h \times h)[U] := \{(h \times h)(x, y) : (x, y) \in U\} = \{(h(x), h(y)) : (x, y) \in U\}$.

We denote **b-UFIL** as category of b-UFIL spaces and buc maps. Similarly, **sb-UFIL** (respectively **LIMsb-UFIL**) as category of symmetric b-UFIL spaces (respectively category of symmetric b-UFIL limit spaces) and buc maps. Furthermore, **BONb-UFIL** is the category of bornological b-UFIL spaces and buc maps.

Definition 2.2. *(cf. [31, 33])*

(i) For given a family of b-UFIL spaces $(X_j, \Theta^{X_j}, \psi_j)_{j \in I}$ and maps $(h_j : X \to X_j)_{j \in I}$. The initial b-UFIL structure on X is represented by (Θ^X, ψ), where $\Theta^X := \{E \subset X : h_j[E] \in \Theta^{X_j},\ \forall j \in I\}$ and $\psi := \{\sigma \in \mathcal{F}(X^2) : (h_j \times h_j)(\sigma) \in \psi_j,\ \forall j \in I\}$ with $X^2 := X \times X$.

(ii) A b-UFIL structure on X is indiscrete if $(\Theta^X, \psi) := (P(X), \mathcal{F}(X^2))$.

(iii) *For given a family of b-UFIL spaces* $(X_j, \Theta^{X_j}, \psi_j)_{j\in I}$ *and maps* $(h_j : X_j \to X)_{j\in I}$. *The final b-UFIL structure on X is represented by* (Θ^X, ψ), *where* $\Theta^X := \{E \subset X : \exists i \in I, \exists E_j \in \Theta^{X_j} : E \subset h_j[E_j]\} \cup D^X := \{\emptyset\} \cup \{\{a\} : a \in X\}$ *and* $\psi := \{\sigma \in \mathcal{F}(X^2) : \exists j \in I, \exists \sigma_j \in \psi_j : (h_j \times h_j)(\sigma_j) \subset \sigma\} \cup \{[x] \times [x] : x \in X\} \cup \{P(X^2)\}$.

(iv) *A b-UFIL structure on X is discrete if* $(\Theta^X, \psi) := (D^X, \psi_{dis})$, *where* $\psi_{dis} := \{[x] \times [x] : x \in X\} \cup \{P(X^2)\}$.

Remark 2.1. *(i) A bornological b-UFIL structure on X is discrete if* $(\Theta^X, \psi) := (D^X_{born}, \psi_{dis})$, *where* $D^X_{born} := \{E \subset X : E$ *is finite*$\}$ *[33].*

(ii) *The category* ***PUConv*** *is isomorphic to* ***DISb-UFIL*** *(category of discrete b-UFIL spaces and buc maps) [31].*

(iii) *The category* ***SUConv*** *is isomorphic to* ***DISsb-UFIL*** *(category of discrete symmetric b-UFIL spaces and buc maps) [31].*

3. Local T_0 and local T_1 bounded uniform filter spaces

In general topology, all the basic concepts including compactness, connectedness, perfectness, soberness, Hausdorffness and closure operators can be defined in terms of closedness. In order to define these notions of closedness in categorical language, Baran [2] introduced local T_0 and local T_1 of topology in topological category using initial, final lifts and discrete objects. Moreover, these notion of closedness (strongly closedness) are used to extend several famous theorems of general topology such as Urysohn lemma and Tietze extension theorem.

In this section, we recall definitions of local T_0 and local T_1 b-UFIL spaces (at some fixed point p). Let X be any set and $p \in X$. We define the wedge product of X at p as the two disjoint copies of X at p and denote it as $X \bigvee_p X$. For a point $x \in X \bigvee_p X$, we write it as x_1 if x belongs to the first component of the wedge product; otherwise, we write x_2 that is in the second component, where X^2 is the cartesian product of X.

Definition 3.1. *(cf. [2])*

(i) *A map* $A_p : X \bigvee_p X \longrightarrow X^2$ *is said to be principal p-axis map provided that*

$$A_p(x_j) := \begin{cases} (x, p), & j = 1, \\ (p, x), & j = 2. \end{cases}$$

(ii) *A map* $S_p : X \bigvee_p X \longrightarrow X^2$ *is said to be skewed p-axis map provided that*

$$S_p(x_j) := \begin{cases} (x, x), & j = 1, \\ (p, x), & j = 2. \end{cases}$$

(iii) *A map* $\nabla_p : X \bigvee_p X \longrightarrow X$ *is said to be fold map at p provided that*

$$\nabla_p(x_j) := x, \qquad j = 1, 2.$$

Assume that $\mathfrak{F} : \mathcal{G} \longrightarrow$ **Set** is a topological functor, $X \in Obj(\mathcal{G})$ with $\mathfrak{F}X = E$ and $p \in E$.

Definition 3.2. *(cf. [2])*

(i) *X is $\overline{T_0}$ at p provided that initial lift of $\mathfrak{F}$-source $\{E \bigvee_p E \xrightarrow{A_p} \mathfrak{F}(X^2) = E^2$ and $E \bigvee_p E \xrightarrow{\nabla_p} \mathfrak{F}DE = E\}$ is discrete.*

(ii) *X is $T_0^{'}$ at p provided that initial lift of $\mathfrak{F}$-source $\{E \bigvee_p E \xrightarrow{id} \mathfrak{F}(X \bigvee_p X) = E \bigvee_p E$ and $E \bigvee_p E \xrightarrow{\nabla_p} \mathfrak{F}DE = E\}$ is discrete, where $X \bigvee_p X$ represents the wedge product in $\mathcal{G}$, i.e., final lift of $\mathfrak{F}$-sink $\{\mathfrak{F}X = E \xrightarrow{i_1,\, i_2} E \bigvee_p E\}$, where i_1 and i_2 denote the canonical injections.*

(iii) *X is T_1 at p provided that initial lift of $\mathfrak{F}$-source $\{E \bigvee_p E \xrightarrow{S_p} \mathfrak{F}(X^2) = E^2$ and $E \bigvee_p E \xrightarrow{\nabla_p} \mathfrak{F}DE = E\}$ is discrete.*

Remark 3.1. *(i) In **Top**, $\overline{T_0}$ and T_0' at p (respectively T_1 at p) are equivalent to the classical T_0 at p (respectively the classical T_1 at p), i.e., for each $x \in X$ with $x \neq p$, there exists a neighborhood N_x of x not containing p or (respectively and) there exists a neighborhood N_p of p not containing x [5].*

(ii) A topological space X is T_0 (respectively T_1) if and only if X is T_0 (respectively T_1) at p for all $p \in X$ [5].

