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Traveling wave solutions to a cubic predator-prey diffusion model with stage structure for the prey

Yujuan Jiao*, **Jinmiao Yang and Hang Zhang**

College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou, 730030, China

* **Correspondence:** Email: jsjyj@xbmu.edu.cn.

Abstract: In this paper, we investigate the traveling wave solutions to a cubic predator-prey diffusion model with stage structure for the prey. Firstly, using the upper and lower solutions method we prove the existence and non-existence of weak traveling wave solutions. Furthermore, we prove that the weak traveling wave solutions are actually traveling wave solutions under additional conditions by using Lyapunov function method and LaSalle's invariance principle.

Keywords: predator-prey model; traveling wave solution; upper and lower solution; Lyapunov function; LaSalle's invariance principle
Mathematics Subject Classification: 34C37, 35C07

1. Introduction

In [2], Cao and Fu considered the following cubic predator-prey diffusion model with stage structure for the prey:

$$\begin{cases} \frac{\partial x_1}{\partial t} = d_1\Delta x_1 + \eta_1 x_2 - r_1 x_1 - \eta_2 x_1 + b_1 x_1^2 - b_2 x_1^3 - b_3 x_1 x_3, \\ \frac{\partial x_2}{\partial t} = d_2\Delta x_2 + \eta_2 x_1 - r_2 x_2, \\ \frac{\partial x_3}{\partial t} = d_3\Delta x_3 - c x_3 + (\alpha x_1 - \beta x_3)\, x_3, \end{cases} \tag{1.1}$$

where x_1 and x_2 denote the densities of the immature and mature prey species, respectively, and x_3 is the density of the predator species. The predators live only on the immature prey species. The constants d_1, d_2, d_3, η_1, η_2, r_1, r_2, b_1, b_2, b_3, c, α, β are positive. d_1, d_2 and d_3 denote diffusion coefficients. η_1 and r_1 represent the birth rate and the mortality rate of the immature prey species, respectively. η_2 is the conversion rate of the immature prey species to the mature prey species. $b_1x_1^2 - b_2x_1^3$ is the density restriction term of the immature prey species. b_3x_1 is the predation rate of the predator to the immature prey population. r_2 and c are the net mortality rate of the mature prey population and the predator population, respectively. αx_1 is the conversion rate of the predator, and βx_3 is the density restriction term of the predator population. For more details on the backgrounds this system, see [2].

Rescaling the system (1.1) such that

$$\frac{\alpha}{r_2}x_1 \mapsto u_1,\ \frac{\alpha}{\eta_2}x_2 \mapsto u_2,\ \frac{\beta}{r_2}x_3 \mapsto v,\ r_2dt \mapsto d\tau,\ \tau \mapsto t$$

yields

$$\begin{cases} u_{1t} = d_1\Delta u_1 + a_0u_2 - a_1u_1 + a_2u_1^2 - a_3u_1^3 - eu_1v, & x \in \mathbb{R}^n, t > 0, \\ u_{2t} = d_2\Delta u_2 + u_1 - u_2, & x \in \mathbb{R}^n, t > 0, \\ v_t = d_3\Delta v + (-b + u_1 - v)v, & x \in \mathbb{R}^n, t > 0, \end{cases} \tag{1.2}$$

where $a_0 = \eta_1\eta_2/(r_2^2)$, $a_1 = (r_1 + \eta_2)/r_2$, $a_2 = b_1/\alpha$, $a_3 = b_2/r_2$, $e = b_3/\beta$, $b = c/r_2$ are positive constants.

If $a_0 > a_1$, $a_2 > e + 2a_3b$, then the system (1.2) has a semi-trivial equilibrium $(K, K, 0)$ and the unique positive constant equilibrium (u_1^*, u_2^*, v^*), where

$$K = \frac{a_2 + \sqrt{a_2^2 + 4a_3(a_0 - a_1)}}{2a_3},$$
$$u_1^* = u_2^* = \frac{(a_2 - e) + \sqrt{(a_2 - e)^2 + 4a_3(eb + a_0 - a_1)}}{2a_3},$$
$$v^* = u_1^* - b.$$

Cao and Fu have obtained the following main conclusions: (1) The asymptotical stability of equilibrium points of the system (1.2) without diffusion; (2) the global existence of solutions and the stability of equilibrium points of system (1.2); (3) the existence of nonnegative classical global solutions and the global asymptotic stability of a unique positive equilibrium point of system (1.2) with cross-diffusion.

A traveling wave solution of the system (1.2) is a special solution $(u_1(x, t), u_2(x, t), v(x, t))$ taking the form

$$u_i(x, t) = u_i(x \cdot \nu + ct)(i = 1, 2), \quad v(x, t) = v(x \cdot \nu + ct),$$

where $\nu \in \mathbb{R}^n$ is a unit vector denoting the direction of wave propagation, $x \cdot \nu$ is the usual inner product in $\mathbb{R}^n$, $c > 0$ is the wave speed, and $(u_1(\xi), u_2(\xi), v(\xi))$ with $\xi = x \cdot \nu + ct$ satisfies the following ODE system:

$$\begin{cases} cu_1' = d_1u_1'' + a_0u_2 - a_1u_1 + a_2u_1^2 - a_3u_1^3 - eu_1v, & \xi \in \mathbb{R}, \\ cu_2' = d_2u_2'' + u_1 - u_2, & \xi \in \mathbb{R}, \\ cv' = d_3v'' + (-b + u_1 - v)\, v, & \xi \in \mathbb{R}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} 0 < u_i(\xi) \le K\ (i = 1, 2),\ \ 0 < v(\xi) \le V_0,\ \forall\ \xi \in \mathbb{R}, \\ (u_1, u_1', u_2, u_2', v, v')(-\infty) = E_0 := (K, 0, K, 0, 0, 0), \\ (u_1, u_1', u_2, u_2', v, v')(\infty) = E^* := (u_1^*, 0, u_2^*, 0, v^*, 0), \end{cases} \tag{1.4}$$

where V_0 positive constant. For convenience, we shall use the variable x to replace ξ and use i to denote the integers $1, 2$ in this paper.

In the past several decades, the existence and non-existence of traveling wave solutions for predator-prey systems have been widely studied by many researchers. Dunbar [4–6] established the existence of traveling wave solutions for a reaction-diffusion system by using Lyapunov function, shooting techniques, invariant manifold theory, etc. Hsu, Yang and Yang [9] obtained the existence of traveling wave solutions for a class of diffusive predator-prey type systems by using the Wazewski theorem, LaSalle's invariance principle and Hopf bifurcation theory. Huang, Guang and Ruan [10] considered the existence of traveling front solutions and small amplitude traveling wave train solutions for a reaction-diffusion system. Ai, Du and Peng [1] studied traveling wave solutions of the generalized Holling-Tanner predator-prey model by the squeezing method and Lyapunov function method. Wang and Fu [15] established the existence of traveling wave solutions to a diffusive generalized Holling-Tanner predator-prey model by constructing the Lyapunov function. For more results, we can see [3, 7, 8, 11–14] and references.

Based on the idea of Ai, Du and Peng [1], in this paper, we are concerned with the existence and non-existence of traveling wave solutions of the system (1.2). We obtain the existence and non-existence of weak traveling wave solutions by using the upper and lower solutions method and the Schauder fixed point theorem. Moreover, we prove that the weak traveling wave solutions are actually traveling wave solutions under additional conditions by the using the Lyapunov function method and LaSalle's invariance principle. Although the idea was used before for other predator-prey systems, the adaptation to our problem harder, and we need more detailed and complicated estimates.

