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Research article

Best proximity points in non-Archimedean fuzzy metric spaces with application to domain of words

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Abstract: This paper deals with the existence and uniqueness of the best proximity points of nonself-mappings in the context of non-Archimedean fuzzy metric spaces. The existence of different proximal quasi-contractive mappings allowed us to generalize some results concerning the existence and uniqueness of the best proximity points in the existing literature. Moreover, an application in computer science, particularly in the domain of words has been provided.

Keywords: best proximity point; quasi-contraction; fuzzy contractive mapping
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1. Introduction and preliminaries

The distinguished Banach fixed point theorem [3] deals with the existence and uniqueness of fixed point of self-mappings defined on a complete metric space and has numerous applications in mathematics and related disciplines, for instance in inverse problems (compare [19, 36]). Due to its vast range of applications, Banach fixed point theorem has attracted several researchers to extend the scope of metric fixed point theory (compare [2, 18, 25, 27]). On the other hand, if C and D are nonempty subsets of a metric space (X, d), then in the case of a nonself-mapping $T : C \to D$, there might not exist a point x in C such that $x = Tx$. In such situations, it is better to find an element x in C such that the distance between x and Tx is minimum, and if such an x in C exists, then it is the best proximity point (shortly BPP) of T.

One of the earlier results regarding the existence of a BPP is attributed to Fan [8], which assures the existence of BPP of a continuous mapping of a nonempty compact convex subset of a Hausdorff locally convex topological vector space. Many extensions of Fan's theorems are available in the literature including Prolla [26], Reich [29], Sehgal and Singh [32]. Eldred and Veeramani [7] discussed the existence and convergence of best proximity points (shortly BPPs) in uniformly convex Banach spaces. Basha [4] presented BPP theorems for proximal contractions of the first and second kinds. Some other types of contractions were introduced by several authors like Bari et al. [6] who coined the idea of cyclic Meir-Keeler contractions and proved the existence and uniqueness of the BPP for cyclic Meir-Keeler contractions. Agarwal and Karpagam [17] discussed BPP results for p-cyclic Meir-Keeler contractions.

Menger [23] introduced probabilistic metric spaces. Zadeh [35] introduced fuzzy sets to deal with uncertainty. Kramosil and Michalek [15] introduced the idea of probabilistic metric spaces to fuzzy metric spaces (shortly FMSs) and it was further modified by George and Veeramani (see [10, 11]), which enabled them to assign to each fuzzy metric space (FMS) a Hausdorff topology. The fuzzy metric fixed point theory was initiated with the paper of Grabiec [12]. Fuzzy contractive mappings were triggered by Gregori and Sapena in [13] and generalized the Banach contraction principle by considering new types of fuzzy contractive mappings in FMSs. Mihet [20] generalized the Banach contraction principle by considering the fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces (shortly N-AFMSs) in [21].

In the context of metric spaces, quasi-contractions were initiated by Ćirić [5]. On the other hand, the generalization of the fuzzy contractive condition of Sapena and Gregori in the form of fuzzy $\mathcal{H}$-contractive mappings was presented by Wardowski in [34]. Amini and Mihet [1] introduced fuzzy $\mathcal{H}$-quasi-contractive mappings using the idea of quasi-contractions by Ćirić and fuzzy $\mathcal{H}$-contractive mappings by Wardowski.

Moreover, results in the context of FMSs have applications in various areas of mathematics and other related disciplines, for instance in computer sciences, particularly in the context of the domain of words (compare [22, 28, 30]).

On the other hand, by using different contractive conditions in an N-AFMS, Vetro and Saleemi [33] discussed the existence and uniqueness of fuzzy best proximity points. In this paper, first we prove the existence and uniqueness of fuzzy best proximity points of more general different proximal contractions on N-AFMSs and then we apply best proximity point results in an N-AFMS to solve a recurrence relation in connection with the domain of words.

We will highlight some basic notions that will be used in the sequel to obtain the main results. Throughout this article, I represent the interval $[0, 1]$. We start with the following definition.

Definition 1.1. *[31] A continuous t-norm is a binary operation* $\diamond : I \times I \to I$ *such that*

$\mathbf{T}_1-$ $\diamond$ *is commutative and associative;*

$\mathbf{T}_2-$ $\diamond$ *is continuous;*

$\mathbf{T}_3-$ $1 \diamond \eta = \eta$ *for every* η *in* I*;*

$\mathbf{T}_4-$ $\eta \diamond \xi \leq \zeta \diamond \upsilon$ *whenever* $\eta \leq \zeta$ *and* $\xi \leq \upsilon$, *for all* $\eta, \zeta, \xi, \upsilon$ *in* I.

The three prototypical t-norms are product, minimum, and Lukasiewicz t-norms defined as

$$\eta \diamond_{prod} \zeta = \eta\zeta,$$
$$\eta \diamond_{\min} \zeta = \min\{\eta, \zeta\},$$
$$\eta \diamond_{L} \zeta = \max\{\eta + \zeta - 1, 0\},$$

respectively. George and Veeramani defined the the FMS as follows.

Definition 1.2. *[10] Let Z be a nonempty set. Then a FMS is a triplet $(Z, \mathcal{F}, \diamond)$ with a continuous t-norm $\diamond$, and a fuzzy set $\mathcal{F}$ defined on $Z \times Z \times (0, \infty)$ satisfying the following conditions for all $\lambda, \mu, \nu \in Z$ and $\upsilon, w \in (0, \infty)$.*

G$_1-$ $\mathcal{F}(\mu, \lambda, \upsilon) > 0$;

G$_2-$ $\mathcal{F}(\mu, \lambda, \upsilon) = 1$ *if and only if* $\mu = \lambda$;

G$_3-$ $\mathcal{F}(\mu, \lambda, \upsilon) = \mathcal{F}(\lambda, \mu, \upsilon)$;

G$_4-$ $\mathcal{F}(\mu, \lambda, \upsilon) \diamond \mathcal{F}(\lambda, \nu, w) \leq \mathcal{F}(\mu, \nu, \upsilon + w)$;

G$_5-$ $\mathcal{F}(\mu, \lambda, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Moreover, the triplet $(Z, \mathcal{F}, \diamond)$ is called an N-AFMS, if **G**$_4$ *is replaced by*

G$_6-$ $\mathcal{F}(\mu, \lambda, \upsilon) \diamond \mathcal{F}(\lambda, \nu, w) \leq \mathcal{F}(\mu, \nu, \max\{\upsilon, w\})$, *or equivalently,*

$$\mathcal{F}(\mu, \lambda, \upsilon) \diamond \mathcal{F}(\lambda, \nu, \upsilon) \leq \mathcal{F}(\mu, \nu, \upsilon) \text{ ([14]).}$$

Note that each N-AFMS is a FMS.

Let $(Z, \mathcal{F}, \diamond)$ be a FMS and C, D be nonempty subsets of $(Z, \mathcal{F}, \diamond)$. Define

$$\begin{aligned} C_0(\upsilon) &= \{\mu \in C : \mathcal{F}(\mu, \lambda, \upsilon) = \mathcal{F}(C, D, \upsilon) \text{ for some } \lambda \in D\}, \\ D_0(\upsilon) &= \{\lambda \in D : \mathcal{F}(\mu, \lambda, \upsilon) = \mathcal{F}(C, D, \upsilon) \text{ for some } \mu \in C\}, \end{aligned}$$

where

$$\mathcal{F}(C, D, \upsilon) = \sup_{\mu \in C, \lambda \in D} \mathcal{F}(\mu, \lambda, \upsilon).$$

Definition 1.3. *[31] Let $(Z, \mathcal{F}, \diamond)$ be a FMS. Then*

(i) *A sequence $\{y_n\}$ in Z converges to y in Z, if and only if*

$$\lim_{n\to\infty} \mathcal{F}(y_n, y, \upsilon) = 1$$

for all $\upsilon > 0$. We denote it as $y_n \to y$ as $n \to \infty$.

(ii) *[10] A sequence $\{y_n\}$ in Z is M-Cauchy if and only if for all $\epsilon \in (0, 1)$ and $\upsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that*

$$\mathcal{F}(y_n, y_m, \upsilon) > 1 - \epsilon$$

for all $m, n \geq n_0$.

(iii) *[13] A sequence $\{y_n\}$ is G-Cauchy if and only if, for all $\epsilon \in (0,1)$, and for all $\upsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \mathcal{F}\left(y_n, y_{n+p}, \upsilon\right) > 1-\epsilon$$

for all $n \geq n_0$ and any integer $p > 0$.