*(iii) Let $\mathcal{U} : \mathcal{G} \to$ **Set** be a topological functor, $X \in Obj(\mathcal{G})$ and $p \in \mathcal{U}(X)$ be a retract of X. If X is $\overline{T_0}$ (respectively T_1) at p, then X is $T_0^{'}$ at p but not conversely in general [3].*

*(iv) Let $\mathcal{U} : \mathcal{G} \to$ **Set** be normalized and $X \in Obj(\mathcal{G})$ with $p \in \mathcal{U}(X)$. If X $PreT_2$ object at p, then X is $\overline{T_0}$ at p iff X is T_1 at p [3, 8].*

Theorem 3.1. *Let (X, Θ^X, ψ) be a b-UFIL space and $p \in X$. Then (X, Θ^X, ψ) is $\overline{T_0}$ at p iff for all $x \in X$ with $x \neq p$, the following hold.*

(i) $\{x, p\} \notin \Theta^X$;

(ii) $[x] \times [p] \notin \psi$ *or* $[p] \times [x] \notin \psi$;

(iii) $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Proof. Let (X, Θ^X, ψ) be $\overline{T_0}$ at p. We shall prove that the above conditions (i)–(iii) hold. Let $\{x, p\} \in \Theta^X$ for $x \neq p$ and $W = \{x_1, x_2\} \in \Theta^{X \vee_p X}$. Since $\nabla_p W = \{x\} \in \mathcal{D}^X$, and $\pi_k A_p W = \{x, p\} \in \Theta^X$ for $k = 1, 2$, where $\pi_k : X^2 \to X$ for $k = 1, 2$ are the projection maps. Since (X, Θ^X, ψ) is $\overline{T_0}$ at p, by the Definitions 2.2 and 3.2, we get a contradiction. Hence, $\{x, p\} \notin \Theta^X$.

Next, suppose that $[x] \times [p] \in \psi$ for some $x \neq p$. Let $\sigma = [x_1] \times [x_2]$. Clearly, $(\nabla_p \times \nabla_p)\sigma = [x] \times [x] \in \psi_{dis}$, $(\pi_1 A_p \times \pi_1 A_p)\sigma = [x] \times [p] \in \psi$ and $(\pi_2 A_p \times \pi_2 A_p)\sigma = [p] \times [x] \in \psi$, a contradiction. It follows that $[x] \times [p] \notin \psi$ or $[p] \times [x] \notin \psi$.

Further, if $([x] \times [x]) \cap ([p] \times [p]) \in \psi$ for some $x \neq p$. Let $\sigma = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$. Since $(\nabla_p \times \nabla_p)\sigma = [x] \times [x] \in \psi_{dis}$, $(\pi_1 A_p \times \pi_1 A_p)\sigma = ([x] \times [x]) \cap ([p] \times [p]) \in \psi$ and $(\pi_2 A_p \times \pi_2 A_p)\sigma = ([p] \times [p]) \cap ([x] \times [x]) \in \psi$, a contradiction since (X, Θ^X, ψ) is $\overline{T_0}$ at p. Thus, $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Conversely, let us assume that the conditions (i)–(iii) hold. Let $(\Theta^{X \vee_p X}, \overline{\psi})$ be the initial structure induced by $A_p : X \vee_p X \to (X^2, \Theta^{X^2}, \psi^2)$ and $\nabla_p : X \vee_p X \to (X, \mathcal{D}^X, \psi_{dis})$, where (Θ^{X^2}, ψ^2) represents the product b-UFIL structure on X^2 and $(\mathcal{D}^X, \psi_{dis})$ the discrete b-UFIL structure on X, respectively. We show that $(\Theta^{X\vee_p X}, \overline{\psi})$ is the discrete b-UFIL structure on $X \vee_p X$. Let $W \in \Theta^{X\vee_p X}$ and $\nabla_p W \in \mathcal{D}^X$.

If $\nabla_p W = \emptyset$, then $W = \emptyset$. Suppose $\nabla_p W \neq \emptyset$, it follows that $\nabla_p W = \{x\}$ for some $x \in X$. If $x = p$, then $W = \{p\}$. Suppose $x \neq p$. It follows that $W = \{x_1\}, \{x_2\}$ or $\{x_1, x_2\}$. The case, $W = \{x_1, x_2\}$ cannot happen since $\pi_k A_p W = \{x, p\} \notin \Theta^X$ $(k = 1, 2)$ by the assumption. Hence, $W = \{x_1\}, \{x_2\}$ and consequently, $\Theta^{X\vee_p X} = \mathcal{D}^{X\vee_p X}$, the discrete b-UFIL structure on $X \vee_p X$.

Next, let $\sigma \in \overline{\psi}$. By Definition 2.2(i), $(\nabla_p \times \nabla_p)\sigma \in \mathcal{D}^X$ and $(\pi_k A_p \times \pi_k A_p)\sigma \in \psi$ for $k = 1, 2$. We need to show that $\sigma = [x_i] \times [x_i]$ $(i = 1, 2)$, $\sigma = [p] \times [p]$ or $\sigma = [\emptyset] = P((X \vee_p X) \times (X \vee_p X))$.

If $(\nabla_p \times \nabla_p)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((X \vee_p X) \times (X \vee_p X))$. Suppose $(\nabla_p \times \nabla_p)\sigma = [x] \times [x]$ for some $x \in X$. If $x = p$, then $\sigma = [p] \times [p]$.

If $x \neq p$, then $(\nabla_p \times \nabla_p)\sigma = [x] \times [x]$, then $\{x_1, x_2\} \times \{x_1, x_2\} \in \sigma$. Thus there is a finite subset N_0 of σ such that $\sigma = [N_0]$. Clearly, $N_0 \subseteq \{x_1, x_2\} \times \{x_1, x_2\}$ and if $i \neq j$, then $\{\{x_i\} \times \{x_j\}\} \neq N_0$ and $\{\{x_1\} \times \{x_1\}, \{x_2\} \times \{x_2\}\} \neq N_0$ since in particular for $k = 1$, $i = 1$, and $j = 2$, $(\pi_1 A_p \times \pi_1 A_p)([x_1] \times [x_2]) = [x] \times [p] \notin \psi$ and $(\pi_1 A_p \times \pi_1 A_p)(([x_1] \times [x_1]) \cap ([x_2] \times [x_2])) = ([x] \times [x]) \cap ([p] \times [p]) \notin \psi$ by using the second and the third conditions respectively.

Therefore, we must have $\sigma = [x_i] \times [x_i]$ $(i = 1, 2)$ and consequently, by Definitions 3.2, 2.1 and 2.2, (X, Θ^X, ψ) is $\overline{T_0}$ at p. □

Theorem 3.2. *Let (X, Θ^X, ψ) be a b-UFIL space and $p \in X$. Then (X, Θ^X, ψ) is T_1 at p iff for all $x \in X$ with $x \neq p$, the followings hold.*

(i) $\{x, p\} \notin \Theta^X$;

(ii) $[x] \times [p] \notin \psi$ *and* $[p] \times [x] \notin \psi$;

(iii) $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Proof. The proof is analogous to the proof of Theorem 3.1 by using the skewed p-axis map S_p instead of the principal p-axis map A_p. □

Example 3.1. Let $X = \{k, l, m\}$ and (Θ^X, ψ) be a b-UFIL structure on X with $\Theta^X = \{\emptyset, \{k\}, \{l\}, \{m\}\}$ and $\psi = \{[\emptyset], [k] \times [k], [l] \times [l], [m] \times [m], [k] \times [l], [k] \times [m]\}$. Then, (X, Θ^X, ψ) is $\overline{T_0}$ at k but not T_1 at k.

