AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16249–16260.
DOI: 10.3934/math.2022887
Received: 22 April 2022
Revised: 25 June 2022
Accepted: 29 June 2022
Published: 04 July 2022

Research article

Decay estimates for three-dimensional nematic liquid crystal system

Xiufang Zhao*

School of Science, Qiqihar University, Qiqihar 161006, China

* **Correspondence:** Email: 01036@qqhar.edu.cn; Tel: +864522738190; Fax: +864522742566.

Abstract: In this paper, we consider the decay rates of the higher-order spatial derivatives of solution to the Cauchy problem of 3D compressible nematic liquid crystals system. The $\dot{H}^s$ ($\frac{1}{2} < s < \frac{3}{2}$) negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates.

Keywords: nematic liquid crystals; Ericksen-Leslie system; energy estimates; negative Sobolev space; decay estimates
Mathematics Subject Classification: 35Q35, 76D03, 35B40

1. Introduction

Nematic liquid crystals are aggregates of molecules which possess same orientational order and are made of elongated, rod-like molecules. Hence, in the study of nematic liquid crystals, one approach is to consider the behavior of the director field d in the absence of the velocity fields. Unfortunately, the flow velocity does disturb the alignment of the molecules. More importantly, the converse is also true, that is, a change in the alignment will induce velocity. This velocity will in turn affect the time evolution of the director field. In this process, we cannot assume that the velocity field will remain small even when we start with zero velocity field.

In the 1960's, Ericksen [3, 4] and Leslie [10, 11] developed the hydrodynamic theory of liquid crystals. The Ericksen-Leslie system consists of the following equations [12]:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t = \rho F_i + \sigma_{ji,j}, \\ \rho_1 (\omega_i)_t = \rho_1 G_i + g_i + \pi_{ji,j}, \end{cases} \tag{1.1}$$

where $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$, represent the conservation of mass, linear momentum and angular momentum respectively. Besides, ρ denotes the fluid density, $u = (u_1, u_2, u_3)$ is the velocity vector and

$d = (d_1, d_2, d_3)$ the direction vector,

$$\begin{cases} \sigma_{ji} = -P\delta_{ij} - \rho\dfrac{\partial F}{\partial d_{k,j}} + \hat{\sigma}'_{ji}, \\ \pi_{ji} = \beta_j d_i + \rho\dfrac{\partial F}{\partial d_{i,j}}, \\ g_i = \gamma d_i - \beta_j d_{i,j} - \rho\dfrac{\partial F}{\partial d_i} + \hat{g}'_i, \end{cases} \tag{1.2}$$

where F_i is the external body force, G_i denotes the external director body force and β, γ come from the restriction of the direction vector $|d| = 1$. The following relations also hold:

$$\begin{cases} 2\rho F = k_{22}d_{i,j}d_{i,j} + (k_{11} - k_{22} - k_{24})d_{i,i}d_{j,j} + (k_{33} - k_{22})d_i d_j d_{k,i} d_{kj} + k_{24}d_{i,j}d_{j,i}, \\ \hat{\sigma}'_{ji} = \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 d_j d_k A_{ki} + \mu_6 d_i d_k A_{kj}, \\ \hat{g}_i = \lambda_1 N_i + \lambda_2 d_j A_{ji}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \omega_i = \dot{d}_i = \dfrac{\partial d_i}{\partial t} + u \cdot \nabla d_i, \\ N_i = \omega_i + \omega_{ki} d_k, \\ N_{ij} = \omega_{i,j} + \omega_{ki} d_{k,j}, \end{cases} \tag{1.4}$$

where

$$2A_{ij} = u_{i,j} + u_{j,i}, \quad 2\omega_{i,j} = u_{i,j} - u_{j,i}.$$

On the basis of the second law of thermodynamics and Onsager reciprocal relation, one obtain

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6 = -(\mu_2 + \mu_3).$$

The nonlinear constraint $|d| = 1$ can also be relaxed by using the Ginzburg-Landau approximation, that is, instead of the restriction $|d| = 1$, we add the term $\frac{1}{\varepsilon^2}(|d|^2 - 1)^2$ in ρF. In addition, to further simplify the calculation, one take $\rho_1 = 0, \beta_j = 0, \gamma = 0, F_i = 0$ and $\rho F = |\nabla d|^2 + \frac{1}{\varepsilon^2}(|d|^2 - 1)^2$, choose the domain $\Omega = \mathbb{R}^3$, obtain the simplified model of nematic liquid crystals:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu\Delta u - (\mu + \lambda)\nabla\nabla \cdot u + \nabla p(\rho) = -\Delta d \cdot \nabla d, \\ d_t + u \cdot \nabla d = \Delta d - f(d), \end{cases} \tag{1.5}$$

with the following initial conditions

$$\rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \quad d(x, 0) = d_0(x), \quad |d_0(x)| = 1, \tag{1.6}$$