This paper is organized as follows. In Section 2, employing the method of upper and lower solutions together with the Schauder fixed point theorem, we prove the existence and non-existence of weak traveling wave solutions for (1.3) with the main Theorem 2.1. In Section 3, we prove that the weak traveling wave solution obtained in Theorem 2.1 is also a traveling wave solution under certain conditions by using the Lyapunov function and LaSalle's invariance principle (Theorem 3.1).

2. Weak traveling wave solutions for the system (1.3)

In this section, we will apply the method of upper and lower solutions together with the Schauder fixed point theorem to study the existence of weak traveling wave solutions for (1.3).

Let

$$F_1(u_1, u_2, v) = a_0u_2 - a_1u_1 + a_2u_1^2 - a_3u_1^3 - eu_1v,$$
$$F_2(u_1, u_2, v) = u_1 - u_2,$$
$$G(u_1, u_2, v) = (-b + u_1 - v)v.$$

We give the following theorem.

Theorem 2.1. *Assume that* $a_0 > a_1$, $a_2 > \max\{e + 2a_3b, 2\sqrt{a_3a_1}\}$ *and* $r = K - b > 0$. *In addition, there exists a constant* $V_0 > 0$ *such that* $1/V_0 \leq 1/r < \min\{d_3/d_2, 1\}$, *and let*

$$\frac{a_2^2 + 4a_0a_3 - 4a_1a_3}{4a_3r} < \min\left\{\frac{d_3}{d_1}, 1\right\}.$$

If there exists a small constant $\delta > 0$,

$$-b + u_1 - v \geq r - 2[(K - u_1) + (K - u_2) + v] \tag{2.1}$$

for any $(u_1, u_2, v) \in [K-\delta, K]^2 \times [0, \delta]$ *holds.*

Then, for arbitrary $c \geq c^* := \sqrt{4d_3 r}$*, the system (1.3) has a solution* (u_1, u_2, v) *satisfying*

$$\begin{cases} 0 < u_i(x) \leq K, \quad 0 < v(x) \leq V_0, & \forall\, x \leq 0, \\ 0 \leq u_i(x) \leq K, \quad 0 \leq v(x) \leq V_0, & \forall\, x > 0, \\ (u_1, u_1', u_2, u_2', v, v')(-\infty) = E_0. \end{cases}$$

Moreover, for any $0 < c < \sqrt{4d_3 r}$*, the system* (1.3) *does not have a solution* $(u_1(x), u_2(x), v(x))$ *connecting* $(K, K, 0)$ *as* $x \to -\infty$ *and satisfying* $v(x) > 0$ *for sufficiently negative* x*.*

Now we give the definition of upper and lower solutions.

Definition 2.1. The continuous functions $(\underline{u}_1, \underline{u}_2, \underline{v})$ and $(\bar{u}_1, \bar{u}_2, \bar{v})$ on $\mathbb{R}$ are called a pair of lower and upper solutions of the system (1.3) if they satisfy

(i)

$$0 \leq \underline{u}_i(x) \leq \bar{u}_i(x) \leq U_{i0}, \qquad 0 \leq \underline{v}(x) \leq \bar{v}(x) \leq V_0, \quad \forall\ x \in \mathbb{R}$$

for some positive constants U_{i0} and V_0.

(ii) There exists a set $\mathbb{D}$ consisting of at most finitely many real numbers such that

(a) $\bar{u}_i,\ \underline{u}_i,\ \bar{v},\ \underline{v}$ are in $C^2(\mathbb{R} \setminus \mathbb{D})$,

(b) The right and left limits of $\underline{u}_i',\ \bar{u}_i',\ \underline{v}',\ \bar{v}'$ all exist at each $x \in \mathbb{D}$ and satisfy

$$\bar{u}_i'(x-) \geq \bar{u}_i'(x+), \quad \underline{u}_i'(x-) \leq \underline{u}_i'(x+), \quad \bar{v}'(x-) \geq \bar{v}'(x+), \quad \underline{v}'(x-) \leq \underline{v}'(x+).$$

(iii) At $\pm\infty$, the first and second derivatives of $\bar{u}_i,\ \bar{v},\ \underline{u}_i,\ \underline{v}$ have at most exponential growth.

(iv) For every pair of continuous functions (u_1, u_2, v) with $\underline{u}_i \leq u_i \leq \bar{u}_i$ and $\underline{v} \leq v \leq \bar{v}$,

$$\begin{cases} d_1\bar{u}_1''(x) - c\bar{u}_1'(x) + a_0 u_2 - a_1\bar{u}_1 + a_2\bar{u}_1^2 - a_3\bar{u}_1^3 - e\bar{u}_1 v \leq 0, \\ d_2\bar{u}_2''(x) - c\bar{u}_2'(x) + u_1 - \bar{u}_2 \leq 0, \\ d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + u_1 - \bar{v})\bar{v} \leq 0, \\ d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0 u_2 - a_1\underline{u}_1 + a_2\underline{u}_1^2 - a_3\underline{u}_1^3 - e\underline{u}_1 v \geq 0, \\ d_2\underline{u}_2''(x) - c\underline{u}_2'(x) + u_1 - \underline{u}_2 \geq 0, \\ d_3\underline{v}''(x) - c\underline{v}'(x) + (-b + u_1 - \underline{v})\underline{v} \geq 0. \end{cases} \quad \forall\ x \in \mathbb{R} \setminus \mathbb{D}.$$

Lemma 2.2. *Assume that* $(\underline{u}_1, \underline{u}_2, \underline{v})$ *and* $(\bar{u}_1, \bar{u}_2, \bar{v})$ *are a pair of lower and upper solutions of* (1.3)*. Then, there is a solution* (u_1, u_2, v) *of the system* (1.3) *satisfying*

$$\underline{u}_i(x) \leq u_i(x) \leq \bar{u}_i(x), \qquad \underline{v}(x) \leq v(x) \leq \bar{v}(x), \quad \forall\, x \in \mathbb{R},$$

and u_i'*,* u_i''*,* v' *and* v'' *are bounded on* $\mathbb{R}$*.*

Proof. Since F_i and G satisfy the Lipschitz condition on $[0, U_{10}] \times [0, U_{20}] \times [0, V_0]$, there is $\Lambda = \max\{(1 + a_1 + V_0(e+1) + 2a_2U_{10} + 3a_3U_{10}^2), b + 2V_0 + U_{10}(e+1), a_0 + 1\}$, so that for any $(u_{1i}, u_{2i}, v_i) \in [0, U_{10}] \times [0, U_{20}] \times [0, V_0]$, we have

$$\begin{aligned} &|F_1(u_{11}, u_{21}, v_1) - F_1(u_{12}, u_{22}, v_2)| + |F_2(u_{11}, u_{21}, v_1) - F_2(u_{12}, u_{22}, v_2)| \\ &+ |G(u_{11}, u_{21}, v_1) - G(u_{12}, u_{22}, v_2)| \end{aligned}$$

$$
\begin{aligned}
\leq & |u_{11}-u_{12}|[1+a_1+a_2(u_{11}+u_{12})+a_3(u_{11}^2+u_{11}u_{12}+u_{12}^2)]+|u_{21}-u_{22}|(a_0+1)\\
&+|v_1-v_2|(v_1+v_2+b)+(e+1)|u_{11}(v_1-v_2)+v_2(u_{11}-u_{12})|\\
\leq & |u_{11}-u_{12}|[1+a_1+v_2(e+1)+a_2(u_{11}+u_{12})+a_3(u_{11}^2+u_{11}u_{12}+u_{12}^2)]+|u_{21}-u_{22}|(a_0+1)\\
&+|v_1-v_2|(v_1+v_2+u_{11}(e+1)+b)\\
\leq & |u_{11}-u_{12}|(1+a_1+V_0(e+1)+2a_2U_{10}+3a_3U_{10}^2)+|u_{21}-u_{22}|(a_0+1)\\
&+|v_1-v_2|[b+2V_0+U_{10}(e+1)]\\
\leq & \Lambda\Big(|u_{11}-u_{12}|+|u_{21}-u_{22}|+|v_1-v_2|\Big). \qquad (2.2)
\end{aligned}
$$