Theorem 3.3. *All b-UFIL spaces are T_0' at p.*

Proof. Suppose (X, Θ^X, ψ) is a b-UFIL space and $p \in X$. By Definition 3.2(ii), we show that for any $W \in \Theta^{X\vee_p X}$, $W \subset i_j(V)$ $(j = 1$ *or* $2)$ for some $V \in \Theta^X$, and $\nabla_p W \in \mathcal{D}^X$, and for any $\sigma \in \mathcal{F}((X \vee_p X) \times (X \vee_p X))$, $\sigma \supset (i_j \times i_j)\alpha$ $(j = 1$ *or* $2)$ for some $\alpha \in \psi$ and $(\nabla_p \times \nabla_p)\sigma \in \psi_{dis}$. Then $W = \emptyset$, $\{p\}$ or $\{x_k\}$ for $k = 1, 2$.

If $\nabla_p W = \emptyset$, then $W = \emptyset$. Let $\nabla_p W \neq \emptyset$. It follows that $\nabla_p W = \{x\}$ for some $x \in X$.

If $x = p$, then $\nabla_p W = \{p\}$, it follows that $W = \{p\}$.

Suppose $x \neq p$, it follows that $W = \{x_1\}$, $\{x_2\}$ or $\{x_1, x_2\}$. If $W = \{x_1, x_2\}$, then $\{x_1, x_2\} \subset i_1(V)$ for some $V \in \Theta^X$ which shows that x_2 should be in first component of the wedge product $X \vee_p X$, a contradiction. In similar manner, $\{x_1, x_2\} \not\subset i_2(V)$ for some $V \in \Theta^X$. Hence, $W \neq \{x_1, x_2\}$. Thus, we must have $W = \{x_k\}$ for $k = 1, 2$ only and consequently, $\Theta^{X\vee_p X} = \mathcal{D}^{X\vee_p X}$, the discrete b-UFIL structure on $X \vee_p X$.

Next, for any $\sigma \in \mathcal{F}((X \vee_p X) \times (X \vee_p X))$, if $(\nabla_p \times \nabla_p)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((X \vee_p X) \times (X \vee_p X))$.

Now, assume that $(\nabla_p \times \nabla_p)\sigma = [x] \times [x]$ for some $x \in X$. If $x = p$, then $(\nabla_p \times \nabla_p)\sigma = [p] \times [p]$, and consequently $\sigma = [p] \times [p]$.

Suppose that $x \neq p$, then $(\nabla_p \times \nabla_p)\sigma = [x]\times[x]$, hence $\{x_1, x_2\}\times\{x_1, x_2\} \in \sigma$. Thus there exists a finite subset M of σ such that $\sigma = [M]$. Clearly, $M \subseteq \{x_1, x_2\} \times \{x_1, x_2\}$ and if $k \neq l$, then $\{\{x_k\} \times \{x_l\}\} \neq M$ and $\{\{x_1\} \times \{x_1\}, \{x_2\} \times \{x_2\}\} \neq M$. Suppose that $M = \{\{x_k\} \times \{x_l\}\}$, then for $k = 1$, $l = 2$, and $j = 1$ (respectively $j = 2$), $[x_1] \times [x_2] \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $(x_1, x_2) \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that x_2 (respectively x_1) is in the first (respectively second) component of the wedge product $X \vee_p X$, a contradiction. In similar manner, if $M = \{\{x_1\} \times \{x_1\}, \{x_2\} \times \{x_2\}\}$, then for $j = 1$ (respectively $j = 2$), $(([x_1] \times [x_1]) \cap ([x_2] \times [x_2])) \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $\{(x_1, x_1), (x_2, x_2)\} \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that x_2 (respectively x_1) is in the first (respectively second) component of the product $X \vee_p X$, a contradiction.

Thus, we must have $\sigma = [x_k] \times [x_k]$ ($k = 1, 2$) and consequently, by Definitions 3.2, 2.2 and Theorems 3.1, 3.2, (X, Θ^X, ψ) is T_0' at p. □

Corollary 3.1. *Let (X, Θ^X, ψ) be a symmetric b-UFIL space and $p \in X$, then the following statements are equivalent:*

(i) X is $\overline{T_0}$ at p.

(ii) X is T_1 at p.

(iii) $\{x, p\} \notin \Theta^X$; $[x] \times [p] \notin \psi$; and $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Proof. The proof of the corollary can be easily deduced from Theorems 3.1, 3.2 and Definition 2.1. □

Corollary 3.2. *Let (X, Θ^X, ψ) be a symmetric b-UFIL limit space and $p \in X$ be any point, then the following statements are equivalent:*

(i) X is $\overline{T_0}$ at p.

(ii) X is T_1 at p.

(iii) $\{x, p\} \notin \Theta^X$; and $[x] \times [p] \notin \psi$.

Proof. The proof of the corollary can be easily deduced from Theorems 3.1, 3.2 and Definition 2.1. □

4. T_0 and T_1 bounded uniform filter spaces

In this section, we recall the definitions of T_0 and T_1 b-UFIL spaces.

To generalize the classical T_0 objects, various approaches have been discussed by Topologists since 1971 such as Brümmer, Marny, Hoffman, Harvey and Baran in [2, 4, 15, 20, 23, 34, 40]. In addition, the relationships between several forms of generalized T_0 objects in topological category have been examined in [4,40]. One of the main reasons for this generalization was to define Hausdorff objects in arbitrary topological categories [2]. But also T_0 is used to define non-T_2 spaces such as sober spaces [11] that are used in theoretical computer science. Further, a generalization of the classical T_1 objects of topology in topological categories has been investigated by Baran [2] in 1991. One of its important uses is to define each of T_3, T_4, regular, completely regular, and normal objects in an abstract topological category [6]. Baran's approach was to characterize these separation axioms in terms of their initial and final lifts, and discreteness.

Since points do not make sense in topos theory, Baran [2] used the generic element method defined by Johnstone [25] to characterize separation axioms that make sense in topos theory as well, where the wedge product $X \bigvee_p X$ at p can be replaced by $X^2 \bigvee_\Delta X^2$ at diagonal Δ.

Definition 4.1. *(cf. [2])*

(i) A map $A : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ is called principal axis map provided that

$$A((x,y)_j) := \begin{cases} (x,y,x), & j=1, \\ (x,x,y), & j=2. \end{cases}$$

(ii) A map $S : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ is called skewed axis map provided that

$$S((x,y)_j) := \begin{cases} (x,y,y), & j=1, \\ (x,x,y), & j=2. \end{cases}$$

(iii) A map $\nabla : X^2 \bigvee_\Delta X^2 \longrightarrow X^2$ is called fold map provided that

$$\nabla((x,y)_j) := (x,y), \qquad j=1,2,$$

where $X^2 \bigvee_\Delta X^2$ is the wedge product of X^2 diagonally intersected with X^2, and any element $(x,y) \in X^2 \bigvee_\Delta X^2$ is written as $(x,y)_1$ (respectively $(x,y)_1$) if it lies in the first (respectively second) component of $X^2 \bigvee_\Delta X^2$. Clearly, $(x,y)_1 = (x,y)_2$ if and only if $x = y$.

Definition 4.2. *Let $\mathfrak{F} : \mathcal{G} \longrightarrow$ **Set** be a topological functor, $X \in Obj(\mathcal{G})$ with $\mathfrak{F}X = E$.*

(i) X is $\overline{T_0}$ provided that the initial lift of the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{A} \mathfrak{F}(X^3) = E^3$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ is discrete [2].