and

$$\rho_0 - \bar{\rho} \in H^N(\mathbb{R}^3), \quad u_0 \in H^N(\mathbb{R}^3), \quad d_0 - \omega_0 \in H^N(\mathbb{R}^3), \tag{1.7}$$

for any integer $N \geq 3$ with a fixed vector $\omega_0 \in \mathbb{S}^2$, that is, $|\omega_0| = 1$. In this paper, we assume that $f(d) = \frac{1}{\varepsilon^2}(|d|^2 - 1)d$ $(\varepsilon > 0)$ is the Ginzburg-Landau approximation and the pressure $p = p(\rho)$ is a smooth function in a neighborhood of $\bar\rho$ with $p'(\bar\rho) > 0$ for $\bar\rho > 0$. Moreover, μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid, respectively. As usual, the following inequalities hold:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

The study of liquid crystals can be traced back to Ericksen [3, 4] and Leslie [10, 11] in the 1960s. Since then, there is a huge amount of literature on this topic. For the incompressible case, we refer the author to [2, 6, 12, 13, 20] and the reference therein. There are also many papers related to the compressible case, see for instance, [1, 5, 7, 8, 17, 19] and the reference cited therein.

In [19], the authors rewrote system (1.5) in the perturbation form as

$$\begin{cases} \varrho_t + \bar\rho\nabla\cdot u = -\varrho\nabla\cdot u - u\cdot\nabla\varrho, \\ u_t - \bar\mu\Delta u - (\bar\mu+\bar\lambda)\nabla\nabla\cdot u + \gamma\bar\rho\nabla\varrho \\ \qquad\qquad = -u\cdot\nabla u - h(\varrho)(\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\nabla\cdot u) - g(\varrho)\nabla\varrho - \phi(\Delta d\cdot\nabla d), \\ d_t + u\cdot\nabla d = \Delta d - f(d), \end{cases} \tag{1.8}$$

where $\varrho = \rho - \bar\rho$, $\bar\mu = \frac{\mu}{\bar\rho}$, $\bar\lambda = \frac{\lambda}{\bar\rho}$, $\gamma = \frac{p'(\bar\rho)}{\bar\rho^2}$ and the nonlinear functions of ϱ are defined by

$$h(\varrho) = \frac{\varrho}{\varrho+\bar\rho}, \quad g(\varrho) = \frac{p'(\varrho+\bar\rho)}{\varrho+\bar\rho} - \frac{p'(\bar\rho)}{\bar\rho}, \quad \phi(\varrho) = \frac{1}{\rho+\bar\rho}.$$

We remark that the functions $h(\varrho)$, $g(\varrho)$ and $\phi(\varrho)$ satisfy (see [19])

$$|h(\varrho)|, |g(\varrho)| \leq C|\varrho|, \quad |\phi^{(l)}(\varrho)|, |h^{(k)}(\varrho)|, |g^{(k)}(\varrho)| \leq C \text{ for any } l \geq 0,\ k \geq 1. \tag{1.9}$$

Wei, Li and Yao [19] obtained the small initial data global well-posedness provided that $\|\varrho_0\|_{H^3} + \|u_0\|_{H^3} + \|d_0 - \omega_0\|_{H^4}$ is sufficiently small. Moreover, the authors also showed the optimal decay rates of higher order spatial derivatives of of strong solutions provided that $(\varrho_0, u_0, \nabla d_0) \in \dot H^{-s}$ for some $s \in [0, \frac{1}{2}]$.

Next, we introduce the main results in [19]:

Lemma 1.1. *(Small initial data global well-posedness [19]) Assume that $N \geq 3$ and $(\varrho_0, u_0, d_0 - \omega_0) \in H^N(\mathbb{R}^3)\times H^N(\mathbb{R}^3)\times H^{N+1}(\mathbb{R}^3)$. Then for a unit vector ω_0, there exists a positive constant δ_0 such that if*

$$\|\varrho_0\|_{H^3} + \|u_0\|_{H^3} + \|d_0 - \omega_0\|_{H^4} \leq \delta_0, \tag{1.10}$$

then problem (1.8) has a unique global solution $(\varrho(t), u(t), d(t))$ satisfying that for all $t \geq 0$,

$$\frac{d}{dt}\left(\|\varrho\|_{H^N}^2 + \|u\|_{H^N}^2 + \|\nabla d\|_{H^N}^2\right) + C_0(\|\nabla\varrho\|_{H^{N-1}}^2 + \|\nabla u\|_{H^N}^2 + \|\nabla\nabla d\|_{H^N}^2) \leq 0. \tag{1.11}$$

Lemma 1.2. *(Decay estimates [19]) Assume that all the assumptions of Lemma 1.1 hold. Then, if $(\varrho_0, u_0, \nabla d_0) \in \dot H^{-s}$ for some $s \in [0, \frac{1}{2}]$, we have*

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d(t)\|_{L^2}^2 \leq C_1, \quad \forall t \geq 0, \tag{1.12}$$

and

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1}d\|_{H^{N-l}} \leq C_2(1+t)^{-\frac{l+s}{2}}, \quad \forall t \geq 0 \text{ and } l = 0, 1, \cdots, N-1. \tag{1.13}$$

The main purpose of this paper is to improve the decay results in [19]. First, we give a remark on the symbol stipulations of this paper.