(iv) *The FMS $(Z, \mathcal{F}, *)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.*

Note that every M-Cauchy sequence is G-Cauchy and hence every G-complete FMS is M-complete (compare [24]).

Definition 1.4. *Suppose Φ denotes the class of all functions, $\varphi : I \to I$ such that φ is continuous, decreasing and $\varphi(w) = 0$ if and only if $w = 1$.*

2. Main results

In this section, we start with the following theorem.

Theorem 2.1. *Let C and D be nonempty closed subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is nonempty for every $\upsilon > 0$ and $T : C \to D$ a nonself-mapping that satisfies:*

(i) *$T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ for all $\upsilon > 0$,*

(ii) *There is a function φ in Φ for which*

$$\left.\begin{array}{l} \mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon) \\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon) \end{array}\right\} \text{ implies } \varphi(\mathcal{F}(\alpha,\beta,\upsilon)) \leq \omega(\upsilon)\mathcal{A}_\varphi(\mu,\lambda,\upsilon) \tag{2.1}$$

holds for all $\alpha, \beta, \lambda, \mu \in C$, and $\upsilon > 0$, where $\omega : (0,\infty) \to (0,1)$ is a function and

$$\mathcal{A}_\varphi(\mu,\lambda,\upsilon) = \max\left\{ \begin{array}{c} \varphi(\mathcal{F}(\mu,\lambda,\upsilon)), \varphi(\mathcal{F}(\mu,\alpha,\upsilon)), \\ \varphi(\mathcal{F}(\lambda,\alpha,\upsilon)), \varphi(\mathcal{F}(\lambda,\beta,\upsilon)) \end{array} \right\},$$

(iii) *For any sequence $\{\lambda_n\}$ in $D_0(\upsilon)$ and $\mu \in C$ satisfying $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ as $n \to \infty$, one has $\mu \in C_0(\upsilon)$.*

Then there exists a unique $\mu^* \in C$ such that

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$$

for every $\upsilon > 0$.

Proof. Pick an arbitrary point $\mu_0 \in C_0(\upsilon)$. As $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$, so there is a $\mu_1 \in C_0(\upsilon)$ such that

$$\mathcal{F}(\mu_1, T\mu_0, \upsilon) = \mathcal{F}(C, D, \upsilon).$$

Recursively, we obtain a sequence $\{\mu_n\}$ in $C_0(\upsilon)$ satisfying

$$\left\{ \begin{array}{l} \mathcal{F}(\mu_n, T\mu_{n-1}, \upsilon) = \mathcal{F}(C, D, \upsilon), \\ \mathcal{F}(\mu_{n+1}, T\mu_n, \upsilon) = \mathcal{F}(C, D, \upsilon) \end{array} \right. \tag{2.2}$$

for all $n \in \mathbb{N}$, and $\upsilon > 0$. Clearly, if for some $n_0 \in \mathbb{N}$, $\mu_{n_0+1} = \mu_{n_0}$, then from (2.2), μ_{n_0} becomes a BPP of T. Hence, we assume $\mu_{n+1} \neq \mu_n$ for all $n \in \mathbb{N}$. Define $\varkappa_n(\upsilon) = \mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)$ for all $n \in \mathbb{N} \cup \{0\}$ and all $\upsilon > 0$. From (2.1), we get

$$\varphi(\varkappa_n(\upsilon)) = \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)) \leq \omega(\upsilon)\mathcal{A}_\varphi(\mu_{n-1}, \mu_n, \upsilon), \tag{2.3}$$

where

$$\begin{aligned}
\mathcal{A}_\varphi(\mu_{n-1}, \mu_n, \upsilon) &= \max\left\{ \begin{array}{l} \varphi(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \varphi(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \\ \varphi(\mathcal{F}(\mu_n, \mu_n, \upsilon)), \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)) \end{array} \right\} \\
&= \max\{\varphi(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \varphi(1), \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon))\} \\
&= \max\{\varphi(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon))\}.
\end{aligned}$$

If

$$\max\{\varphi(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon))\} = \varphi(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)),$$

then

$$\varphi(\varkappa_n(\upsilon)) \leq \omega(\upsilon)\varphi(\varkappa_n(\upsilon)) < \varphi(\varkappa_n(\upsilon)),$$

a contradiction as $0 < \omega(\upsilon) < 1$. Hence,

$$\varphi(\varkappa_n(\upsilon)) \leq \omega(\upsilon)\varphi(\varkappa_{n-1}) < \varphi(\varkappa_{n-1}(\upsilon)).$$

This implies that $\varkappa_n(\upsilon)$ is an increasing sequence that is bounded above by 1. Let $\lim_{n\to\infty} \varkappa_n(\upsilon) = \varkappa(\upsilon)$. Now, we claim that $\varkappa(\upsilon) = 1$ for all $\upsilon > 0$. On the contrary, if $0 < \varkappa(\upsilon_0) < 1$ for some $\upsilon_0 > 0$, then

$$\begin{aligned}
\varphi(\varkappa(\upsilon_0)) &= \lim_{n\to\infty} \varphi(\varkappa_n(\upsilon_0)) \leq \omega(\upsilon_0) \lim_{n\to\infty} \varphi(\varkappa_{n-1}(\upsilon_0)) \\
&\leq \omega(\upsilon_0)\varphi(\varkappa(\upsilon_0)) < \varphi(\varkappa(\upsilon_0))
\end{aligned}$$

a contradiction. Hence,

$$\lim_{n\to\infty} \varkappa_n(\upsilon) = 1 \tag{2.4}$$

for all $\upsilon > 0$. If $\{\mu_n\}$ is not a Cauchy sequence, then there is an $\epsilon \in (0, 1)$ and $\upsilon_0 > 0$ such that for every $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$\mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) \leq 1 - \epsilon. \tag{2.5}$$

Let m_k be the least integer greater than n_k satisfying (2.5), that is,

$$\mathcal{F}(\mu_{m_k-1}, \mu_{n_k}, \upsilon_0) > 1 - \epsilon,$$

which implies

$$\begin{aligned}
1 - \epsilon &\geq \mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) \geq \mathcal{F}(\mu_{m_k-1}, \mu_{m_k}, \upsilon_0) \diamond \mathcal{F}(\mu_{m_k-1}, \mu_{n_k}, \upsilon_0) \\
&> \varkappa_{m_k-1}(\upsilon_0) \diamond (1 - \epsilon).
\end{aligned}$$

Consequently, we get

$$\lim_{k\to\infty} \mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) = 1 - \epsilon. \tag{2.6}$$

Further

$$\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)\geq\mathcal{F}(\mu_{m_k+1},\mu_{m_k},\upsilon_0)\diamond\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0)\diamond\mathcal{F}(\mu_{n_k},\mu_{n_k+1},\upsilon_0).$$

On taking limit as $k\to\infty$, we get

$$\lim_{k\to\infty}\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)\geq 1-\epsilon. \tag{2.7}$$

Now, from (2.4) and (2.6) we get

$$\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0)\geq\mathcal{F}(\mu_{m_k},\mu_{m_k+1},\upsilon_0)\diamond\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)\diamond\mathcal{F}(\mu_{n_k+1},\mu_{n_k},\upsilon_0),$$

which implies

$$\lim_{k\to\infty}\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)=1-\epsilon.$$

Further

$$\mathcal{F}(\mu_{m_k},\mu_{n_k+1},\upsilon_0)\geq\mathcal{F}(\mu_{m_k},\mu_{m_k+1},\upsilon_0)\diamond\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)$$

implies

$$\lim_{k\to\infty}\mathcal{F}(\mu_{m_k},\mu_{n_k+1},\upsilon_0)\geq 1-\epsilon.$$

Similarly,

$$\lim_{k\to\infty}\mathcal{F}(\mu_{n_k},\mu_{m_k+1},\upsilon_0)\geq 1-\epsilon.$$

Now,

$$\begin{cases}\mathcal{F}(\mu_{m_k+1},T\mu_{m_k},,\upsilon_0)=\mathcal{F}(C,D,\upsilon_0),\\ \mathcal{F}(\mu_{n_k+1},T\mu_{n_k},\upsilon_0)=\mathcal{F}(C,D,\upsilon_0)\end{cases}$$

implies

$$\begin{aligned}&\varphi(\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0))\leq\omega(\upsilon_0)\mathcal{A}_\varphi(\mu_{m_k},\mu_{n_k},\upsilon_0)\\ &\leq\omega(\upsilon_0)\max\left\{\begin{array}{c}\varphi(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0)),\varphi(\mathcal{F}(\mu_{m_k},\mu_{m_k+1},\upsilon_0)),\\ \varphi(\mathcal{F}(\mu_{n_k},\mu_{m_k+1},\upsilon_0)),\varphi(\mathcal{F}(\mu_{n_k},\mu_{n_{k+1}},\upsilon_0))\end{array}\right\}.\end{aligned}$$