Define

$$
\hat{F}_i(u_1,u_2,v):=F_i(u_1,u_2,v)+\Lambda u_i,
$$
$$
\hat{G}(u_1,u_2,v)=G(u_1,u_2,v)+\Lambda v.
$$

According to (2.2), we derive that $\hat{F}_1(u_1,u_2,v)$ is nondecreasing in $u_1\in[0,U_{10}]$ for each fixed $(u_2,v)\in[0,U_{20}]\times[0,V_0]$, $\hat{F}_2(u_1,u_2,v)$ is nondecreasing in $u_2\in[0,U_{20}]$ for each fixed $(u_1,v)\in[0,U_{10}]\times[0,V_0]$, $\hat{G}(u_1,u_2,v)$ nondecreasing in $v\in[0,V_0]$ for each fixed $(u_1,u_2)\in[0,U_{10}]\times[0,U_{20}]$, and the system (1.3) can be written as

$$
\begin{cases}
d_1u_1''-cu_1'-\Lambda u_1+\widehat{F}_1(u_1,u_2,v)=0, & x\in\mathbb{R}^n,\\
d_2u_2''-cu_2'-\Lambda u_2+\widehat{F}_2(u_1,u_2,v)=0, & x\in\mathbb{R}^n,\\
d_3v''-cv'-\Lambda v+\widehat{G}(u_1,u_2,v)=0, & x\in\mathbb{R}^n.
\end{cases}
$$

Now, let

$$
X=\Big\{(u_1,u_2,v)\in[C(\mathbb{R})]^3:\underline{u}_i(x)\leq u_i(x)\leq\bar{u}_i(x),\underline{v}(x)\leq v(x)\leq\bar{v}(x),\forall x\in\mathbb{R}\Big\},
$$

and define the map $T=(T_1,T_2,T_3):X\to[C(\mathbb{R})]^3$ by

$$
T_i(u_1,u_2,v)(x)=\frac{1}{d_i(\lambda_i^+-\lambda_i^-)}\left(\int_{-\infty}^{x}e^{\lambda_i^-(x-y)}+\int_x^{\infty}e^{\lambda_i^+(x-y)}\right)\hat{F}_i(u_1,u_2,v)(y)\,dy,
$$

$$
T_3(u_1,u_2,v)(x)=\frac{1}{d_3(\lambda_3^+-\lambda_3^-)}\left(\int_{-\infty}^{x}e^{\lambda_3^-(x-y)}+\int_x^{\infty}e^{\lambda_3^+(x-y)}\right)\hat{G}(u_1,u_2,v)(y)\,dy,
$$

where

$$
\lambda_i^{\pm}=\frac{1}{2d_i}\left(c\pm\sqrt{c^2+4d_i\Lambda}\right),\quad \lambda_3^{\pm}=\frac{1}{2d_3}\left(c\pm\sqrt{c^2+4d_3\Lambda}\right).
$$

By rather standard arguments similarly to those in the reference [1] that $(U_1,U_2,V)=T(u_1,u_2,v)$ for each $(u_1,u_2,v)\in X$ is the unique bounded solution of the linear equation

$$
\begin{cases}
d_1U_1''-cU_1'-\Lambda U_1+\widehat{F}_1(u_1,u_2,v)=0,\\
d_2U_2''-cU_2'-\Lambda U_2+\widehat{F}_2(u_1,u_2,v)=0,\\
d_3V''-cV'-\Lambda V+\widehat{G}(u_1,u_2,v)=0,
\end{cases}
$$

and any fixed point of T in X gives a solution of the system (1.3). Therefore, it suffices to show by the Schauder fixed point theorem that T has a fixed point in X. To do so, we define the Banach space

$$
C_\rho(\mathbb{R},\mathbb{R}^3)=\{(u_1,u_2,v)\in[C(\mathbb{R})]^3:\|(u_1,u_2,v)\|_\rho<\infty\}
$$

with the exponentially weighted norm

$$\|(u_1,u_2,v)\|_\rho = \sup_{x\in\mathbb{R}} |(u_1(x),u_2(x),v(x))|e^{-\rho|x|} := \sup_{x\in\mathbb{R}}[|u_1(x)|+|u_2(x)|+|v(x)|]e^{-\rho|x|},$$

where $0<\rho<\min\{|\lambda_1^-|,|\lambda_2^-|,|\lambda_3^-|\}$, and it follows that X is a bounded, closed and convex subset of $C_\rho(\mathbb{R},\mathbb{R}^3)$.

It is easy to check that T is completely continuous on X. By applying the Schauder fixed point theorem, we conclude that T has a fixed point (u_1,u_2,v) in X, which gives a solution of the system (1.3).

Note that for $x\in\mathbb{R}$,

$$u_i'(x) = \frac{1}{d_i(\lambda_i^+-\lambda_i^-)}\left(\lambda_i^-\int_{-\infty}^x e^{\lambda_i^-(x-y)} + \lambda_i^+\int_x^\infty e^{\lambda_i^+(x-y)}\right)\hat{F}_i(u_1,u_2,v)(y)\,dy.$$

$$v'(x) = \frac{1}{d_3(\lambda_3^+-\lambda_3^-)}\left(\lambda_3^-\int_{-\infty}^x e^{\lambda_3^-(x-y)} + \lambda_3^+\int_x^\infty e^{\lambda_3^+(x-y)}\right)\hat{G}(u_1,u_2,v)(y)\,dy.$$

It follows that $\left|u_i'(x)\right| \le M_0/[d_i(\lambda_i^+-\lambda_i^-)]$, and $\left|v'(x)\right| \le M_0/[d_3(\lambda_3^+-\lambda_3^-)]$ for $x\in\mathbb{R}$, where $M_0 = \max\{|a_0u_2-a_1u_1+a_2u_1^2-a_3u_1^3-eu_1v+\Lambda u_1|, |u_1-u_2+\Lambda u_2|, |(-b+u_1-v)v+\Lambda v| : 0\le u_1\le U_{10}, 0\le u_2\le U_{20}, 0\le v\le V_0\}$. This shows that u_i' and v' are bounded on $\mathbb{R}$, and then using the equations in the system (1.3) yields the boundedness of u_i'' and v'' as well. This completes the proof of Lemma 2.2. $\square$

In the following two subsections, we will construct the upper and lower solutions of the system (1.3) under $c>c^* := \sqrt{4d_3r}$, $c=c^*$, respectively.

2.1. Upper and lower solutions with super-critical wave speed

In this subsection, we always assume that $c>c^*$, $\lambda = (c-\sqrt{c^2-4d_3r})/(2d_3)$, and Λ is the constant in (2.2). We will construct the upper and lower solutions with super-critical wave speed for the system (1.3).

Now, we introduce the non-negative, continuous and bounded functions $\bar{u}_i(x)$, $\underline{u}_i(x)$, $\bar{v}(x)$ and $\underline{v}(x)$ on $\mathbb{R}$ by

$$\bar{u}_1(x)=\bar{u}_2(x)=K,\quad \underline{u}_1(x)=\underline{u}_2(x)=\begin{cases} K-\beta e^{\gamma x}, & \forall\ x\le a_1,\\ 0, & \forall\, x>a_1,\end{cases}$$

$$\bar{v}(x)=\begin{cases} e^{\lambda x}, & \forall\, x\le a_2,\\ V_0, & \forall\, x>a_2,\end{cases}\qquad \underline{v}(x)=\begin{cases} e^{\lambda x}(1-Ae^{\eta x}), & \forall\, x\le a_0,\\ 0, & \forall\, x>a_0,\end{cases}$$

where

$$a_0=-\frac{1}{\eta}\ln A,\qquad a_1=-\frac{1}{\gamma}\ln\frac{\beta}{K},\qquad a_2=\frac{1}{\lambda}\ln V_0.$$

Next, in Lemmas 2.3 and 2.4, we give that $(\bar{u}_1,\bar{u}_2,\bar{v})$ and $(\underline{u}_1,\underline{u}_2,\underline{v})$ constructed above are a pair of upper and lower solutions of the system (1.3) with super-critical wave speed.