(ii) X is T_0' provided that the initial lift of the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{id} \mathfrak{F}(E^2 \bigvee_\Delta E^2)' = E^2 \bigvee_\Delta E^2$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ is discrete, where $(E^2 \bigvee_\Delta E^2)'$ is the final lift of the $\mathfrak{F}$-sink $\{\mathfrak{F}(X^2) = E^2 \xrightarrow{i_1,\, i_2} E^2 \bigvee_\Delta E^2\}$ [2, 4].

(iii) X is called T_0 property provided that X doesn't contain an indiscrete subspace with at least two points [34, 40].

(iv) X is T_1 provided that the initial lift of the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{S} \mathfrak{F}(X^3) = E^3$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ is discrete [2].

Remark 4.1. *(i) In **Top**, all T_0, $\overline{T_0}$ and T_0' (respectively T_1) are equivalent to the classical T_0 (respectively the classical T_1), i.e., for each $x, y \in X$ with $x \neq y$, there exists a neighborhood N_x of x not containing y or (respectively and) there exists a neighborhood N_y of y not containing x [5].*

(ii) *In any topological category, $\overline{T_0} \implies T_0'$ but not conversely in general. Also, each of $\overline{T_0}$ and T_0' objects has no relation with T_0 object [4]. For Example, $\overline{T_0}$ could be all objects such as in* ***Born*** *[4], and $\overline{T_0} \implies T_0 \implies T_0'$ such as in* **SUConv** *[9] and $\overline{T_0} = T_0 \implies T_0'$ such as in* ***Lim*** *[4].*

(iii) *Let $\mathcal{U} : \mathcal{G} \to$* ***Set*** *be a topological functor and $X \in Obj(\mathcal{G})$. If $X \in \mathbf{PreT_2}(\mathcal{G})$, then X is $\overline{T_0}$ iff X is T_1, where $\mathbf{PreT_2}(\mathcal{G})$ is the category of $PreT_2$ objects [8].*

Theorem 4.1. *Let (X, Θ^X, ψ) be a b-UFIL space. (X, Θ^X, ψ) is $\overline{T_0}$ iff for all $x, y \in X$ with $x \neq y$, the following hold:*

(i) $\{x, y\} \notin \Theta^X$;

(ii) $[x] \times [y] \notin \psi$ *or* $[y] \times [x] \notin \psi$;

(iii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. Let (X, Θ^X, ψ) be $\overline{T_0}$. We shall prove that the above conditions (i)–(iii) hold. Let $\{x, y\} \in \Theta^X$ for $x \neq y$ and $W = \{(x, y)_1, (x, y)_2\} \in \Theta^{X^2 \vee_\Delta X^2}$. Since $\nabla W = \{(x, y)\} \in \mathcal{D}^{X^2}$, $\pi_1 AW = \{x\} \in \Theta^X$, and $\pi_k AW = \{x, y\} \in \Theta^X$ for $k = 2, 3$, where $\pi_k : X^3 \to X^2$ for $k = 1, 2, 3$ are the projection maps. Since (X, Θ^X, ψ) is $\overline{T_0}$, by the Definitions 2.2 and 4.2, we get a contradiction. Hence, $\{x, y\} \notin \Theta^X$.

Next, suppose that $[x] \times [y] \in \psi$ for some $x \neq y$. Let $\sigma = [(x, y)_1] \times [(x, y)_2]$. Clearly, $(\nabla \times \nabla)\sigma = [(x, y)] \times [(x, y)] \in \psi^2_{dis}$, $(\pi_1 A \times \pi_1 A)\sigma = [x] \times [x] \in \psi$, $(\pi_2 A \times \pi_2 A)\sigma = [y] \times [x] \in \psi$, and $(\pi_3 A \times \pi_3 A)\sigma = [x] \times [y] \in \psi$, a contradiction. It follows that $[x] \times [y] \notin \psi$ or $[y] \times [x] \notin \psi$.

Further, if $([x]\times[x])\cap([y]\times[y]) \in \psi$ for some $x \neq y$. Let $\sigma = ([(x, y)_1]\times[(x, y)_1])\cap([(x, y)_2]\times[(x, y)_2])$. Since $(\nabla \times \nabla)\sigma = [(x, y)] \times [(x, y)] \in \psi^2_{dis}$, $(\pi_1 A \times \pi_1 A)\sigma = [x] \times [x] \in \psi$, $(\pi_2 A \times \pi_2 A)\sigma = ([y] \times [y]) \cap ([x] \times [x]) \in \psi$, and $(\pi_3 A \times \pi_3 A)\sigma = ([x] \times [x]) \cap ([y] \times [y]) \in \psi$, a contradiction since (X, Θ^X, ψ) is $\overline{T_0}$. Thus, $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Conversely, suppose that the conditions (i)–(iii) hold. Let $(\Theta^{X^2 \vee_\Delta X^2}, \overline{\psi})$ be the initial structure induced by $A : X^2 \vee_\Delta X^2 \to (X^3, \Theta^{X^3}, \psi^3)$ and $\nabla : X^2 \vee_\Delta X^2 \to (X^2, \mathcal{D}^{X^2}, \psi^2_{dis})$, where (Θ^{X^3}, ψ^3) represents the product b-UFIL structure on X^3 and $(\mathcal{D}^{X^2}, \psi^2_{dis})$ the discrete b-UFIL structure on X^2, respectively. We show that $(\Theta^{X^2 \vee_\Delta X^2}, \overline{\psi})$ is a discrete b-UFIL structure on $X^2 \vee_\Delta X^2$. Let $W \in \Theta^{X^2 \vee_\Delta X^2}$ and $\nabla W \in \mathcal{D}^{X^2}$.

If $\nabla W = \emptyset$, then $W = \emptyset$. Suppose $\nabla W \neq \emptyset$, it follows that $\nabla W = \{(x, y)\}$ for some $(x, y) \in X^2$. Suppose $x \neq y$. It follows that $W = \{(x, y)_1\}$ or $\{(x, y)_2\}$ or $\{(x, y)_1, (x, y)_2\}$. The case, $W = \{(x, y)_1, (x, y)_2\}$ cannot happen since $\pi_1 AW = \{x\} \in \Theta^X$ but $\pi_k AW = \{x, y\} \notin \Theta^X$ ($k = 2, 3$) by the assumption. Hence, $W = \{(x, y)_1\}$ or $\{(x, y)_2\}$ and consequently, $\Theta^{X^2 \vee_\Delta X^2} = \mathcal{D}^{X^2 \vee_\Delta X^2}$, the discrete b-UFIL structure on $X^2 \vee_\Delta X^2$.

Next, let $\sigma \in \overline{\psi}$. By Definition 2.2(i), $(\nabla \times \nabla)\sigma \in \mathcal{D}^{X^2}$ and $(\pi_k A \times \pi_k A)\sigma \in \psi$ for $k = 1, 2, 3$. We need to prove that $\sigma = [(x, y)_i] \times [(x, y)_i]$ ($i = 1, 2, 3$), or $\sigma = [\emptyset] = P((X^2 \vee_\Delta X^2) \times (X^2 \vee_\Delta X^2))$.