As k tends to ∞ in above, we get

$$\begin{aligned}\varphi(1-\epsilon)&\leq\omega(\upsilon_0)\max\{\varphi(1-\epsilon),\varphi(1),\varphi(1-\epsilon),\varphi(1)\}\\ &=\omega(\upsilon_0)\varphi(1-\epsilon).\end{aligned}$$

If $\varphi(1-\epsilon)=0$, then $\epsilon=0$, a contradiction. If $\varphi(1-\epsilon)>0$, then

$$\varphi(1-\epsilon)\leq\omega(\upsilon_0)\varphi(1-\epsilon)<\varphi(1-\epsilon),$$

a contradiction, as $0<\omega(\upsilon_0)<1$. Hence, $\{\mu_n\}$ is a Cauchy sequence. The completeness of $(Z,\mathcal{F},\diamond)$ implies $\{\mu_n\}$ converges to some $\mu^*\in Z$, that is,

$$\lim_{n\to\infty}\mathcal{F}(\mu_n,\mu^*,\upsilon)=1\text{ for all }\upsilon>0. \tag{2.8}$$

Moreover

$$\mathcal{F}(C,D,\upsilon)\;=\;\mathcal{F}(\mu_{n+1},T\mu_n,\upsilon)$$

$$
\begin{aligned}
&\geq \mathcal{F}(\mu_{n+1},\mu^*,\upsilon)\diamond\mathcal{F}(\mu^*,T\mu_n,\upsilon)\\
&\geq \mathcal{F}(\mu_{n+1},\mu^*,\upsilon)\diamond\mathcal{F}(\mu^*,\mu_{n+1},\upsilon)\diamond\mathcal{F}(\mu_{n+1},T\mu_n,\upsilon)\\
&= \mathcal{F}(\mu_{n+1},\mu^*,\upsilon)\diamond\mathcal{F}(\mu^*,\mu_{n+1},\upsilon)\diamond\mathcal{F}(C,D,\upsilon).
\end{aligned}
$$

By taking the limit as n tends to ∞, we get

$$
\begin{aligned}
\mathcal{F}(C,D,\upsilon) &= \lim_{n\to\infty}\mathcal{F}(\mu^*,T\mu_n,\upsilon)\\
&\geq \lim_{n\to\infty}\mathcal{F}(\mu_{n+1},\mu^*,\upsilon)\diamond\lim_{n\to\infty}\mathcal{F}(\mu^*,\mu_{n+1},\upsilon)\diamond\mathcal{F}(C,D,\upsilon)\\
&= \mathcal{F}(C,D,\upsilon)
\end{aligned}
$$

implies

$$
\lim_{n\to\infty}\mathcal{F}(\mu^*,T\mu_n,\upsilon)=\mathcal{F}(C,D,\upsilon). \tag{2.9}
$$

Now, we show that T has a BPP. Note that (iii) and (2.8) implies $\mu^*\in C_0(\upsilon)$ and hence $T\mu^*\in T(C_0(\upsilon))$. As $T(C_0(\upsilon))\subseteq D_0(\upsilon)$ ensures that there is a $\xi\in C_0(\upsilon)$ for which

$$
\mathcal{F}(\xi,T\mu^*,\upsilon)=\mathcal{F}(C,D,\upsilon). \tag{2.10}
$$

We claim that $\xi=\mu^*$. On the contrary, assume that $\xi\neq\mu^*$. By (2.1), (2.2) and (2.10), we obtain

$$
\begin{aligned}
\varphi(\mathcal{F}(\xi,\mu_{n+1},\upsilon)) &\leq \omega(\upsilon)\mathcal{A}_\varphi(\mu_n,\mu^*,\upsilon)\\
&\leq \omega(\upsilon)\max\left\{\begin{array}{l}\varphi(\mathcal{F}(\mu_n,\mu^*,\upsilon)),\varphi(\mathcal{F}(\mu^*,\xi,\upsilon)),\\ \varphi(\mathcal{F}(\mu_n,\xi,\upsilon)),\varphi(\mathcal{F}(\mu_n,\mu_{n+1},\upsilon))\end{array}\right\}.
\end{aligned}
$$

Upon taking limit as n tends to ∞ in above, we get

$$
\begin{aligned}
\varphi(\mathcal{F}(\xi,\mu^*,\upsilon)) &\leq \omega(\upsilon)\max\{\varphi(\mathcal{F}(\mu^*,\mu^*,\upsilon)),\varphi(\mathcal{F}(\mu^*,\xi,\upsilon)),\varphi(1)\}\\
&= \omega(\upsilon)\varphi(\mathcal{F}(\mu^*,\xi,\upsilon))<\varphi(\mathcal{F}(\mu^*,\xi,\upsilon)),
\end{aligned}
$$

a contradiction, as $0<\omega(\upsilon)<1$. Hence, $\xi=\mu^*$ and consequently

$$
\mathcal{F}(\mu^*,T\mu^*,\upsilon)=\mathcal{F}(C,D,\upsilon),
$$

that is, μ^* is the BPP of T. If r is another BPP of T such that $r\neq\mu^*$, then $0<\mathcal{F}(\mu^*,r,\upsilon)<1$ for all $\upsilon>0$ and

$$
\mathcal{F}(\mu^*,T\mu^*,\upsilon)=\mathcal{F}(C,D,\upsilon)\ \text{ and }\ \mathcal{F}(r,Tr,\upsilon)=\mathcal{F}(C,D,\upsilon).
$$

Then we have

$$
\begin{aligned}
\varphi(\mathcal{F}(\mu^*,r,\upsilon)) &\leq \omega(\upsilon)\mathcal{A}_\varphi(\mu^*,r,\upsilon)\\
&\leq \omega(\upsilon)\max\left\{\begin{array}{l}\varphi(\mathcal{F}(\mu^*,r,\upsilon)),\varphi(\mathcal{F}(\mu^*,\mu^*,\upsilon)),\\ \varphi(\mathcal{F}(r,\mu^*,\upsilon)),\varphi(\mathcal{F}(r,r,\upsilon))\end{array}\right\}\\
&= \omega(\upsilon)\varphi(\mathcal{F}(\mu^*,r,\upsilon))<\varphi(\mathcal{F}(\mu^*,r,\upsilon))
\end{aligned}
$$

a contradiction. Hence, the BPP of T is unique.

If we consider $C_0(\upsilon)$ a nonempty and closed set, then we can relax some conditions in Theorem 2.1 as follows.

Theorem 2.2. *Let C and D be nonempty subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is a closed subset of $(Z, \mathcal{F}, \diamond)$ for every $\upsilon > 0$ and $T : C \to D$ a nonself-mapping satisfying the following:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ *for all* $\upsilon > 0$*,*

(ii) *There exists* $\varphi \in \Phi$ *for which*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \varphi(\mathcal{F}(\alpha, \beta, \upsilon)) \leq \omega(\upsilon)\mathcal{A}_\varphi(\mu, \lambda, \upsilon), \tag{2.11}$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ *and* $\upsilon > 0$*, where* $\omega : (0, \infty) \to (0, 1)$ *a function and*

$$\mathcal{A}_\varphi(\mu, \lambda, \upsilon) = \max\left\{\begin{array}{l}\varphi(\mathcal{F}(\mu, \lambda, \upsilon)), \varphi(\mathcal{F}(\mu, \alpha, \upsilon)),\\ \varphi(\mathcal{F}(\lambda, \alpha, \upsilon)), \varphi(\mathcal{F}(\lambda, \beta, \upsilon))\end{array}\right\}.$$

Then there is a unique $\mu^* \in C$ *for which* $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$ *for all* $\upsilon > 0$*.*

Proof. Construct a Cauchy sequence $\{\mu_n\}$ in $C_0(\upsilon)$ same as in the proof of Theorem 2.1. As $C_0(\upsilon)$ is a closed so the completeness of $(Z, \mathcal{F}, \diamond)$ ensures that the sequence $\{\mu_n\}$ is convergent to some μ^* in $C_0(\upsilon)$. The remaining part of the proof is same as the proof of Theorem 2.1.

In the next theorem, we use the different contraction condition in comparison with the above results.