Lemma 2.3. *Suppose all the assumptions of Theorem 2.1 are satisfied. For all $x\in\mathbb{R}$, choose*

$$\max\left\{\gamma_1^-,\gamma_2^-\right\}<\gamma<\min\left\{\lambda,\gamma_1^+,\gamma_2^+\right\},$$

$$\beta > \max\left\{\left(\frac{\Lambda K^{\lambda/\gamma-1}}{\beta_1}\right)^{\gamma/\lambda}, \left(\frac{\Lambda K^{\lambda/\gamma-1}}{\beta_2}\right)^{\gamma/\lambda}, K\left(\frac{1}{V_0}\right)^{\gamma/\lambda}\right\},$$

where

$$\gamma_i^{\pm} = \frac{1}{2d_i}\left(c \pm \sqrt{c^2 - 4d_i(M_{i1} + M_{i2})}\right), \ \beta_i = c\gamma - d_i\gamma^2 - (M_{i1} + M_{i2}), \ \ i = 1, 2,$$

where

$$M_{11} = \frac{a_2^2}{4a_3} - a_1 \geq 0, M_{12} = a_0, M_{21} = 1, M_{22} = 0, \tag{2.3}$$

and then we can obtain that the following inequalities

$$\begin{aligned}
d_1\bar{u}_1''(x) - c\bar{u}_1'(x) + a_0u_2(x) - a_1\bar{u}_1(x) + a_2\bar{u}_1^2(x) - a_3\bar{u}_1^3(x) - e\bar{u}_1(x)v(x) \leq 0,\\
d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \geq 0,\\
d_2\bar{u}_2''(x) - c\bar{u}_2'(x) + u_1(x) - \bar{u}_2(x) \leq 0,\\
d_2\underline{u}_2''(x) - c\underline{u}_2'(x) + u_1(x) - \underline{u}_2(x) \geq 0
\end{aligned}$$

hold.

Proof. According to (2.3) and the assumptions of Theorem 2.1, we have $M_{i1} + M_{i2} < min\{d_3/d_i, 1\}r$, so

$$\begin{aligned}
a_0u_2 - a_1u_1 + a_2u_1^2 - a_3u_1^3 \geq& M_{11}(u_1 - K) + M_{12}(u_2 - K)\\
\geq& -M_{11}(K - u_1) - M_{12}(K - u_2).
\end{aligned} \tag{2.4}$$

In addition, we also obtain

$$\frac{1}{2d_i}\left(c - \sqrt{c^2 - 4d_i(M_{i1} + M_{i2})}\right) < \lambda = \frac{1}{2d_3}\left(c - \sqrt{c^2 - 4d_3r}\right),$$

and then γ is well defined. If $d_3/d_i < 1$, then we have $M_{i1} + M_{i2} < (d_3/d_i)r$, so this inequality is clearly true. If $d_3/d_i \geq 1$, then $M_{i1} + M_{i2} < r \leq (d_3/d_i)r$, which implies that an equivalent inequality

$$\frac{M_{i1} + M_{i2}}{c + \sqrt{c^2 - 4d_i(M_{i1} + M_{i2})}} < \frac{r}{c + \sqrt{c^2 - 4d_3r}}$$

holds. Since the choice of γ, we have $\beta_i = c\gamma - d_i\gamma^2 - (M_{i1} + M_{i2}) > 0$, so β is well defined.

According to the definitions of a_1, $\underline{u}_i$ are continuous at a_1, and by the assumptions on γ, we have

$$\underline{u}_i(x) < \bar{u}_i(x), \ \ \underline{u}_i'(a_1-) = -\gamma K < 0 = \underline{u}_i'(a_1+), \ \forall x \in \mathbb{R}.$$

Since $\bar{u}_1 \equiv K$, it follows that

$$\begin{aligned}
&d_1\bar{u}_1''(x) - c\bar{u}_1'(x) + a_0u_2(x) - a_1\bar{u}_1(x) + a_2\bar{u}_1^2(x) - a_3\bar{u}_1^3(x) - e\bar{u}_1(x)v(x)\\
&= a_0u_2(x) - a_1K + a_2K^2 - a_3K^3 - eKv\\
&\leq a_0K - a_1K + a_2K^2 - a_3K^3 - eKv\\
&= 0, \ \ \forall x \in \mathbb{R}.
\end{aligned}$$

Choose $\beta > K(1/V_0)^{\gamma/\lambda}$ such that $a_1 < a_2$, and then we have for any $x < a_1$ that

$$\underline{u}_1(x) = K - \beta e^{\gamma x}, \quad \bar{v}(x) = e^{\lambda x}.$$

According to (2.2) and (2.4), we have

$$\begin{aligned}
& d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \\
\geq\ & d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - \Lambda v(x) \\
\geq\ & d_1\underline{u}_1''(x) - c\underline{u}_1'(x) - [M_{11}(K - \underline{u}_1(x)) + M_{12}(K - u_2(x))] - \Lambda\bar{v}(x) \\
\geq\ & d_1\underline{u}_1''(x) - c\underline{u}_1'(x) - [M_{11}(K - \underline{u}_1(x)) + M_{12}(K - \underline{u}_2(x))] - \Lambda\bar{v}(x) \\
=\ & -d_1\beta\gamma^2e^{\gamma x} + c\beta\gamma e^{\gamma x} - (M_{11} + M_{12})\beta e^{\gamma x} - \Lambda e^{\lambda x} \\
=\ & \beta e^{\gamma x}\left[\gamma(c - d_1\gamma) - (M_{11} + M_{12}) - \frac{1}{\beta}\Lambda e^{(\lambda-\gamma)x}\right] \\
\geq\ & \beta e^{\gamma x}\left[\gamma(c - d_1\gamma) - (M_{11} + M_{12}) - \frac{1}{\beta}\Lambda e^{(\lambda-\gamma)a_1}\right] \\
=\ & \beta e^{\gamma x}\left[\gamma(c - d_1\gamma) - (M_{11} + M_{12}) - \Lambda K^{\lambda/\gamma-1}\beta^{-\lambda/\gamma}\right] \\
\geq\ & 0, \ \forall x < a_1,
\end{aligned}$$

where the last inequality is guaranteed by the assumptions on γ and β.

For $x > a_1$, since $\underline{u}_1(x) = 0$, we also have

$$\begin{aligned}
& d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \\
=\ & a_0u_2(x) \geq 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& d_2\bar{u}_2''(x) - c\bar{u}_2'(x) + u_1(x) - \bar{u}_2(x) \leq 0, \ \forall x \in \mathbb{R}, \\
& d_2\underline{u}_2''(x) - c\underline{u}_2'(x) + u_1(x) - \underline{u}_2(x) \geq 0, \ \forall x \in \mathbb{R}.
\end{aligned}$$

The proof is completed. □

Lemma 2.4. *Let the assumptions of Theorem 2.1 hold and γ satisfy Lemma 2.3, and choose*

$$\beta > \max\left\{K\left(\frac{1}{V_0}\right)^{\gamma/\lambda}, \ K\right\},$$

$$0 < \eta < \gamma, \quad -d_3(\lambda+\eta)^2 + c(\lambda+\eta) - r > 0,$$

$$A > \max\left\{\left(\frac{\beta}{K}\right)^{\eta/\gamma}, \left(\frac{1}{\delta}\right)^{\eta/\lambda}, \left(\frac{\beta}{\delta}\right)^{\eta/\gamma}, \ \frac{2(1+2\beta)}{-d_3(\lambda+\eta)^2 + c(\lambda+\eta) - r}\right\}.$$

Then, for all $x \in \mathbb{R}$, the inequalities

$$d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + u_1(x) - \bar{v}(x))\bar{v}(x) \leq 0,$$

$$d_3\underline{v}''(x) - c\underline{v}'(x) + (-b + u_1(x) - \underline{v}(x))\underline{v}(x) \geq 0$$

hold.