If $(\nabla\times\nabla)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((X^2 \vee_\Delta X^2)\times(X^2 \vee_\Delta X^2))$. Suppose $(\nabla\times\nabla)\sigma = [(x, y)]\times[(x, y)]$ for some $(x, y) \in X^2$. If $x \neq y$, then $(\nabla\times\nabla)\sigma = [(x, y)]\times[(x, y)]$, hence $\{(x, y)_1, (x, y)_2\}\times\{(x, y)_1, (x, y)_2\} \in \sigma$. Thus, there is a finite subset N_0 of σ such that $\sigma = [N_0]$. Clearly, $N_0 \subseteq \{(x, y)_1, (x, y)_2\}\times\{(x, y)_1, (x, y)_2\}$ and if $i \neq j$, then it can be easily seen that $N_0 \neq \{\{(x, y)_i\} \times \{(x, y)_j\}\}$ by the second condition and that by the third condition $N_0 \neq \{\{(x, y)_1\} \times \{(x, y)_1\}, \{(x, y)_2\} \times \{(x, y)_2\}\}$.

Therefore, we must have $\sigma = [(x, y)_i] \times [(x, y)_i]$ ($i = 1, 2$) and consequently, by Definitions 4.2, 2.1 and 2.2, (X, Θ^X, ψ) is $\overline{T_0}$. □

Theorem 4.2. *Let (X, Θ^X, ψ) be a b-UFIL space. (X, Θ^X, ψ) is T_0 iff for all $x, y \in X$ with $x \neq y$, the following hold:*

(i) $\{x, y\} \notin \Theta^X$;

(ii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. Let (X, Θ^X, ψ) be T_0, and $\{x, y\} \in \Theta^X$, and $[\{x, y\}] \times [\{x, y\}] \in \psi$ for some $x, y \in X$ with $x \neq y$. Suppose that $W = \{x, y\}$. Note that (W, Θ^W, ψ_W) is the subspace of (X, Θ^X, ψ), where (Θ^W, ψ_W) is the initial b-UFIL structure on W induced by the inclusion map $i : W \to X$. By Definition 2.2(i), for any $B \subset W$, $B \in \Theta^W$ precisely when $i(B) = B \in \Theta^X$, and for $\alpha \in \mathcal{F}(W \times W)$, $\alpha \in \psi_W$ precisely when $(i \times i)\alpha = \alpha \in \psi$. Specifically, for $B = W = \{x, y\}$, $i(W) = W \in \Theta^X$ and for $\alpha = [W] \times [W] = [\{x, y\}] \times [\{x, y\}] = ([x] \times [x]) \cap ([y] \times [y])$, $(i \times i)\alpha = [W] \times [W] \in \psi$, by the assumption. It follows that $(\Theta^W, \psi_W) = (P(W), \mathcal{F}(W \times W))$, the indiscrete b-UFIL structure on W, a contradiction. Therefore, $\{x, y\} \notin \Theta^X$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Conversely, suppose that $\{x, y\} \notin \Theta^X$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$ for all $x, y \in X$ with $x \neq y$. We show that the initial structure (Θ^W, ψ_W) is not an indiscrete b-UFIL structure on W. Assume that $W = \{x, y\} \subset X$. By the assumption and using the Definition 2.2(i), $\{x, y\} \notin \Theta^W$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi_W$. Thus, (W, Θ^W, ψ_W) is not an indiscrete bounded uniform subspace of (X, Θ^X, ψ) and therefore by the Definition 4.2(iii), (X, Θ^X, ψ) is T_0. □

Theorem 4.3. *Let (X, Θ^X, ψ) be a b-UFIL space. (X, Θ^X, ψ) is T_1 iff for all $x, y \in X$ with $x \neq y$, the following hold:*

(i) $\{x, y\} \notin \Theta^X$;

(ii) $[x] \times [y] \notin \psi$ *and* $[y] \times [x] \notin \psi$;

(iii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. The proof is analogous to the proof of Theorem 4.1 by using the skewed axis map S instead of the principal axis map A. □

Example 4.1. Let $X = \{a, b, c\}$ and (Θ^X, ψ) be a b-UFIL structure on X with $\Theta^X = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ and $\psi = \{[\emptyset], [a] \times [a], [b] \times [b], [c] \times [c], [a] \times [b], [a] \times [c], [b] \times [c]\}$. Then, (X, Θ^X, ψ) is $\overline{T_0}$ but not T_1 b-UFIL space.

Theorem 4.4. *All b-UFIL spaces are T_0'.*

Proof. Suppose (X, Θ^X, ψ) is a b-UFIL space. By Definition 4.2(ii), we show that for any $W \in \Theta^{X^2 \vee_\Delta X^2}$, $W \subset i_j(V)$ ($j = 1$ *or* 2) for some $V \in \Theta^{X^2}$, and $\nabla W \in \mathcal{D}^{X^2}$, and for any $\sigma \in \mathcal{F}((X^2 \vee_\Delta X^2) \times (X^2 \vee_\Delta X^2))$, $\sigma \supset (i_j \times i_j)\alpha$ ($j = 1$ *or* 2) for some $\alpha \in \psi^2$ and $(\nabla \times \nabla)\sigma \in \psi^2_{dis}$. Then $W = \emptyset$, $\{p\}$ or $\{x_k\}$ for $k = 1, 2$.

If $\nabla_p W = \emptyset$, then $W = \emptyset$. Let $\nabla_p W \neq \emptyset$. It follows that $\nabla_p W = \{(x, y)\}$ for some $x \in X$.

Suppose $x \neq y$, it follows that $W = \{(x, y)_1\}$ or $\{(x, y)_2\}$ or $\{(x, y)_1, (x, y)_2\}$. If $W = \{(x, y)_1, (x, y)_2\}$, then $\{(x, y)_1, (x, y)_2\} \subset i_1(V)$ for some $V \in \Theta^{X^2}$ which shows that $(x, y)_2$ must be in the first component of $X^2 \vee_\Delta X^2$, a contradiction. Similarly, $\{(x, y)_1, (x, y)_2\} \not\subset i_2(V)$ for some $V \in \Theta^{X^2}$. Hence, $W \neq \{(x, y)_1, (x, y)_2\}$. Thus, we must have $W = \{(x, y)_k\}$ for $k = 1, 2$ only and consequently, $\Theta^{X^2 \vee_\Delta X^2} = \mathcal{D}^{X^2 \vee_\Delta X^2}$, the discrete b-UFIL structure on $X^2 \vee_\Delta X^2$.