Theorem 2.3. *Let C and D be nonempty closed subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is nonempty for every $\upsilon > 0$ and $T : C \to D$ a nonself-mapping satisfying the following:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ *for all* $\upsilon > 0$*,*

(ii) *There is a continuous function* $\rho : I \to I$*, with* $\rho(s) > 0$ *for every* $s \in (0, 1]$*, for which*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \mathcal{F}(\alpha, \beta, \upsilon) \geq \mathcal{B}(\mu, \lambda, \upsilon) + \mathcal{A}_\rho(\mu, \lambda, \upsilon), \tag{2.12}$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ *and* $\upsilon > 0$*, where*

$$\mathcal{A}_\rho(\mu, \lambda, \upsilon) = \min\left\{\begin{array}{l}\rho(\mathcal{F}(\mu, \lambda, \upsilon)), \rho(\mathcal{F}(\mu, \alpha, \upsilon))\\ \rho(\mathcal{F}(\lambda, \alpha, \upsilon)), \rho(\mathcal{F}(\lambda, \beta, \upsilon))\end{array}\right\},$$
$$\mathcal{B}(\mu, \lambda, \upsilon) = \min\{\mathcal{F}(\mu, \lambda, \upsilon), \mathcal{F}(\lambda, \alpha, \upsilon)\}.$$

(iii) *For any sequence* $\{\lambda_n\}$ *in* $D_0(\upsilon)$ *and* $\mu \in C$ *satisfying* $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ *as* $n \to \infty$*, one has* $\mu \in C_0(\upsilon)$*.*

Then there is a unique $\mu^* \in C$ *for which* $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$ *for all* $\upsilon > 0$*.*

Proof. As $C_0(\upsilon)$ is nonempty for every $\upsilon > 0$, so we pick a μ_0 in $C_0(\upsilon)$. Since $T\mu_0 \in T(C_0(\upsilon)) \subseteq D_0(\upsilon)$, we can find $\mu_1 \in C_0(\upsilon)$ such that

$$\mathcal{F}(\mu_1, T\mu_0, \upsilon) = \mathcal{F}(C, D, \upsilon).$$

Recursively, we obtain a sequence $\{\mu_n\}$ in $C_0(\upsilon)$ satisfying

$$\begin{aligned} &\mathcal{F}(\mu_n, T\mu_{n-1}, \upsilon) = \mathcal{F}(C, D, \upsilon) \text{ and} \\ &\mathcal{F}(\mu_{n+1}, T\mu_n, \upsilon) = \mathcal{F}(C, D, \upsilon) \end{aligned} \tag{2.13}$$

for all $n \in \mathbb{N}$, $\upsilon > 0$. From (2.12) and (2.13), we obtain

$$\begin{aligned} &\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon) \geq \mathcal{B}(\mu_{n-1}, \mu_n, \upsilon) + \mathcal{A}_\rho(\mu_{n-1}, \mu_n, \upsilon) \\ &\geq \min\{\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon), \mathcal{F}(\mu_n, \mu_n, \upsilon)\} + \\ &\min\left\{\begin{array}{l} \rho(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \rho(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \\ \rho(\mathcal{F}(\mu_n, \mu_n, \upsilon)), \rho(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)) \end{array}\right\} \\ &= \mathcal{F}(\mu_{n-1}, \mu_n, \upsilon) + \min\{\rho(\mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)), \rho(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon))\} \end{aligned} \tag{2.14}$$

which implies

$$\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon) \geq \mathcal{F}(\mu_{n-1}, \mu_n, \upsilon)$$

that is, $\{\mathcal{F}(\mu_{n+1}, T\mu_n, \upsilon)\}$ is an increasing sequence in $(0, 1]$ which is bounded above by 1. So, there is $j(\upsilon) \in (0, 1]$ for which

$$\lim_{n\to\infty} \mathcal{F}(\mu_n, \mu_{n+1}, \upsilon) = j(\upsilon)$$

for all $\upsilon > 0$. We claim that $j(\upsilon) = 1$ for all $\upsilon > 0$. On contrary, assume that there is $\upsilon_0 > 0$ for which $0 < j(\upsilon_0) < 1$. Taking the limit as n tends to ∞ in (2.14) implies

$$j(\upsilon_0) \geq j(\upsilon_0) + \min\{\rho(j(\upsilon_0)), \rho(1)\}.$$

If $\min\{\rho(j(\upsilon_0)), \rho(1)\} = \rho(j(\upsilon_0))$, then we get $j(\upsilon_0) \geq j(\upsilon_0) + \rho(j(\upsilon_0))$ implies that $\rho(j(\upsilon_0)) = 0$, which is a contradiction. If $\min\{\rho(j(\upsilon_0)), \rho(1)\} = \rho(1)$, then we get $j(\upsilon_0) \geq j(\upsilon_0) + \rho(1)$ implies that $\rho(1) = 0$, which is a contradiction. This shows that $j(\upsilon) = 1$ for all $\upsilon > 0$. Next we show that $\{\mu_n\}$ is a Cauchy sequence. If we suppose on contrary that $\{\mu_n\}$ is not a Cauchy sequence, then there is an $\epsilon \in (0, 1)$ and $\upsilon_0 > 0$, so that for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$\mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) \leq 1 - \epsilon. \tag{2.15}$$

Let m_k be the least integer greater than n_k satisfying (2.15), that is,

$$\mathcal{F}(\mu_{m_k-1}, \mu_{n_k}, \upsilon_0) > 1 - \epsilon.$$

On similar lines as in the proof of Theorem 2.1, we get

$$\begin{aligned} \lim_{k\to\infty} \mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) &= 1 - \epsilon, \ \lim_{k\to\infty} \mathcal{F}(\mu_{m_k+1}, \mu_{n_k+1}, \upsilon_0) = 1 - \epsilon, \\ \lim_{k\to\infty} \mathcal{F}(\mu_{m_k}, \mu_{n_k+1}, \upsilon_0) &\geq 1 - \epsilon \text{ and } \lim_{k\to\infty} \mathcal{F}(\mu_{n_k}, \mu_{m_k+1}, \upsilon_0) \geq 1 - \epsilon. \end{aligned}$$

From (2.13) we get

$$\mathcal{F}(\mu_{m_k+1}, T\mu_{m_k}, \upsilon_0) = \mathcal{F}(C, D, \upsilon_0) \text{ and } \mathcal{F}(\mu_{n_k+1}, T\mu_{n_k}, \upsilon_0) = \mathcal{F}(C, D, \upsilon_0).$$

Hence, (2.12) implies

$$\begin{aligned}
&\mathcal{F}(\mu_{m_k+1}, \mu_{n_k+1}, \upsilon_0) \geq \mathcal{B}(\mu_{m_k}, \mu_{n_k}, \upsilon_0) + \mathcal{A}_\rho(\mu_{m_k}, \mu_{n_k}, \upsilon_0) \\
&\geq \min\{\mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0), \mathcal{F}(\mu_{n_k}, \mu_{m_{k+1}}, \upsilon_0)\} + \\
&\min\left\{\begin{array}{l} \rho(\mathcal{F}(\mu_{m_k}, \mu_{n_k}, \upsilon_0)), \rho(\mathcal{F}(\mu_{m_k}, \mu_{m_k+1}, \upsilon_0)), \\ \rho(\mathcal{F}(\mu_{n_k}, \mu_{m_{k+1}}, \upsilon_0)), \rho(\mathcal{F}(\mu_{n_k}, \mu_{n_k+1}, \upsilon_0)) \end{array}\right\}.
\end{aligned}$$

As k tends to ∞ in above, we get

$$1 - \epsilon \geq (1 - \epsilon) + \min\{\rho(1 - \epsilon), \rho(1)\}.$$

That is

$$1 - \epsilon \geq 1 - \epsilon + \min\{\rho(1 - \epsilon), \rho(1)\}.$$

Consequently,

$$0 \geq \min\{\rho(1 - \epsilon), \rho(1)\}.$$

Hence, either $\rho(1 - \epsilon) = 0$ or $\rho(1) = 0$, a contradiction in both cases. This implies that $\{\mu_n\}$ is a Cauchy sequence. The completeness of $(Z, \mathcal{F}, \diamond)$ implies $\{\mu_n\}$ converges to some μ^* in Z. That is,

$$\lim_{n\to\infty} \mathcal{F}(\mu_n, \mu^*, \upsilon) = 1 \text{ for all } \upsilon > 0. \tag{2.16}$$

Now, we show that T has a BPP. On the similar lines as in Theorem 2.1, we get $\mu^* \in C_0(\upsilon)$. As $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ for all $\upsilon > 0$ ensures that there is $\xi \in C_0(\upsilon)$ such that

$$\mathcal{F}(\xi, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon). \tag{2.17}$$