Proof. We first point out that by the assumptions on γ, β, η, A and the definitions of $\underline{u}_i$, $\bar{v}$, $\underline{v}$, $a_0 < a_1 < \min\{0, a_2\}$, and

$$\underline{v}(x) < \bar{v}(x), \quad \forall x \in \mathbb{R},$$
$$\underline{v}'(a_0-) = -\eta e^{\lambda a_0} < 0 = \underline{v}'(a_0+),$$
$$\bar{v}'(a_2-) = \lambda V_0 > 0 = \bar{v}'(a_2+).$$

For $x < a_2$, we have $\bar{v}(x) = e^{\lambda x}$, and then

$$\begin{aligned}
&d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + u_1(x) - \bar{v}(x))\bar{v}(x)\\
\le\ & d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + K - \bar{v}(x))\bar{v}(x)\\
\le\ & d_3\bar{v}'' - c\bar{v}' + (-b + K)\bar{v}(x)\\
=\ & d_3\bar{v}''(x) - c\bar{v}'(x) + r\bar{v}(x)\\
=\ & (d_3\lambda^2 - c\lambda + r)e^{\lambda x} = 0.
\end{aligned}$$

For $x > a_2$, since $\bar{v}(x) = V_0$, we can obtain

$$\begin{aligned}
&d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + u_1(x) - \bar{v}(x))\bar{v}(x)\\
=\ & (-b + u_1(x) - V_0)V_0\\
\le\ & (-b + K - r)r\\
=\ & 0.
\end{aligned}$$

For $x < a_0$, since $a_0 < 0 < a_1 < \min\{0, a_2\}$, $0 < \eta < \gamma$ and by the choice of A, we have

$$\underline{v}(x) = e^{\lambda x} - Ae^{(\lambda+\eta)x},\ \bar{v}(x) = e^{\lambda x},\ K - \underline{u}_i(x) = \beta e^{\gamma x},$$

$$\underline{v}(x) \le \bar{v}(x) \le e^{\lambda a_0} < \delta,\ K - u_i(x) \le K - \underline{u}_i(x) \le \beta e^{\gamma a_0} < \delta.$$

By (2.1), one can obtain that

$$\begin{aligned}
&d_3\underline{v}''(x) - c\underline{v}'(x) + (-b + u_1(x) - \underline{v}(x))\underline{v}(x)\\
\ge\ & d_3\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - 2\left[(K - u_1(x)) + (K - u_2(x)) + \underline{v}(x)\right]\underline{v}(x)\\
\ge\ & d_3\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - 2\left[(K - u_1(x)) + (K - u_2(x)) + \bar{v}(x)\right]\bar{v}(x)\\
\ge\ & d_3\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - 2\left(\beta e^{\gamma x} + \beta e^{\gamma x} + e^{\lambda x}\right)\bar{v}(x)\\
=\ & d_3\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - 2\left(2\beta + e^{(\lambda-\gamma)x}\right)e^{\gamma x}e^{\lambda x}\\
\ge\ & d_3\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - 2(2\beta + 1)e^{\gamma x}e^{\lambda x}\\
=\ & e^{(\lambda+\eta)x}\left\{A\left[-d_3(\lambda + \eta)^2 + c(\lambda + \eta) - r\right] - 2(2\beta + 1)e^{(\gamma-\eta)x}\right\}\\
\ge\ & 0.
\end{aligned}$$

For $x > a_0$, we have $\underline{v}(x) = 0$, and thus

$$d_3\underline{v}''(x) - c\underline{v}'(x) + (-b + u_1(x) - \underline{v}(x))\underline{v}(x) = 0.$$

Combining the above, we have proved the assertions of Lemma 2.4. □

2.2. Upper and lower solutions with critical wave speed

In this subsection, we always assume that $c := \sqrt{4d_3r}$, $\lambda = c/(2d_3)$, $M = \lambda eV_0$ and Λ is the constant in (2.2). We will construct the upper and lower solutions with super-critical wave speed for the system (1.3).

There exists a large $R_0 > 0$ so that for all $R \geq R_0$, we define

$$\bar{u}_1 = \bar{u}_2 \equiv K, \qquad \underline{u}_1(x) = \underline{u}_2(x) = \begin{cases} K - \beta e^{\gamma x}, & x \leq a_1, \\ 0, & x > a_1, \end{cases}$$

$$\bar{v}(x) = \begin{cases} M|x|e^{\lambda x}, & x \leq a_2, \\ v_0, & x > a_2, \end{cases} \qquad \underline{v}(x) = \begin{cases} \left(M|x| - R\sqrt{|x|}\right)e^{\lambda x}, & x \leq a_0, \\ 0, & x > a_0, \end{cases}$$

where

$$a_2 = -\frac{1}{\lambda}, \qquad a_1 = \frac{1}{\gamma}\ln\frac{1}{\beta}, \qquad a_0 = -\frac{R^2}{M^2}.$$

Next, in Lemmas 2.5 and 2.6, we will prove $(\bar{u}_1, \bar{u}_2, \bar{v})$ and $(\underline{u}_1, \underline{u}_2, \underline{v})$ constructed above are a pair of upper and lower solutions of the system (1.3) with critical wave speed.

Lemma 2.5. *Let the assumptions of Theorem 2.1 hold, and M_{i1} and M_{i2} are fixed points and satisfy Lemma 2.3. For all $x \in \mathbb{R}$, choose auxiliary constants γ and β such that*

$$\max\{\gamma_1^-, \gamma_2^-\} < \gamma < \min\left\{\lambda, \gamma_1^+, \gamma_2^+\right\},$$
$$\beta > \max\left\{e^{\frac{\gamma}{\lambda-\gamma}}, \ \frac{M\Lambda}{\beta_1(\lambda-\gamma)e}, \ \frac{M\Lambda}{\beta_2(\lambda-\gamma)e}\right\}.$$

Then, for all $x \in \mathbb{R}$, the inequalities

$$d_1\bar{u}_1''(x) - c\bar{u}_1'(x) + a_0u_2(x) - a_1\bar{u}_1(x) + a_2\bar{u}_1^2(x) - a_3\bar{u}_1^3(x) - e\bar{u}_1(x)v(x) \leq 0,$$
$$d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \geq 0,$$
$$d_2\bar{u}_2''(x) - c\bar{u}_2'(x) + u_1(x) - \bar{u}_2(x) \leq 0,$$
$$d_2\underline{u}_2''(x) - c\underline{u}_2'(x) + u_1(x) - \underline{u}_2(x) \geq 0$$

hold.