Next, for $\sigma \in \mathcal{F}((X^2 \vee_\Delta X^2) \times (X^2 \vee_\Delta X^2))$, if $(\nabla \times \nabla)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((X^2 \vee_\Delta X^2) \times (X^2 \vee_\Delta X^2))$. Now, assume that $(\nabla \times \nabla)\sigma = [(x, y)] \times [(x, y)]$ for some $(x, y) \in X^2$. Suppose that $x \neq y$, then $\{(x, y)_1, (x, y)_2\} \times \{(x, y)_1, (x, y)_2\} \in \sigma$. Thus, there exists a finite subset M of σ such that $\sigma = [M]$. Clearly, $M \subseteq \{(x, y)_1, (x, y)_2\} \times \{(x, y)_1, (x, y)_2\}$ and if $k \neq l$, then $\{\{(x, y)_k\} \times \{(x, y)_l\}\} \neq M$ and $\{\{(x, y)_1\} \times \{(x, y)_1\}, \{(x, y)_2\} \times \{(x, y)_2\}\} \neq M$. Suppose that $M = \{\{(x, y)_k\} \times \{(x, y)_l\}\}$, then for $k = 1$, $l = 2$, and $j = 1$ (resp. $j = 2$), $[(x, y)_1] \times [(x, y)_2] \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $((x, y)_1, (x, y)_2) \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that $(x, y)_2$ (resp. $(x, y)_1$) is in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction. Similarly, if $M = \{\{(x, y)_1\} \times \{(x, y)_1\}, \{(x, y)_2\} \times \{(x, y)_2\}\}$, then for $j = 1$ (resp. $j = 2$), $(([(x, y)_1] \times [(x, y)_1]) \cap ([(x, y)_2] \times [(x, y)_2])) \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $\{((x, y)_1, (x, y)_1), ((x, y)_2, (x, y)_2)\} \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that $(x, y)_2$ (resp. $(x, y)_1$) is in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction.

Thus, we must have $\sigma = [(x, y)_k] \times [(x, y)_k]$ $(k = 1, 2)$ and consequently, by the Definitions 4.2, 2.2, and Theorems 4.1, 4.3, (X, Θ^X, ψ) is T_0'. □

Remark 4.2. *Let X be a b-UFIL space.*

(i) By Theorems 3.1 and 4.1, X is $\overline{T_0}$ iff X is $\overline{T_0}$ at p, for all $p \in X$.

(ii) By Theorems 3.2 and 4.3, X is T_1 iff X is T_1 at p, for all $p \in X$.

(iii) By Theorem 3.3 and 4.4, X is T_0' iff X is T_0' at p, for all $p \in X$.

(iv) By Theorems 4.1–4.4, $T_1 \implies \overline{T_0} \implies T_0 \implies T_0'$, but the converse does not hold in general.

Corollary 4.1. *Let (X, Θ^X, ψ) be a bornological b-UFIL space. Then, (X, Θ^X, ψ) is $\overline{T_0}$ iff for all $x, y \in X$ with $x \neq y$, the following hold:*

(i) $[x] \times [y] \notin \psi$ or $[y] \times [x] \notin \psi$;

(ii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. By using the similar argument in Theorem 4.1, and by applying Remark 2.1(i), we obtain the claim. □

Corollary 4.2. *Let (X, Θ^X, ψ) be a bornological b-UFIL space. Then, (X, Θ^X, ψ) is T_1 iff for all $x, y \in X$ with $x \neq y$, the following hold:*

(i) $[x] \times [y] \notin \psi$ and $[y] \times [x] \notin \psi$;

(ii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. By using the similar argument in Theorem 4.1, applying Remark 2.1(i), and replacing the mapping S by mapping A, we get the results. □

Corollary 4.3. *Let (X, Θ^X, ψ) be a symmetric b-UFIL space, then the following statements are equivalent:*

(i) X is $\overline{T_0}$.

(ii) X is T_1.

(iii) For all $x, y \in X$ with $x \neq y$, $\{x, y\} \notin \Theta^X$; $[x] \times [y] \notin \psi$; and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. The proof of the corollary can be easily deduced from Theorems 4.1, 4.3 and Definition 2.1. □

Corollary 4.4. *Let* (X, Θ^X, ψ) *be a symmetric b-UFIL limit space, then the following statements are equivalent:*

(i) X *is* $\overline{T_0}$.

(ii) X *is* T_1.

(iii) For all $x, y \in X$ *with* $x \neq y$, $\{x, y\} \notin \Theta^X$*; and* $[x] \times [y] \notin \psi$.

Proof. The proof of the corollary can be easily deduced from Theorems 4.1, 4.3 and Definition 2.1. □

Definition 4.3. *(Set convergence) (cf. [32]) For an arbitrary set X, let* $\Theta^X \subset P(X)$ *be a non-empty boundedness of X and* $q \subset \Theta^X \times \mathcal{F}(X)$*. A pair* (Θ^X, q) *is called a set-convergence on X and the triplet* (X, Θ^X, q) *is known as a set-convergence space (or S-Conv space) if the following axioms hold:*
(S-Conv1) $E \in \Theta^X$ *implies* $(E, [E]) \in q$;
(S-Conv2) $(\emptyset, \xi) \in q$ *implies* $\xi = P(X)$;
(S-Conv3) $(E, \xi) \in q$ *and* $\xi \subset \xi' \in \mathcal{F}(X)$ *implies* $(E, \xi') \in q$.

A set-convergence space (X, Θ^X, q) is called a reordered set-convergence space (or ROS-Conv space) provided that the following axiom holds:
(S-Conv4) If $E \in \Theta^X$ and $(E, \xi) \in q$ then for $E' \subset E \in \Theta^X$ with $E' \neq \emptyset$ implies $(E', \xi) \in q$.

Let (X, Θ^X, q) and (Y, Θ^Y, p) be a pair of S-Conv spaces and $h : X \to Y$ be a map. Then h is called bounded continuous (or b-continuous) map if h is bounded and h transfers convergent filters.

We denote **S-Conv** (respectively **ROS-Conv**) as the category of S-Conv spaces (respectively reordered S-Conv spaces) and b-continuous maps. Also, we write $\xi \; q \; E$ for $(E, \xi) \in q$.

Note that if we restrict Θ^X to be the discrete bounded structure on X, then many point-convergence spaces in the classical sense, such as, limit spaces, Kent-convergence spaces, topological spaces etc., can be embedded into **ROS-Conv** spaces. Also note that the category **ROS-Conv** can be regarded as a full subcategory of **b-UFIL** as mentioned in [32].

For a b-UFIL space (X, Θ^X, μ), the corresponding ROS-Conv structure (Θ^X, q_μ) can be achieved provided that the following axioms hold:

(i) $\xi \; q_\mu \; \emptyset$ iff $\xi = P(X)$;

(ii) $\xi \; q_\mu \; E$ iff $[E] \times \xi \in \mu$, $\forall E \in \Theta^X \backslash \{\emptyset\}$.

Definition 4.4. *(cf. [32]) Let* (X, Θ^X, q) *be a ROS-Conv space. A reordered set-convergence pair* (Θ^X, q) *is said to be:*

(1) T_0 *set- convergence iff the following condition holds, i.e.*

(T_0) $\forall a, b \in X$, $[a] \; q \; \{b\}$ *and* $[b] \; q \; \{a\}$ *implies that* $a = b$;

(2) T_1 *set- convergence iff the following condition holds, i.e.*

(T_1) $\forall a, b \in X$ *and* $[a] \; q \; \{b\}$ *implies that* $a = b$.