We claim that $\xi = \mu^*$. On the contrary, assume that $\xi \neq \mu^*$. So from (2.13) and (2.17) we get

$$\mathcal{F}(\mu_{n+1}, \xi, \upsilon) \geq \mathcal{B}(\mu_n, \mu^*, \upsilon) + \mathcal{A}_\rho(\mu_n, \mu^*, \upsilon)$$

which implies

$$\begin{aligned}
\mathcal{F}(\mu_{n+1}, \xi, \upsilon) \geq\; & \min\{\mathcal{F}(\mu_n, \mu^*, \upsilon), \mathcal{F}(\mu^*, \mu_{n+1}, \upsilon)\} + \\
& \min\left\{\begin{array}{l} \rho(\mathcal{F}(\mu_n, \mu^*, \upsilon)), \rho(\mathcal{F}(\mu_n, \mu_{n+1}, \upsilon)), \\ \rho(\mathcal{F}(\mu^*, \mu_{n+1}, \upsilon)), \rho(\mathcal{F}(\mu^*, \xi, \upsilon)) \end{array}\right\}.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, we get

$$\mathcal{F}(\mu^*, \xi, \upsilon) \geq 1 + \min\{\rho(1), \rho(\mathcal{F}(\mu^*, \xi, \upsilon))\},$$

so $1 \geq \mathcal{F}(\mu^*, \xi, \upsilon) \geq 1$, which implies $\mathcal{F}(\mu^*, \xi, \upsilon) = 1$, for all $\upsilon > 0$, that is $\mu^* = \xi$ and $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$. To show the uniqueness of μ^* which is the BPP of T, let r be another BPP of T such that $r \neq \mu^*$, that is, $0 < \mathcal{F}(\mu^*, r, \upsilon) < 1$ for all $\upsilon > 0$. As

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon) \text{ and } \mathcal{F}(r, Tr, \upsilon) = \mathcal{F}(C, D, \upsilon),$$

so from (2.12), we have

$$\begin{aligned}
&\mathcal{F}(\mu^*, r, \upsilon) \geq \mathcal{B}(\mu^*, r, \upsilon) + \mathcal{A}_\rho(\mu^*, r, \upsilon)\\
&\geq \min\{\mathcal{F}(\mu^*, r, \upsilon), \mathcal{F}(r, \mu^*, \upsilon)\}\\
&+\min\left\{\begin{array}{l}\rho(\mathcal{F}(\mu^*, r, \upsilon)), \rho(\mathcal{F}(\mu^*, \mu^*, \upsilon)),\\ \rho(\mathcal{F}(r, \mu^*, \upsilon)), \rho(\mathcal{F}(r, r, \upsilon))\end{array}\right\}\\
&= \mathcal{F}(\mu^*, r, \upsilon) + \rho(1).
\end{aligned}$$

Hence,

$$\mathcal{F}(\mu^*, r, \upsilon) \geq \mathcal{F}(\mu^*, r, \upsilon) + \min\{\rho(\mathcal{F}(\mu^*, r, \upsilon)), \rho(1)\},$$

which implies $\rho(\mathcal{F}(\mu^*, r, \upsilon)) = 0$ or $\rho(1) = 0$, which is a contradiction in both cases as $\rho(s) > 0$ for all $s \in (0, 1]$. Therefore, $\mathcal{F}(\mu^*, r, \upsilon) = 1$ for every $\upsilon > 0$ and so $\mu^* = r$.

In the next Theorem, we use another contraction condition involving a function $\zeta : I \to [1, \infty)$.

Theorem 2.4. *Let C and D be nonempty closed subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is nonempty for all $\upsilon > 0$ and $T : C \to D$ a nonself-mapping satisfying the following:*

(i) *$T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ for all $\upsilon > 0$,*

(ii) *There is a function $\zeta : I \to [1, \infty)$ such that for any sequence $\{s_n\} \subseteq I$ of positive real numbers, $\zeta(s_n) \to 1$ as $n \to +\infty$ implies $s_n \to 1$ as $n \to +\infty$ and*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \mathcal{F}(\alpha, \beta, \upsilon) \geq \zeta(\mathcal{F}(\mu, \lambda, \upsilon))\mathcal{B}(\mu, \lambda, \upsilon), \tag{2.18}$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ and $\upsilon > 0$, where

$$\mathcal{B}(\mu, \lambda, \upsilon) = \min\{\mathcal{F}(\mu, \lambda, \upsilon), \mathcal{F}(\lambda, \alpha, \upsilon)\},$$

(iii) *For any sequence $\{\lambda_n\}$ in $D_0(\upsilon)$ and $\mu \in C$ satisfying $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ as $n \to \infty$, one has $\mu \in C_0(\upsilon)$.*

Then there is a unique $\mu^* \in C$ such that

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$$

for every $\upsilon > 0$.

Proof. As $C_0(\upsilon)$ is nonempty for every $\upsilon > 0$, so we pick a μ_0 in $C_0(\upsilon)$. Since $T\mu_0 \in T(C_0(\upsilon)) \subseteq D_0(\upsilon)$, we can find $\mu_1 \in C_0(\upsilon)$ such that

$$\mathcal{F}(\mu_1, T\mu_0, \upsilon) = \mathcal{F}(C, D, \upsilon).$$

Recursively, we obtain a sequence $\{\mu_n\}$ in $C_0(\upsilon)$ satisfying

$$\begin{aligned}
\mathcal{F}(\mu_n, T\mu_{n-1}, \upsilon) &= \mathcal{F}(C, D, \upsilon),\\
\mathcal{F}(\mu_{n+1}, T\mu_n, \upsilon) &= \mathcal{F}(C, D, \upsilon).
\end{aligned} \tag{2.19}$$

Using (2.18) and (2.19) we get

$$\begin{aligned}\mathcal{F}(\mu_n,\mu_{n+1},\upsilon) &\geq \zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon))\mathcal{B}(\mu_{n-1},\mu_n,\upsilon)\\ &= \zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon))\min\{\mathcal{F}(\mu_{n-1},\mu_n,\upsilon),\mathcal{F}(\mu_n,\mu_n,\upsilon)\}\\ &= \zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon))\mathcal{F}(\mu_{n-1},\mu_n,\upsilon),\end{aligned} \tag{2.20}$$

which implies

$$\mathcal{F}(\mu_n,\mu_{n+1},\upsilon) \geq \mathcal{F}(\mu_{n-1},\mu_n,\upsilon).$$

Hence, $\{\mathcal{F}(\mu_{n+1},T\mu_n,\upsilon)\}$ is an increasing sequence in $(0,1]$, which is bounded above by 1. This implies that there is $j(\upsilon)\in(0,1]$ such that

$$\lim_{n\to\infty}\mathcal{F}(\mu_n,\mu_{n+1},\upsilon) = j(\upsilon)$$

for all $\upsilon>0$. We claim that $j(\upsilon)=1$ for all $\upsilon>0$. On the contrary, assume that there is $\upsilon_0>0$ such that $0<j(\upsilon_0)<1$. Taking the limit as $n\to\infty$, in (2.20) we get

$$\lim_{n\to\infty}\mathcal{F}(\mu_n,\mu_{n+1},\upsilon_0) \geq \lim_{n\to\infty}\zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0))\lim_{n\to\infty}\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0),$$

which implies

$$1=\frac{\lim_{n\to\infty}\mathcal{F}(\mu_n,\mu_{n+1},\upsilon_0)}{\lim_{n\to\infty}\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0)} \geq \lim_{n\to\infty}\zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0)\geq 1.$$

That is,

$$\lim_{n\to\infty}\zeta(\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0)=1 \text{ implies } \lim_{n\to\infty}\mathcal{F}(\mu_{n-1},\mu_n,\upsilon_0)=1.$$

Hence, $j(\upsilon)=1$. Now, we prove that $\{\mu_n\}$ is a Cauchy sequence. Suppose on contrary $\{\mu_n\}$ is not a Cauchy sequence, that is, there is an $\epsilon\in(0,1)$ and $\upsilon_0>0$ such that for every $k\in\mathbb{N}$, there are $n_k,m_k\in\mathbb{N}$ with $m_k>n_k\geq k$ and

$$\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0)\leq 1-\epsilon. \tag{2.21}$$

Let m_k be the least integer greater than n_k satisfying (2.21), that is,

$$\mathcal{F}(\mu_{m_k-1},\mu_{n_k},\upsilon_0) > 1-\epsilon.$$

On similar lines as in the proof of Theorem 2.1, we get

$$\begin{aligned}\lim_{k\to\infty}\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0) &= 1-\epsilon,\ \lim_{k\to\infty}\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0)=1-\epsilon,\\ \lim_{k\to\infty}\mathcal{F}(\mu_{m_k},\mu_{n_k+1},\upsilon_0) &\geq 1-\epsilon \text{ and } \lim_{k\to\infty}\mathcal{F}(\mu_{n_k},\mu_{m_k+1},\upsilon_0)\geq 1-\epsilon.\end{aligned}$$