Proof. Similar to Lemma (2.3), it is easy to get

$$d_1\bar{u}_1''(x) - c\bar{u}_1'(x) + a_0u_2(x) - a_1\bar{u}_1(x) + a_2\bar{u}_1^2(x) - a_3\bar{u}_1^3(x) - e\bar{u}_1(x)v(x) \leq 0,$$

$$d_2\bar{u}_2''(x) - c\bar{u}_2'(x) + u_1(x) - \bar{u}_2(x) \leq 0.$$

For $x < a_1$, we can obtain $a_1 < a_2$ from $\beta > e^{\gamma/\lambda-\gamma}$, and then

$$\underline{u}_1(x) = K - \beta e^{\gamma x}, \ \bar{v}(x) = M|x|e^{\lambda x},$$

$$\underline{u}_1''(x) = -\beta\gamma^2 e^{\gamma x}, \ \underline{u}_1''(x) = -\beta\gamma e^{\gamma x}.$$

By (2.2) and (2.4), we have

$$\begin{aligned}
& d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \\
\geq\ & d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - \Lambda v(x) \\
\geq\ & d_1\underline{u}_1''(x) - c\underline{u}_1'(x) - [M_{11}(K - \underline{u}_1(x)) + M_{12}(K - \underline{u}_2(x))] - \Lambda\bar{v}(x) \\
=\ & -d_1\beta\gamma^2 e^{\gamma x} + c\beta\gamma e^{\gamma x} - (M_{11} + M_{12})\beta e^{\gamma x} - \Lambda M|x|e^{\lambda x} \\
=\ & \beta e^{\gamma x}\left[\gamma(c - d_1\gamma) - (M_{11} + M_{12}) - \frac{1}{\beta}\Lambda M|x|e^{(\lambda-\gamma)}\right].
\end{aligned}$$

Since $|x|e^{(\lambda-\gamma)x}$ is monotone increasing over $(-\infty, -1/(\lambda-\gamma))$, and $a_1 < -1/(\lambda-\gamma)$, it follows that $|x|e^{(\lambda-\gamma)x} \leq 1/((\lambda-\gamma)e)$ for $x < a_1$, and then by the choice of β, we have

$$\begin{aligned}
& d_1\underline{u}_1''(x) - c\underline{u}_1'(x) + +a_0u_2(x) - a_1\underline{u}_1(x) + a_2\underline{u}_1^2(x) - a_3\underline{u}_1^3(x) - e\underline{u}_1(x)v(x) \\
\geq\ & \beta e^{\gamma x}\left[\gamma(c - d_1\gamma) - (M_{11} + M_{12}) - \frac{M\Lambda}{\beta(\lambda-\gamma)e}\right] \\
\geq\ & 0, \ \forall x < a_1.
\end{aligned}$$

Moreover,

$$\underline{u}_1'(a_1-) = -\beta\gamma e^{\gamma a_1} \leq 0 = \underline{u}_1'(a_1+).$$

Similarly, we have

$$d_2\underline{u}_2''(x) - c\underline{u}_2'(x) + u_1(x) - \underline{u}_2(x) \geq 0,$$

and

$$\underline{u}_2'(a_1-) = -\beta\gamma e^{\gamma a_1} \leq 0 = \underline{u}_2'(a_1+).$$

This ends the proof. □

Lemma 2.6. *Let the hypothesis of Theorem 2.1 and Lemma 2.5 be satisfied. For all $x \in \mathbb{R}$, the inequalities*

$$d_3\bar{v}''(x) - c\bar{v}'(x) + (-b + u_1(x) - \bar{v}(x))\bar{v}(x) \leq 0,$$
$$d_3\underline{v}''(x) - c\underline{v}'(x) + (-b + u_1(x) - \underline{v}(x))\underline{v}(x) \geq 0$$

hold.

Proof. For $x < a_0$, we have

$$(\underline{v} + Mxe^{\lambda x})' = \left(\frac{R}{2\sqrt{-x}} - \lambda R\sqrt{-x}\right)e^{\lambda x} = R\left(\frac{1}{2\sqrt{-x}} - \sqrt{-x}\lambda\right)e^{\lambda x},$$

$$(\underline{v} + Mxe^{\lambda x})'' = R\left(-\frac{1}{4x\sqrt{-x}} + \frac{1}{\sqrt{-x}}\lambda - \sqrt{-x}\lambda^2\right)e^{\lambda x},$$

and so

$$\begin{aligned}
& d_3\underline{v}'' - c\underline{v}' + r\underline{v} \\
=\ & R\left(\frac{-d_3}{4x\sqrt{-x}} + \frac{d_3}{\sqrt{-x}}\lambda - d_3\sqrt{-x}\lambda^2 - \frac{c}{2\sqrt{-x}} + c\sqrt{-x}\lambda\right)e^{\lambda x} - rR\sqrt{-x}e^{\lambda x}
\end{aligned}$$

$$= \frac{-d_3R}{4x\sqrt{-x}}e^{\lambda x}.$$

Since $a_0 < 0$, and R is large enough to ensure that $a_0 < a_1 < a_2$ and $a_0 < -1$, for $x < a_0$, we have

$$\underline{v}(x) = (M|x| - R\sqrt{|x|})e^{\lambda x},\ \bar{v}(x) = M|x|e^{\lambda x},\ K - \underline{u}_i(x) = \beta e^{\gamma x},$$

and

$$\underline{v}(x) \le \bar{v}(x) < \delta,\ K - u_i(x) \le K - \underline{u}_i(x) \le \beta e^{\gamma a_0} < \delta.$$

Hence, for such R and x,

$$\begin{aligned}
&(-b + u_1(x) - \underline{v}(x))\underline{v}(x)\\
&\ge r\underline{v}(x) - 2[(K - u_1(x)) + (K - u_2(x)) + \underline{v}(x)]\underline{v}(x)\\
&\ge r\underline{v}(x) - 2[(K - \underline{u}_1(x)) + (K - \underline{u}_2(x)) + \bar{v}(x)]\bar{v}(x)\\
&= r\underline{v}(x) - 2\left(\beta e^{\gamma x} + \beta e^{\gamma x} + M|x|e^{\lambda x}\right)\bar{v}(x)\\
&= r\underline{v}(x) - 2M\left(2\beta + M|x|e^{(\lambda-\gamma)x}\right)e^{\gamma x}|x|e^{\lambda x}\\
&\ge r\underline{v}(x) - 2M(2\beta + M|x|)|x|e^{\gamma x}e^{\lambda x}.
\end{aligned}$$

So,

$$\begin{aligned}
&d_3\underline{v}'' - c\underline{v}' + (-b + u_1(x) - \underline{v}(x))\underline{v}(x)\\
&\ge \left[\frac{-d_3R}{4x\sqrt{-x}} - 2M\Big(2\beta + M|x|\Big)|x|e^{\gamma x}\right]e^{\lambda x}\\
&= \frac{1}{4|x|\sqrt{|x|}}\left[d_3R - 2M\Big(2\beta + M|x|\Big)4x^2\sqrt{|x|}e^{\gamma x}\right]e^{\lambda x}\\
&\ge \frac{1}{4|x|\sqrt{|x|}}\Big[d_3R - 2M(2\beta + M)x^4e^{\gamma x}\Big]e^{\lambda x}\\
&\ge 0.
\end{aligned}$$

Using

$$\underline{v}'(x) = \left[-M + \frac{R}{2\sqrt{-x}} + \lambda(-Mx - R\sqrt{-x})\right]e^{\lambda x},$$

we have

$$\underline{v}'(a_0-) = \left[-M + \frac{R}{2\sqrt{|a_0|}}\right]e^{\lambda a_0} = -\frac{M}{2}e^{\lambda a_0} < 0 = \underline{v}'(a_0+).$$