Remark 4.3. *(cf. [32]) Let (X, Θ^X, μ) be a b-UFIL space. The pair (Θ^X, μ) of b-UFIL structure on X is said to be T_0 (respectively T_1) iff the corresponding pair (Θ^X, q_μ) is T_0 (respectively T_1) set-convergence. Note that we refer it as usual.*

Corollary 4.5. *Let (X, Θ^X, ψ) be a discrete symmetric b-UFIL limit space, then the following statements are equivalent.*

(i) X is $\overline{T_0}$;

(ii) X is T_1;

(iii) X is T_0 (in the usual sense);

(iv) X is T_1 (in the usual sense);

(v) For all $x, y \in X$ with $x \neq y$, $[x] \times [y] \notin \psi$.

Proof. It follows from Corollary 4.4, Remark 4.3 and Definition 2.2(iii). □

Corollary 4.6. *The following categories are isomorphic.*

*(i) $\overline{T_0}$**DISb-UFIL**;*

*(ii) $\overline{T_0}$**PUConv**;*

*(iii) $\overline{T_0}$**BONb-UFIL**.*

Proof. It follows from Theorem 4.1, Corollary 4.1, Definition 2.1 and Theorem 3.1.10 of [30]. □

Corollary 4.7. *The following categories are isomorphic.*

*(i) $\overline{T_0}$**DISsb-UFIL**;*

*(ii) $\overline{T_0}$**SUConv**;*

*(iii) T_1**SUConv**;*

*(iv) $\overline{T_0}$**BONsb-UFIL**;*

*(v) T_1**BONsb-UFIL**.*

Proof. It follows from Corollaries 4.1–4.3, Theorems 4.4 and 4.6 of [9]. □

5. Quotient-reflective subcategories of the category of b-UFIL spaces

Definition 5.1. *(cf. [32]) Given a topological functor $\mathfrak{F} : \mathcal{G} \to$ **Set**, and a full and isomorphism-closed subcategory $\mathcal{H}$ of $\mathcal{G}$, we say that $\mathcal{H}$ is:*

(1) Epireflective in $\mathcal{G}$ if and only if $\mathcal{H}$ is closed under the formation of products and extremal subobjects (i.e., subspaces).

(2) Quotient-reflective in $\mathcal{G}$ if and only if $\mathcal{H}$ is epireflective in $\mathcal{G}$ and closed under finer structures (i.e., if $X \in \mathcal{H}$, $Y \in \mathcal{G}$, $\mathfrak{F}(X) = \mathfrak{F}(Y)$, and $id : X \to Y$ is a $\mathcal{G}$-morphism, then $Y \in \mathcal{H}$).

Theorem 5.1. *(i) Every* $\overline{T_0}$***b-UFIL*** *(resp.* T_0***b-UFIL***, T_1***b-UFIL****) is a quotient-reflective subcategory of* ***b-UFIL****.*

(ii) T_0'***b-UFIL*** *is a cartesian closed and hereditary topological construct.*

Proof. (i) Let $\mathcal{G} = \overline{T_0}$**b-UFIL** and $(A, \Theta^A, \mu_A) \in \mathcal{G}$. It can be easily verified that A is full subcategory, isomorphism-closed and closed under finer structures. We are left to show that it is also closed under extremal sub-objects and closed under the formation of products.

Let $X \subset A$ and (Θ^X, μ_X) be the sub-b-UFIL structure on X induced by $i : X \to A$. We show that (X, Θ^X, μ_X) is a $\overline{T_0}$b-UFIL space. Suppose that $\{x, y\} \in \Theta^X$ for any $x, y \in X$ with $x \neq y$. Then $i(\{x, y\}) = \{i(x), i(y)\} = \{x, y\} \in \Theta^A$, a contradiction by Theorem 4.1. Thus, $\{x, y\} \notin \Theta^X$. Similarly, let $[x] \times [y] \in \mu_X$ and $([x] \times [x]) \cap ([y] \times [y]) \in \mu_X$, then $(i \times i)([x] \times [y]) = [x] \times [y] \in \mu_A$ and $(i \times i)(([x] \times [x]) \cap ([y] \times [y])) = ([x] \times [x]) \cap ([y] \times [y]) \in \mu_A$, again a contradiction. Thus $[x] \times [y] \notin \mu_X$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \mu_X$. Hence, A is closed under extremal subobjects.

Next, suppose that $A = \prod_{j \in I} A_j$, where $(\Theta^{A_j}, \mu_{A_j})$ are the $\overline{T_0}$b-UFIL structures on A_j induced by $\pi_j : A \to A_j$ for all $j \in I$, i.e., $(A_j, \Theta^{A_j}, \mu_{A_j}) \in \mathcal{G}$. We show that (A, Θ^A, μ_A) is a $\overline{T_0}$b-UFIL space. Let $\{x, y\} \in \Theta^A$ for any $x, y \in A$ with $x \neq y$. Then $\pi_j(\{x, y\}) = \{\pi_j(x), \pi_j(y)\} = \{x_j, y_j\} \in \Theta^{A_j}$, a contradiction to Theorem 4.1. Thus $\{x, y\} \notin \Theta^A$. Similarly, suppose that $[x] \times [y] \in \mu_A$ and $([x] \times [x]) \cap ([y] \times [y]) \in \mu_A$, then there exists $j \in I$ for which $x_j \neq y_j \in A_j$, such that $(\pi_j \times \pi_j)([x] \times [y]) = [x_j] \times [y_j] \in \mu_{A_j}$ and $(\pi_j \times \pi_j)(([x] \times [x]) \cap ([y] \times [y])) = ([x_j] \times [x_j]) \cap ([y_j] \times [y_j]) \in \mu_{A_j}$, a contradiction. Thus $[x] \times [y] \notin \mu_A$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \mu_A$. Hence, A is closed under the formation of products.

Therefore, the category $\overline{T_0}$**b-UFIL** is a quotient-reflective subcategory of **b-UFIL**.

Analogous to the above argument, setting $\mathcal{G} = T_0$**b-UFIL** or T_1**b-UFIL**, the proof can be easily deduced by using Theorem 4.2 or Theorem 4.3, respectively.

(ii) By Theorem 4.4, both **b-UFIL** and T_0'**b-UFIL** are isomorphic categories, and consequently, by Theorems 2.9.4 and 2.9.5 of [33], T_0'**b-UFIL** is a cartesian closed and hereditary topological construct. □

6. Conclusions

First of all, we characterized local $\overline{T_0}$, local T_0' and local T_1 b-UFIL spaces, and showed that every local T_1 b-UFIL is local $\overline{T_0}$ b-UFIL but converse is not true in general. Moreover, we characterized $\overline{T_0}$, T_0', T_0 and T_1 in the category **b-UFIL**, and showed that $T_1 \implies \overline{T_0} \implies T_0 \implies T_0'$, but the converse does not hold in general and provided some related results. Furthermore, we showed that under $\overline{T_0}$ condition, $\overline{T_0}$**DISb-UFIL** $\cong$ $\overline{T_0}$**PUConv** $\cong$ $\overline{T_0}$**BONb-UFIL** which is not isomorphic in general. Also, we showed the isomorphic relation among $\overline{T_0}$**DISsb-UFIL**, $\overline{T_0}$**SUConv**, T_1**SUConv**, $\overline{T_0}$**BONsb-UFIL** and T_1**BONsb-UFIL**, and examined their relationships with the usual ones. Finally, we examined that $\overline{T_0}$**b-UFIL**, T_0**b-UFIL** and T_1**b-UFIL** are quotient-reflective subcategories of **b-UFIL**, and T_0'**b-UFIL** is a hereditary and cartesian closed topological category. In light of this study, the following can be examined in **b-UFIL** as a future research work:

(i) Can one characterize closed and strongly closed objects in **b-UFIL**, and what would be their corresponding closure operators using the notion of closedness in **b-UFIL**?