From (2.19), we get

$$\mathcal{F}(\mu_{m_k+1},T\mu_{m_k},\upsilon_0)=\mathcal{F}(C,D,\upsilon_0) \text{ and } \mathcal{F}(\mu_{n_k+1},T\mu_{n_k},\upsilon_0)=\mathcal{F}(C,D,\upsilon_0).$$

So, by applying (2.18), we get

$$\begin{aligned}\mathcal{F}(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0) &\geq \zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\mathcal{B}(\mu_{m_k},\mu_{n_k},\upsilon_0)\\ &= \zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\min\{\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0),\mathcal{F}(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0)\}.\end{aligned} \tag{2.22}$$

If

$$\min\left\{\mathcal{F}\left(\mu_{m_k},\mu_{n_k},\upsilon_0\right),\mathcal{F}\left(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0\right)\right\}=\mathcal{F}\left(\mu_{m_k},\mu_{n_k},\upsilon_0\right),$$

then from (2.22),

$$\mathcal{F}\left(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0\right)\geq\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\mathcal{F}\left(\mu_{m_k},\mu_{n_k},\upsilon_0\right),$$

which implies

$$\frac{\mathcal{F}\left(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0\right)}{\mathcal{F}\left(\mu_{m_k},\mu_{n_k},\upsilon_0\right)}\geq\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\geq 1$$

and taking limit as $k\to\infty$, above inequality gives

$$\lim_{k\to\infty}\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))=1,$$

which implies

$$\lim_{k\to\infty}\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0)=1.$$

So, $\epsilon=0$, a contradiction. If

$$\min\left\{\mathcal{F}\left(\mu_{m_k},\mu_{n_k},\upsilon_0\right),\mathcal{F}\left(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0\right)\right\}=\mathcal{F}\left(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0\right),$$

then from (2.22),

$$\mathcal{F}\left(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0\right)\geq\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\mathcal{F}\left(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0\right),$$

which implies

$$1=\frac{1-\epsilon}{1-\epsilon}\geq\frac{\mathcal{F}\left(\mu_{m_k+1},\mu_{n_k+1},\upsilon_0\right)}{\mathcal{F}\left(\mu_{n_k},\mu_{m_{k+1}},\upsilon_0\right)}\geq\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))\geq 1, \tag{2.23}$$

which implies

$$\lim_{k\to\infty}\zeta(\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))=1 \text{ and}$$
$$\lim_{k\to\infty}\mathcal{F}(\mu_{m_k},\mu_{n_k},\upsilon_0))=1.$$

So, $\epsilon=0$, a contradiction again. Thus, $\{\mu_n\}$ is a Cauchy sequence. As $(Z,\mathcal{F},\diamond)$ is a complete N-AFMS, therefore the sequence $\{\mu_n\}$ converges to some $\mu^*\in Z$, that is, $\lim_{n\to\infty}\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)=1$ for all $\upsilon>0$. Since $T\left(C_0\left(\upsilon\right)\right)\subseteq D_0\left(\upsilon\right)$ for all $\upsilon>0$, so there is $\xi\in C_0\left(\upsilon\right)$ such that

$$\mathcal{F}\left(\xi,T\mu^*,\upsilon\right)=\mathcal{F}\left(C,D,\upsilon\right).$$

So, by (2.18) it is evident that

$$\begin{aligned}\mathcal{F}\left(\mu_{n+1},\xi,\upsilon\right)&\geq\zeta\left(\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)\right)\mathcal{B}\left(\mu_n,\mu^*,\upsilon\right)\\&=\zeta\left(\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)\right)\min\left\{\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right),\mathcal{F}\left(\mu^*,\mu_{n+1},\upsilon\right)\right\}.\end{aligned} \tag{2.24}$$

If

$$\min\left\{\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right),\mathcal{F}\left(\mu^*,\mu_{n+1},\upsilon\right)\right\}=\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right),$$

then from (2.24),

$$\mathcal{F}\left(\mu_{n+1},\xi,\upsilon\right)\geq\zeta\left(\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)\right)\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)\geq\mathcal{F}\left(\mu_n,\mu^*,\upsilon\right)$$

and applying the limit to the above inequality as $n \to \infty$, we have $\mathcal{F}(\mu^*, \xi, \upsilon) = 1$ for all $\upsilon > 0$, that is, $\mu^* = \xi$ and $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$. If

$$\min\{\mathcal{F}(\mu_n, \mu^*, \upsilon), \mathcal{F}(\mu^*, \mu_{n+1}, \upsilon)\} = \mathcal{F}(\mu^*, \mu_{n+1}, \upsilon),$$

then from (2.24),

$$\mathcal{F}(\mu_{n+1}, \xi, \upsilon) \geq \zeta(\mathcal{F}(\mu_n, \mu^*, \upsilon))\mathcal{F}(\mu^*, \mu_{n+1}, \upsilon) \geq \mathcal{F}(\mu^*, \mu_{n+1}, \upsilon)$$

and applying the limit to the above inequality as $n \to \infty$ we have $\mathcal{F}(\mu^*, \xi, \upsilon) = 1$ for all $\upsilon > 0$, that is, $\mu^* = \xi$ and $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$. Now, we show that μ^* is the only BPP of T. If r is the another BPP of T, then

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon) \text{ and } \mathcal{F}(r, Tr, \upsilon) = \mathcal{F}(C, D, \upsilon).$$

From (2.18) we get

$$1 = \frac{\mathcal{F}(\mu^*, r, \upsilon_0)}{\mathcal{F}(\mu^*, r, \upsilon_0)} \geq \zeta(\mathcal{F}(\mu^*, r, \upsilon_0)) \geq 1,$$

which implies that $\mathcal{F}(\mu^*, r, \upsilon_0) = 1$. Hence, $\mu^* = r$.

Now, we present an example to illustrate Theorem 2.1.

Example 2.5. *Let* $Z = \{1, 2, 3, \cdots, 10\}$, $C = \{1, 3, 5, 7\}$*,* $D = \{2, 4, 6, 8\}$ *and* $\mathcal{F} : Z \times Z \times (0, \infty) \to (0, 1]$ *be a N-AFMS which is defined by*

$$\mathcal{F}(\mu, \lambda, \upsilon) = \begin{cases} \frac{\mu}{\lambda}, & \text{if } \mu \leq \lambda \\ \frac{\lambda}{\mu}, & \text{if } \lambda < \mu \end{cases}$$

for all $\upsilon > 0$*. Note that* $(Z, \mathcal{F}, \diamond)$ *is complete with* $\mu \diamond \lambda = \mu\lambda$*,* $\mathcal{F}(C, D, \upsilon) = \frac{7}{8}$ *and* C *and* D *are nonempty closed subsets of* Z*. Define* $T : C \to D$ *as*

$$T(x) = \begin{cases} 8, & \text{if } x = 7 \\ x + 7 & \text{otherwise.} \end{cases}$$

Since

$$\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon) = \frac{7}{8}$$

implies $(\alpha, \mu) = (7, 7)$ *or* $(\alpha, \mu) = (7, 1)$*, therefore*

$$\begin{aligned} \mathcal{F}(7, T7, \upsilon) &= \mathcal{F}(7, 8, \upsilon) = \frac{7}{8} = \mathcal{F}(C, D, \upsilon) \text{ and} \\ \mathcal{F}(7, T1, \upsilon) &= \mathcal{F}(7, 8, \upsilon) = \frac{7}{8} = \mathcal{F}(C, D, \upsilon) \end{aligned}$$

for all $\upsilon > 0$*. Also, note that*

$$C_0(\upsilon) = \{7\}, \ D_0(\upsilon) = \{8\} \text{ and}$$
$$T(C_0(\upsilon)) = \{7\} \subseteq D_0(\upsilon) = \{8\}.$$

Now, consider the function $\varphi \in \Phi$ defined by

$$\varphi(r) = 1 - r \text{ for all } r \in [0, 1].$$

From (2.1), we have

$$\mathcal{F}(\alpha, \beta, \upsilon) = \mathcal{F}(7, 7, \upsilon) = 1 \text{ which implies } \varphi(\mathcal{F}(\alpha, \beta, \upsilon)) = 1 - 1 = 0$$

which shows that

$$\varphi(\mathcal{F}(\alpha, \beta, \upsilon)) \le \omega(\upsilon)\, \mathcal{A}_{\varphi}(\mu, \lambda, \upsilon)$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ and for every $\upsilon > 0$ and $\omega(\upsilon) \in (0, 1)$. For any sequence $\{\lambda_n\}$ in $D_0(\upsilon)$ and μ in C, $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(A, B, \upsilon)$ as $n \to +\infty$, we have $\lambda_n = 8$ for all n and $\mu = 7 \in C_0(\upsilon)$. Thus, all the conditions of Theorem 2.1 are satisfied, and so there exists a unique $\mu^* \in C$ such that $\mathcal{F}(\mu^*, T\mu^*, t) = \mathcal{F}(C, D, \upsilon)$ for all $\upsilon > 0$. Here, $\mu^* = 7$.