For $x < a_2 = -1/\lambda$, we have

$$\bar{v}(x) = -Mxe^{\lambda x},\ \bar{v}'(x) = -M(1 + \lambda x)e^{\lambda x},$$

so that

$$\bar{v}(a_2) = M/(\lambda e) = v_0,\ \bar{v}'(a_2-) = 0 = \bar{v}'(a_2+),$$

and then we have

$$d_3\bar{v}'' - c\bar{v}' + (-b + u_1(x) - \underline{v}(x))\underline{v}(x)$$

$$\begin{aligned}
&\le d_3\bar{v}'' - c\bar{v}' + (-b + K - \underline{v}(x))\underline{v}(x)\\
&\le d_3\bar{v}'' - c\bar{v}' + (-b + K)\underline{v}(x)\\
&= d_3\bar{v}'' - c\bar{v}' + rv(x)\\
&= 0.
\end{aligned}$$

We thus conclude the proof of Lemma 2.6. □

Proof of Theorem 2.1. We first show that for any $c \ge c^* := \sqrt{4d_3r}$, the system (1.3) has a solution (u_1, u_2, u_3) satisfying Theorem 2.1. Applying all the Lemmas above with $U_{10} = U_{20} = K$ yields the existence of a solution (u_1, u_2, v) to the system (1.3), satisfying $\underline{u}_i \le u_i \le \bar{u}_i$ and $\underline{v} \le v \le \bar{v}$. The definitions of $\underline{u}_i$, $\bar{u}_i$, $\underline{v}$ and $\bar{v}$ imply that $(u_1, u_2, v)(x) \to (K, K, 0)$ as $x \to -\infty$, that, after a translation in x, $0 < u_i(x) \le K$ and $0 < v(x) \le V_0$ for $x \le 0$, that $0 \le u_i(x) \le K$ and $0 \le v(x) \le V_0$ for $x > 0$. Using the expressions

$$u_i'(x) = e^{\frac{cx}{d_i}} u_i'(0) + \int_x^0 e^{\frac{c(x-y)}{d_i}} F_i(u_1(y), u_2(y), v(y))\, dy,$$
$$v'(x) = e^{\frac{cx}{d_3}} u'(0) + \frac{1}{d}\int_x^0 e^{\frac{c(x-y)}{d_3}} G(u_1(y), u_2(y), v(y))\, dy,$$

and L'Hospital's rule, we get $(u_1'(x), u_2'(x), v'(x)) \to 0$ as $x \to -\infty$. Therefore, (u_1, u_2, v) is a weak traveling wave solution to the system (1.3).

Next, we prove that for any $0 < c < \sqrt{4d_3r}$, there is no weak traveling wave solution of the system (1.3). Under the assumptions, we can write in a neighborhood of $(K, K, 0)$ the v equation in (1.3) as

$$d_3v'' - cv' + rv + (g(u_1, u_2, v) - r)\, v = 0,$$

where $g(K, K, 0) = -b + K = r$, and $g(u_1, u_2, v) - r \to 0$ as $(u_1, u_2, v) \to (K, K, 0)$. The characteristic equation $d_3\lambda^2 - c\lambda + r = 0$ has a pair of complex roots $\lambda = \left(c \pm i\sqrt{4d_3r - c^2}\right)/(2d_3)$. Assume by contradiction there is a solution (u_1, u_2, v) of (1.3) satisfying $(u_1(x), u_2(x), v(x)) \to (K, K, 0)$ as $x \to -\infty$ and $v(x) > 0$ for sufficiently negative x. Then, using the variation of constants formula, one can show that, for sufficiently negative x_0 and x,

$$v(x) = e^{\alpha(x-x_0)} \left\{ v(x_0)\cos\beta(x - x_0) + \frac{1}{\beta}(v'(x_0) - \alpha v(x_0))\sin\beta(x - x_0) \right\} (1 + R(x, x_0)),$$

where $\lim_{x_0\to-\infty} \sup_{x<x_0} |R(x, x_0)| = 0$. (See [1] for details.) This asymptotic expression shows that $v(x)$ changes signs infinitely many times as $x \to -\infty$, a contradiction.

The proof is completed. □

3. Traveling wave solutions for the system (1.3)

In this section, we prove that under certain conditions, the weak traveling wave solution obtained in Theorem 2.1 is also a traveling wave solution by using the Lyapunov function and LaSalle's invariance principle.

Theorem 3.1. *Assume that all conditions in Theorem 2.1 are satisfied, and $a_3(eb + a_0 - a_1) > a_2e$ holds. Then, the system (1.3) has a traveling wave solution (u_1, u_2, v) satisfying (1.4) for every $c \ge \sqrt{4d_3r}$.*

Proof. Let (u_1, u_2, v) be a weak traveling wave solution of the system (1.3). By Theorem 2.1, there are $\delta > 0$ and $x_0 > 0$ such that $u_i(x) > \delta$ for $x \in \mathbb{R}$ and $v(x) > \delta$ for $x > x_0$. This implies that the orbit $(u_1, u_1', u_2, u_2', v, v')(x)$ lies in the set $\Omega_\delta =: ([\delta, K] \times \mathbb{R})^2 \times [\delta, V_0] \times \mathbb{R}$ for $x > x_0$. To show that $(u_1, u_1', u_2, u_2', v, v')(x) \to (u_1^*, 0, u_1^*, 0, v^*, 0)$ as $x \to \infty$, it is necessary to define a Lyapunov function L on $((0, K] \times \mathbb{R})^2 \times (0, V_0] \times \mathbb{R}$ by

$$L(u_1, u_1', u_2, u_2', v, v') = cH(u_1, u_2, v) - d_1 \frac{\partial H}{\partial u_1} u_1' - d_2 \frac{\partial H}{\partial u_2} u_2' - d_3 \frac{\partial H}{\partial v} v',$$

where

$$H(u_1, u_2, v) = \alpha_1 \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*} \right) + \alpha_2 \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*} \right) + \alpha_3 \left(v - v^* - v^* \ln \frac{v}{v^*} \right),$$

and α_1, α_2, α_3 are non-negative constants. Then, along the orbits of the system (1.3) with $x > x_0$, we have

$$\begin{aligned} \frac{d}{dx} L = & \left(\frac{\partial H}{\partial u_1} F_1(u_1, u_2, v) + \frac{\partial H}{\partial u_2} F_2(u_1, u_2, v) + \frac{\partial H}{\partial v} G(u_1, u_2, v) \right) \\ & - d_1 \frac{\partial^2 H}{\partial u_1^2} (u_1')^2 - d_2 \frac{\partial^2 H}{\partial u_2^2} (u_2')^2 - d_3 \frac{\partial^2 H}{\partial v^2} (v')^2, \end{aligned} \tag{3.1}$$

where

$$F_1(u_1, u_2, v) = a_0 u_2 - a_1 u_1 + a_2 u_1^2 - a_3 u_1^3 - e u_1 v,\ F_2(u_1, u_2, v) = u_1 - u_2,\ G(u_1, u_2, v) = (-b + u_1 - v)v.$$

Then, we have

$$\begin{aligned} & \frac{\partial H}{\partial u_1} F_1(u_1, u_2, v) + \frac{\partial H}{\partial u_2} F_2(u_1, u_2, v) + \frac{\partial H}{\partial v} G(u_1, u_2, v) \\ = & \alpha_1 \frac{u_1 - u_1^*}{u_1} \\ & \times \left\{ \frac{a_0}{u_1^*} \left[u_1 \left(u_2 - u_2^* \right) - u_2 \left(u_1 - u_1^* \right) \right] - u_1 \left(u_1 - u_1^* \right) \left(-a_2 + a_3 u_1 + a_3 u_1^* \right) - e u_1 \left(v - v^* \right) \right\} \\ & + \alpha_2 \frac{u_2 - u_2^*}{u_2} \frac{1}{u_2^*} \left[u_2 \left(u_1 - u_1^* \right) - u_1 \left(u_2 - u_2^* \right) \right] + \alpha_3 v \left[\left(u_1 - u_1^* \right) - \left(v - v^* \right) \right] \frac{v - v^*}{v} \\ = & - \alpha_1 (-a_2 + a_3 u_1 + a_3 u_1^*)(u_1 - u_1^*)^2 - \alpha_3 (v - v^*)^2 \\ & + (\alpha_3 - k\alpha_1) \left(u_1 - u_1^* \right) \left(v - v^* \right) + a_0 \alpha_1 \left(u_1 - u_1^* \right) \times \left[-\frac{u_2}{u_1 u_1^*} \left(u_1 - u_1^* \right) + \frac{1}{u_1^*} \left(u_2 - u_2^* \right) \right] \\ & + \alpha_2 \left(u_2 - u_2^* \right) \frac{\left(u_1 - u_1^* \right) u_2 - \left(u_2 - u_2^* \right) u_1}{u_2 u_2^*}. \end{aligned}$$