(ii) How can one characterize irreducibility, soberness, connectedness and hyperconnectedness in the category **b-UFIL**? Can Urysohn Lemma and Tietze Extension Theorem be extended in the category of **b-UFIL**?

(iii) How one can define pre-Hausdorff, Hausdorff, regular and normal objects in **b-UFIL**, and what would be their relation to the classical ones?

Conflict of interest

We declare that we have no conflicts of interest.

References

1. J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and concrete categories*, New York: Wiley, 1990.
2. M. Baran, Separation properties, *Indian J. Pure Appl. Math.*, **23** (1991), 333–341.
3. M. Baran, Generalized local separation properties, *Indian J. Pure Appl. Math.*, **25** (1994), 615–620.
4. M. Baran, H. Altındis, T_0-objects in topological categories, *J. Univ. Kuwait (Sci.)*, **22** (1995), 123–127.
5. M. Baran, Separation properties in topological categories, *Math. Balkanica*, **10** (1996), 39–48.
6. M. Baran, Completely regular objects and normal objects in topological categories, *Acta Math. Hung.*, **80** (1998), 211–224. https://doi.org/10.1023/A:1006550726143
7. M. Baran, Compactness, perfectness, separation, minimality and closedness with respect to closure operators, *Appl. Categor. Struct.*, **10** (2002), 403–415. https://doi.org/10.1023/A:1016388102703
8. M. Baran, PreT_2 objects in topological categories, *Appl. Categor. Struct.*, **17** (2009), 591–602. https://doi.org/10.1007/s10485-008-9161-4
9. M. Baran, S. Kula, A. Erciyes, T_0 and T_1 semiuniform convergence spaces, *Filomat*, **27** (2013), 537–546.
10. M. Baran, S. Kula, T. M. Baran, M. Qasim, Closure operators in semiuniform convergence spaces, *Filomat*, **30** (2016), 131–140.
11. M. Baran, H. Abughalwa, Sober spaces, *Turkish J. Math.*, **46** (2022), 299–310. https://doi.org/10.3906/mat-2109-95
12. T. M. Baran, Closedness, separation and connectedness in pseudo-quasi-semi metric spaces, *Filomat*, **34** (2020), 4757–4766. https://doi.org/10.2298/FIL2014757B
13. T. M. Baran, A. Erciyes, T_4, Urysohn's lemma and Tietze extension theorem for constant filter convergence spaces, *Turkish J. Math.*, **45** (2021), 843–855. https://doi.org/10.3906/mat-2012-101
14. T. M. Baran, M. Kula, Separation axioms, Urysohn's lemma and Tietze extension theorem for extended pseudo-quasi-semi metric spaces, *Filomat*, **36** (2022), 703–713. https://doi.org/10.2298/FIL2202703B
15. G. C. L. Brümmer, *A categorial study of initiality in uniform topology*, University of Cape Town, 1971.

16. D. Dikranjan, E. Giuli, Closure operators I, *Topology Appl.*, **27** (1987), 129–143. https://doi.org/10.1016/0166-8641(87)90100-3

17. D. B. Doitchinov, A unified theory of topological spaces, proximity spaces and uniform spaces, *Soviet Math. Dokl.*, **5** (1964), 595–598.

18. A. Erciyes, T. M. Baran, M. Qasim, Closure operators in constant filter convergence spaces, *Konuralp J. Math.*, **8** (2020), 185–191.

19. F. Peter, W. F. Lindgren, *Quasi-uniform spaces*, 1982.

20. J. M. Harvey, T_0-separation in topological categories, *Quaest. Math.*, **2** (1977), 177–190. https://doi.org/10.1080/16073606.1977.9632541

21. G. T. Herman, On topology as applied to image analysis, *Comput. Vis. Graph. Image Process.*, **52** (1990), 409–415. https://doi.org/10.1016/0734-189X(90)90084-9

22. H. Herrlich, Topological structures, *Math. Centre Tracts*, **52** (1974), 59–122.

23. R. E. Hoffmann, *(E, M)-universally topological functors*, Habilitationsscheift, Universität Düsseldorf, 1974.

24. G. Janelidze, Light morphisms for generalized T_0-reflections, *Topology Appl.*, **156** (2009), 2109–2115. https://doi.org/10.1016/j.topol.2009.03.031

25. P. T. Johnstone, *Stone spaces*, New York: Academic Press, 1977.

26. M. Katětov, On continuity structures and spaces of mappings, *Comment. Math. Univ. Carolin.*, **6** (1965), 257–278.

27. D. C. Kent, Convergence functions and their related topologies, *Fund. Math.*, **54** (1964), 125–133.

28. V. A. Kovalevsky, Finite topology as applied to image analysis, *Comput. Vis. Graph. Image Process.*, **46** (1989), 141–161. https://doi.org/10.1016/0734-189X(89)90165-5

29. V. Kovalevsky, R. Kopperman, Some topology-based image processing algorithms, *Ann. New York Acad. Sci.*, **728** (1994), 174–182. https://doi.org/10.1111/j.1749-6632.1994.tb44143.x

30. S. Kula, *Closedness and T_0, T_1 objects in the category of preuniform convergence spaces*, Ph. D. thesis, Turkey: Erciyes University, 2014.

31. D. Leseberg, Z. Vaziry, The quasitopos of b-uniform filter spaces, *Math. Appl.*, **7** (2018), 155–171. https://doi.org/10.13164/ma.2018.13

32. D. Leseberg, Z. Vaziry, On the completeness of non-symmetrical uniform convergence with some links to approach spaces, *Math. Appl.*, **8** (2019), 37–57. https://doi.org/10.13164/ma.2019.04

33. D. Leseberg, Z. Vaziry, *Bounded topology*, Lap Lambert Academic Publishing, 2019.

34. T. Marny, *Rechts-Bikategoriestrukturen in topologischen Kategorien: Inaugural-dissertation*, 1973.

35. G. Preuß, Semiuniform convergence spaces, *Math. Japn.*, **41** (1995), 465–491.

36. G. Preuss, *Foundations of topology: An approach to convenient topology*, Dordrecht: Springer Science & Business Media, 2002. https://doi.org/10.1007/978-94-010-0489-3

37. M. Qasim, M. Baran, H. Abughalwa, Closure operators in convergence approach spaces, *Turkish J. Math.*, **45** (2021), 139–152. https://doi.org/10.3906/mat-2008-65

38. A. Salibra, A continuum of theories of lambda calculus without semantics, In: *Proceedings 16th Annual IEEE Symposium on Logic in Computer Science*, 2001, 334–343. https://doi.org/10.1109/LICS.2001.932509

39. J. E. Stoy, *Denotational semantics: the Scott-Strachey approach to programming language theory*, MIT Press, 1981.

40. S. Weck-Schwarz, T_0-objects and separated objects in topological categories, *Quaest. Math.*, **14** (1991), 315–325. https://doi.org/10.1080/16073606.1991.9631649