Remark 2.6. *Theorems 2.1, 2.2 and 2.4 are generalizations of [33, Theorems 1, 2 and 4]. Theorem 2.3 is a partial generalization of [33, Theorem 3] as we considered $\rho(s) > 0$ for every $s \in (0, 1]$ instead of for every $s \in (0, 1)$ in order to use more general contraction condition.*

Now, we give some important corollaries of the main results.

Corollary 2.7. *[33, Theorem 1] Let C and D be nonempty closed subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is nonempty for every $\upsilon > 0$ and $T : C \to D$ a nonself-mapping that satisfies:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ for all $\upsilon > 0$,
(ii) There is a function φ in Φ for which

$$\left.\begin{array}{l} \mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon) \\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon) \end{array}\right\} \text{ implies } \varphi(\mathcal{F}(\alpha, \beta, \upsilon)) \le \omega(\upsilon)\varphi(\mathcal{F}(\mu, \lambda, \upsilon))$$

holds for all $\alpha, \beta, \lambda, \mu \in C$, and $\upsilon > 0$, where $\omega : (0, \infty) \to (0, 1)$ is a function and
(iii) For any sequence $\{\lambda_n\}$ in $D_0(\upsilon)$ and $\mu \in C$ satisfying $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ as $n \to \infty$, one has $\mu \in C_0(\upsilon)$.

Then there exists a unique $\mu^* \in C$ such that

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$$

for every $\upsilon > 0$.

Proof. Consider $\mathcal{A}_{\varphi}(\mu, \lambda, \upsilon) = \varphi(\mathcal{F}(\mu, \lambda, \upsilon))$ in Theorem 2.1.

Corollary 2.8. *[33, Theorem 2] Let C and D be nonempty subsets of a complete N-AFMS $(Z, \mathcal{F}, \diamond)$. Suppose that $C_0(\upsilon)$ is a closed subset of $(Z, \mathcal{F}, \diamond)$ for every $\upsilon > 0$ and $T : C \to D$ a nonself-mapping satisfying the following:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ for all $\upsilon > 0$,

(ii) *There exists* $\varphi \in \Phi$ *for which*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \varphi(\mathcal{F}(\alpha, \beta, \upsilon)) \leq \omega(\upsilon)\varphi(\mathcal{F}(\mu, \lambda, \upsilon))$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ *and* $\upsilon > 0$, *where* $\omega : (0, \infty) \to (0, 1)$ *a function.*

Then there is a unique $\mu^* \in C$ *for which* $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$ *for all* $\upsilon > 0$.

Proof. Consider $\mathcal{A}_\varphi(\mu, \lambda, \upsilon) = \varphi(\mathcal{F}(\mu, \lambda, \upsilon))$ in Theorem 2.2.

Corollary 2.9. *[33, Theorem 3] Let C and D be nonempty closed subsets of a complete N-AFMS* $(Z, \mathcal{F}, \diamond)$. *Suppose that* $C_0(\upsilon)$ *is a nonempty for every* $\upsilon > 0$ *and* $T : C \to D$ *a nonself-mapping satisfying the following:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ *for all* $\upsilon > 0$,
(ii) *There is a continuous function* $\rho : I \to I$, *with* $\rho(s) > 0$ *for every* $s \in (0, 1]$, *for which*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \mathcal{F}(\alpha, \beta, \upsilon) \geq \mathcal{F}(\mu, \lambda, \upsilon) + \rho(\mathcal{F}(\mu, \lambda, \upsilon))$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ *and* $\upsilon > 0$,
(iii) *For any sequence* $\{\lambda_n\}$ *in* $D_0(\upsilon)$ *and* $\mu \in C$ *satisfying* $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ *as* $n \to \infty$, *one has* $\mu \in C_0(\upsilon)$.

Then there is a unique $\mu^* \in C$ *for which* $\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$ *for all* $\upsilon > 0$.

Proof. Consider $\mathcal{A}_\rho(\mu, \lambda, \upsilon) = \rho(\mathcal{F}(\mu, \lambda, \upsilon))$ and $\mathcal{B}(\mu, \lambda, \upsilon) = \mathcal{F}(\mu, \lambda, \upsilon)$ in Theorem 2.3.

Corollary 2.10. *[33, Theorem 4] Let C and D be nonempty closed subsets of a complete N-AFMS* $(Z, \mathcal{F}, \diamond)$. *Suppose that* $C_0(\upsilon)$ *is a nonempty for all* $\upsilon > 0$ *and* $T : C \to D$ *a nonself-mapping satisfying the following:*

(i) $T(C_0(\upsilon)) \subseteq D_0(\upsilon)$ *for all* $\upsilon > 0$,
(ii) *There is a function* $\zeta : I \to [1, \infty)$ *such that for any sequence* $\{s_n\} \subseteq I$ *of positive real numbers,* $\zeta(s_n) \to 1$ *as* $n \to +\infty$ *implies* $s_n \to 1$ *as* $n \to +\infty$ *and*

$$\left.\begin{array}{l}\mathcal{F}(\alpha, T\mu, \upsilon) = \mathcal{F}(C, D, \upsilon)\\ \mathcal{F}(\beta, T\lambda, \upsilon) = \mathcal{F}(C, D, \upsilon)\end{array}\right\} \text{ implies } \mathcal{F}(\alpha, \beta, \upsilon) \geq \zeta(\mathcal{F}(\mu, \lambda, \upsilon))\mathcal{F}(\mu, \lambda, \upsilon)$$

holds for all $\alpha, \beta, \lambda, \mu \in C$ *and* $\upsilon > 0$,
(iii) *For any sequence* $\{\lambda_n\}$ *in* $D_0(\upsilon)$ *and* $\mu \in C$ *satisfying* $\mathcal{F}(\mu, \lambda_n, \upsilon) \to \mathcal{F}(C, D, \upsilon)$ *as* $n \to \infty$, *one has* $\mu \in C_0(\upsilon)$.

Then there is a unique $\mu^* \in C$ *such that*

$$\mathcal{F}(\mu^*, T\mu^*, \upsilon) = \mathcal{F}(C, D, \upsilon)$$

for every $\upsilon > 0$.

Proof. Consider $\mathcal{B}(\mu, \lambda, \upsilon) = \mathcal{F}(\mu, \lambda, \upsilon)$ in Theorem 2.4.

The following corollaries are the fixed point version of Theorem 2.4 and will be used in the sequel.

Corollary 2.11. *Let $(Z, \mathcal{F}, \diamond)$ be a complete N-AFMS and $T : Z \to Z$ a self-mapping satisfying*

$$\mathcal{F}(T\mu, T\lambda, \upsilon) \geq \zeta(\mathcal{F}(\mu, \lambda, \upsilon))\mathcal{B}(\mu, \lambda, \upsilon)$$

for all $\mu, \lambda \in Z$ and all $\upsilon > 0$, where

$$\mathcal{B}(\mu, \lambda, \upsilon) = \min\{\mathcal{F}(\mu, \lambda, \upsilon), \mathcal{F}(\lambda, T\mu, \upsilon)\}$$

and $\zeta : I \to [1, \infty)$ a function such that for any sequence $\{s_n\} \subseteq I$ of positive real numbers, $\zeta(s_n) \to 1$ as $n \to +\infty$ implies $s_n \to 1$ as $n \to +\infty$. Then there is a unique $\lambda^* \in C$ such that $\lambda^* = T\lambda^*$.

Proof. Put $C = D = Z$ in Theorem 2.4.

If $\mathcal{B}(\mu, \lambda, \upsilon) = \mathcal{F}(\mu, \lambda, \upsilon)$ in the corollary 2.11, then we get the following result.