Let $\alpha_2 = a_0 \alpha_1$, $\alpha_3 = e\alpha_1$. Thus,

$$\begin{aligned} & \frac{\partial H}{\partial u_1} F_1(u_1, u_2, v) + \frac{\partial H}{\partial u_2} F_2(u_1, u_2, v) + \frac{\partial H}{\partial v} G(u_1, u_2, v) \\ = & - \alpha_1 \left(-a_2 + a_3 u_1 + a_3 u_1^* \right) \left(u_1 - u_1^* \right)^2 - \alpha_3 \left(v - v^* \right)^2 \\ & - a_0 \alpha_1 \frac{1}{u_1^*} \left[\sqrt{\frac{u_2}{u_1}} \left(u_1 - u_1^* \right) - \sqrt{\frac{u_1}{u_2}} \left(u_2 - u_2^* \right) \right]^2. \end{aligned}$$

We observe that $a_3(eb + a_0 - a_1) > a_2 e$ is a sufficient condition of $-a_2 + a_3 u_1 + a_3 u_1^* > 0$. So, when $a_3(eb + a_0 - a_1) > a_2 e$ holds, we have

$$\frac{\partial H}{\partial u_1} F_1(u_1, u_2, v) + \frac{\partial H}{\partial u_2} F_2(u_1, u_2, v) + \frac{\partial H}{\partial v} G(u_1, u_2, v) < 0.$$

Since

$$d_i \frac{\partial^2 H}{\partial u_i^2}(u_i')^2 = d_i \frac{u_i^*}{u_i^2}(u_i')^2 = d_i u_i^* \left(\frac{u_i'}{u_i}\right)^2, \quad d_3 \frac{\partial^2 H}{\partial v^2}(v')^2 = d_3 \frac{v^*}{v^2}(v')^2 = d_3 v^* \left(\frac{v'}{v}\right)^2, \tag{3.2}$$

we can obtain that $dL(u_1, u_1', u_2, u_2', v, v')/dx \leq 0$ for $(u_1, u_1', u_2, u_2', v, v') \in \Omega_\delta$.

Let $\rho_i = u_i'/u_i$, $\rho_3 = v'/v$. Next, we prove that $|\rho_i| \leq \rho_i^+$, $|\rho_3| \leq \rho_3^+$.

Since $0 < u_i \leq K$ and $0 < v \leq V_0$, there exists a positive constant $\overline{M} > 0$ such that $\max\{a_1 - a_2 u_1 + a_3 u_1^2 + ev, b - u_1 + v, 1\} \leq \overline{M}$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned}
\rho_1' &= \frac{u_1'' u_1 - (u_1')^2}{u_1^2} = \frac{u_1''}{u_1} - \rho_1^2 \\
&= \frac{c}{d_1}\rho_1 - \frac{a_0 u_2}{d_1 u_1} + \frac{1}{d_1}(a_1 - a_2 u_1 + a_3 u_1^2 + ev) - \rho_1^2 \\
&\leq -\rho_1^2 + \frac{c}{d_1}\rho_1 + \frac{\overline{M}}{d_1}.
\end{aligned}$$

Let ρ_1^+ be a positive constant solution of $\rho' = -\rho^2 + (c/d_1) + \overline{M}/d_1$. According to the comparison theorem, we have $\rho_1(x) < \rho_1^+$ for all $x \in \mathbb{R}$. Similarly, if $\rho_1 < -\rho_1^+$ occurs at some x_0, then, letting $\rho(x)$ be the solution of $\rho' = -\rho^2 + (c/d_1)\rho + \overline{M}/d_1$ with $\rho(x_0) = \rho_1(x_0)$, it follows from the comparison theorem that $\rho_1(x) \leq \rho(x)$ for $x \geq x_0$.

Note that

$$-\rho^2(x_0) + \frac{c}{d_1}\rho(x_0) - \frac{\overline{M}}{d_1} < -(-\rho_1^+)^2 + \frac{c}{d_1}(-\rho_1^+) + \frac{\overline{M}}{d_1} < 0,$$

implies $\rho(x) \to -\infty$ as $x \to x_1$ for some finite value $x_1 > x_0$. It follows that $\rho_1(x) \to -\infty$ as $x \to x_2$ for some $x_2 \in (x_0, x_1]$, contradicting the fact that $\rho_1(x)$ is defined for all $x \in \mathbb{R}$.

Similarly,

$$\begin{aligned}
\rho_2' &= \frac{u_2'' u_2 - (u_2')^2}{u_2^2} \\
&= \frac{c}{d_2}\rho_2 - \frac{u_1}{d_2 u_2} + \frac{1}{d_2} - \rho_2^2 \\
&\leq -\rho_2^2 + \frac{c}{d_2}\rho_2 + \frac{\overline{M}}{d_2},
\end{aligned}$$

and

$$\rho_3' = \frac{v'' v - (v')^2}{v^2}$$

$$
\begin{aligned}
&= \frac{c}{d_3}\rho_3 - \frac{1}{d_3}(-b + u_1 - v) - \rho_3^2 \\
&\leq -\rho_3^2 + \frac{c}{d_3}\rho_3 + \frac{\overline{M}}{d_3},
\end{aligned}
$$

and there exist constants $\rho_2^+ > 0, \rho_3^+ > 0$ such that $|\rho_2| \leq \rho_2^+$, $|\rho_3| \leq \rho_3^+$ for $\forall(u_1, u_2, v) \in (0, K]^2 \times (0, V_0] \setminus \{(u_1^*, u_2^*, v^*)\}$.

Since equality holds only at E^*, we derive that $(u_1, u_1', u_2, u_2', v, v')(x) \to E^*$ as $x \to \infty$ by LaSalle's invariance principle. This proves Theorem 3.1. □

4. Conclusions

The main results of this work can be summarized as follows: This paper is concerned with traveling wave solutions to a cubic predator-prey diffusion model with stage structure for the prey given by system (1.2). First, employing the method of upper and lower solutions together with the Schauder fixed point theorem, we give a sharp existence result on weak traveling wave solutions for system (1.2), with minimal speed explicitly determined. Such a weak traveling wave $(u_1(\xi), u_2(\xi), v(\xi))$ connects the semi-trivial equilibrium $(K, K, 0)$ at $\xi = -\infty$ but needs not connect the coexistence equilibrium (u_1^*, u_2^*, v^*) at $\xi = \infty$ (see Theorem 2.1). Then, we use the Lyapunov function method and LaSalle's invariance principle to prove that, under additional conditions, the weak traveling wave solutions of (1.2) established in Theorem 2.1 are actually traveling wave solutions; namely, they converge to the coexistence equilibrium (u_1^*, u_2^*, v^*) as $\xi \to \infty$ (see Theorem 3.1). To the best of the authors' knowledge, the results in Theorems 2.1 and 3.1 are new.

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Conflict of interest

All authors declare that they have no competing interests.

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