Corollary 2.12. *Let $(Z, \mathcal{F}, \diamond)$ be a complete N-AFMS and $T : Z \to Z$ a self-mapping satisfying*

$$\mathcal{F}(T\mu, T\lambda, \upsilon) \geq \zeta(\mathcal{F}(\mu, \lambda, \upsilon))\mathcal{F}(\mu, \lambda, \upsilon)$$

for all $\mu, \lambda \in Z$ and all $\upsilon > 0$, where $\zeta : I \to [1, \infty)$ is a function such that for any sequence $\{s_n\} \subseteq I$ of positive real numbers, $\zeta(s_n) \to 1$ as $n \to +\infty$ implies $s_n \to 1$ as $n \to +\infty$. Then there is a unique $\lambda^* \in C$ such that $\lambda^* = T\lambda^*$.

3. Application in the domain of words

Let a nonempty set of alphabets be denoted by Σ and the set of all finite and infinite words over Σ denoted by Σ_∞. Note that Σ_∞ contains the empty sequence (word) which is denoted by ϕ. Let the prefix order on Σ_∞ be denoted by $\circledcirc$ and defined as

$$a \circledcirc b \text{ if and only if } a \text{ is prefix of } b.$$

For every nonempty (word) $a \in \Sigma_\infty$, the length of a is $\Omega(a) \in [1, \infty]$ and $\Omega(\phi) = 0$. Further, if $a \in \Sigma_\infty$ is finite, then $n < \infty$ and we write

$$a = a_1 a_2, ..., a_n,$$

otherwise we write

$$a = a_1 a_2, ...$$

Now, for $a, b \in \Sigma_\infty$, then the common prefix of a and b is represented by $a * b$. It is to be noted that $a = b$ if and only if $a \circledcirc b$ and $b \circledcirc a$ and $\Omega(a) = \Omega(b)$. Define $\mathcal{S}_\circledcirc : \Sigma_\infty \times \Sigma_\infty \to [0, \infty)$ by

$$\mathcal{S}_\circledcirc(a, b) = \begin{cases} 0, & \text{iff } a = b \\ 2^{-\Omega(a)}, & \text{iff } a \circledcirc b \\ 2^{-\Omega(b)}, & \text{iff } b \circledcirc a \\ 2^{-\Omega(a*b)}, & \text{otherwise.} \end{cases}$$

If $a \circledcirc b$, then $a * b = a$ and if $b \circledcirc a$, then $b * a = b$. Therefore, for all $a, b \in \Sigma_\infty$, we can write

$$\mathcal{S}_{\circledcirc}(a,b) = \begin{cases} 0, & \text{iff } a = b \\ 2^{-\Omega(a*b)}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{S}_{\circledcirc}$ is a Baire metric [30] which is a complete metric on Σ_∞. Assign a fuzzy metric on Σ_∞ by

$$\mathcal{F}_{\mathcal{S}_{\circledcirc}}(a,b,\upsilon) = e^{-\frac{\mathcal{S}_{\circledcirc}(a,b)}{\upsilon}}.$$

Then $(\Sigma_\infty, \mathcal{F}, \diamond)$ represents a complete N-AFMS, where the t-norm is $a \diamond b = ab$. The Quicksort algorithm gives the recurrence relation

$$\begin{aligned} a_1 &= 0, \text{ for } m = 1, \\ a_m &= \frac{2(m-1)}{m} + \frac{m+1}{m} a_{m-1}, \text{ for } m \geq 2. \end{aligned}$$

For more on Quicksort algorithm and its applications, we refer the reader to [9, 16]. For $\Sigma = [0, \infty)$, in correspondence to the above sequence, we define the functional $\eta : \Sigma_\infty \to \Sigma_\infty$ that assigns

$$\eta(a) := \eta((a))_1 \eta((a))_2, ...$$

to $a := a_1 a_2, ...$ and is defined by

$$\begin{cases} \eta((a))_1 = 0, \text{ for } m = 1, \\ \eta((a))_m = \dfrac{2(m-1)}{m} + \dfrac{m+1}{m} a_{m-1}, \text{ for } m \geq 2. \end{cases}$$

Note that

$$\Omega(\eta((a))) = \Omega(a) + 1$$

for all $a \in \Sigma_\infty$ and in particular

$$\Omega(\eta((a))) = \infty,$$

whenever $\Omega(a) = \infty$. By definition of η, we have

$$a \circledcirc b \Leftrightarrow \eta(a) \circledcirc \eta(b)$$

and this implies that

$$\eta(a * b) \circledcirc \eta(a) * \eta(b)$$

for all $a, b \in \Sigma_\infty$. Hence,

$$\Omega(\eta(a * b)) \leq \Omega(\eta(a) * \eta(b))$$

for all $a, b \in \Sigma_\infty$. We apply Corollary 2.12 and prove that the functional η has a fixed point. Let $\zeta : I \to [1, \infty)$ be defined as $\zeta(t) = 1$ for all $t \in I$. Then there are two cases:

Case 1: If $a = b$, then

$$\mathcal{F}_{\mathcal{S}_{\circledcirc}}(\eta(a), \eta(a), \upsilon) = 1 = \mathcal{F}(a, a, \upsilon).$$

Case 2: If $a \neq b$, then for all $\upsilon > 0$, we have

$$\Omega\left(\eta\left(a\right) * \eta\left(b\right)\right) \geq \Omega\left(\eta\left(a * b\right)\right),$$

that is,

$$\frac{-2^{-\Omega(\eta(a)*\eta(b))}}{\upsilon} \geq \frac{-2^{-\Omega(\eta(a*b))}}{\upsilon},$$

which further implies

$$e^{\frac{-2^{-\Omega(\eta(a)*\eta(b))}}{\upsilon}} \geq e^{\frac{-2^{-\Omega(\eta(a*b))}}{\upsilon}}.$$

Now,

$$\begin{aligned}
&\mathcal{F}_{\mathcal{S}_{\circledcirc}}\left(\eta\left(a\right), \eta\left(b\right), \upsilon\right) = e^{\frac{-2^{-\Omega(\eta(a)*\eta(b))}}{\upsilon}} \\
&\geq e^{\frac{-2^{-\Omega(\eta(a*b))}}{\upsilon}} = e^{\frac{-2^{-\Omega(a*b)-1}}{\upsilon}} \\
&= e^{\frac{-2^{-\Omega(a*b)}.2^{-1}}{\upsilon}} = \left(e^{\frac{-2^{-\Omega(a*b)}}{\upsilon}}\right)^{2^{-1}} \\
&= \sqrt{e^{\frac{-2^{-\Omega(a*b)}}{\upsilon}}} \geq e^{\frac{-2^{-\Omega(a*b)}}{\upsilon}} \\
&\geq e^{\frac{-2^{-\Omega(a*b)}}{\upsilon}} = \zeta\left(\mathcal{F}(a, b, \upsilon)\right)\mathcal{F}(a, b, \upsilon).
\end{aligned}$$

Hence,

$$\mathcal{F}_{\mathcal{S}_{\circledcirc}}\left(\eta\left(a\right), \eta\left(b\right), \upsilon\right) \geq \zeta\left(\mathcal{F}(\mu, \lambda, \upsilon)\right)\mathcal{F}(\mu, \lambda, \upsilon)$$

for all $\mu, \lambda \in Z$ and all $\upsilon > 0$. Thus, all conditions of Corollary 2.12 are satisfied and η has a fixed point $\xi = \xi_1\xi_2...$, which is the solution of the recurring relation for T. Hence, we obtain

$$\begin{cases} \xi_1 = 0, \\ \xi_n = \dfrac{2(n-1)}{n} + \dfrac{n+1}{n}a_{n-1}, \text{ for } n \geq 2. \end{cases}$$

Remark 3.1. *The prefix order $\circledcirc$ on Σ_∞ defined as above is a partial order on Σ_∞ (domain of words) which is associated with the graph via the relation*

$$a \circledcirc b \text{ if and only if } (a, b) \in E(G),$$

where $E(G)$ is the set of edges of G and the graph $G = (V(G), E(G))$ with $V(G) = \Sigma_\infty$. Domain of words problem can be considered in connection with the graphs as well to solve some problems related to networks.

4. Conclusions

In this paper, we proved the existence of best proximity points for various different proximal quasi-contractive nonself-mappings of non-Archimedean fuzzy metric spaces. Moreover, we were able to present an example to illustrate the main result and an application in computer science, particularly in the domain of words as well. As fuzzy quasi metric spaces are linked in a very natural way with applications in computer sciences (see [30]), so the results in this paper can be investigated in connection with fuzzy quasi metric spaces with some applications.

Conflict of interest

The authors declare that they do not have any conflict of interests regarding this paper.

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