Remark 1.3. In this paper, we use $H^k(\mathbb{R}^3)$ ($k \in \mathbb{R}$), to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$, and $L^p(\mathbb{R}^3)$ ($1 \leq p \leq \infty$) to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. We also introduce the homogeneous negative index Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$:

$$\dot{H}^{-s}(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3) : \||\xi|^{-s}\hat{f}(\xi)\|_{L^2} < \infty\}$$

endowed with the norm $\|f\|_{\dot{H}^{-s}} := \||\xi|^{-s}\hat{f}(\xi)\|_{L^2}$. The symbol ∇^l with an integer $l \geq 0$ stands for the usual spatial derivatives of order l. For instance, we define

$$\nabla^l z = \{\partial_x^\alpha z_i \| \alpha| = l, i = 1,2,3\}, \quad z = (z_1, z_2, z_3).$$

If $l < 0$ or l is not a positive integer, ∇^l stands for Λ^l defined by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\hat{f}$ is the Fourier transform of f. Besides, C and C_i ($i = 0,1,2,\cdots$) will represent generic positive constants that may change from line to line even if in the same inequality. The notation $A \lesssim B$ means that $A \leq CB$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem.

It is worth pointing out that in [19], the authors consider problem (1.5) in 3D case, the negative Sobolev norms were shown to be preserved along time evolution and enhance the decay rates. However, because the Ginzburg-Landau approximation term is difficulty to control, only $s \in [0, \frac{1}{2}]$ were considered in [19]. In this paper, we ovcome the difficult caused by Ginzburg-Landau approximation, assume that $s \in [0, \frac{3}{2})$, obtain the optimal decay rates of higher order spatial derivatives of strong solutions for problem (1.5). Our main results are stated in the following theorem.

Theorem 1.4. *Assume that all the assumptions of Lemma 1.1 hold. Then, if* $(\varrho_0, u_0, \nabla d_0) \in \dot{H}^{-s}$ *for some* $s \in [0, \frac{3}{2})$, *we have*

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d(t)\|_{L^2}^2 \leq C_0, \quad \forall t \geq 0, \tag{1.14}$$

and

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1} d\|_{H^{N-l}} \leq C(1+t)^{-\frac{l+s}{2}}, \quad \text{for } l = 0,1,\cdots,N-1, \ \forall t \geq 0. \tag{1.15}$$

Note that the Hardy-Littlewood-Sobolev theorem implies that for $p \in (1,2]$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{3}{2})$. Then, on the basis of Lemma 1.2 and Theorem 1.4, we obtain the optimal decay estimates for system (1.8).

Corollary 1.5. *Under the assumptions of Lemma 1.2 and Theorem 1.4, if we replace the* $\dot{H}^{-s}(\mathbb{R}^3)$ *assumption by*

$$(\varrho, u_0, \nabla d_0) \in L^p(\mathbb{R}^3), \quad 1 < p \leq 2,$$

then for $l = 0,1,\cdots,N-1$,, *the following decay estimate holds:*

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1} d\|_{H^{N-l}} \leq C(1+t)^{-\left[\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{l}{2}\right]}, \quad \forall t \geq 0. \tag{1.16}$$

Remark 1.6. Lemma 1.1 shows the global well-posedness of strong solutions for system (1.5) provided that the smallness assumption (1.10) holds. One can use the energy method to obtain the higher order energy estimates for the solution to prove this lemma (see [19]). We remark that the negative Sobolev norm estimates did not appear in the proving process of Lemma 1.1, it is only used in the decay estimates. Hence, the value of s in Lemma 1.2 and Theorem 1.4 do not affect the energy estimates (1.10) and (1.11). And those two estimates hold for both Lemma 1.2 and Theorem 1.4.

Remark 1.7. The main purpose of this paper is to prove Theorem 1.4 and Corollary 1.5 on the asymptotic behavior of strong solutions for a compressible Ericksen-Leslie system. We remark that the global well-posedness and asymptotic behavior of solutions are important for the study of nematic liquid crystals system. Thanks to the above properties of solutions, one can understand the model more profoundly. Our results maybe useful for the study of nematic liquid crystals.

The structure of this paper is organized as follows. In Section 2, we introduce some preliminary results. The proof of Theorem 1.4 is postponed in Section 3.

2. Preliminaries

We first show a useful Sobolev embedding theorem in the following Lemma 2.1:

Lemma 2.1. *([15]) If* $0 \le s < \frac{3}{2}$*, one have*

$$\|u\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)} \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^3)} \quad \text{for all } \ u \in \dot{H}^s(\mathbb{R}^3). \tag{2.1}$$

In [14], the author proved the following Gagliardo-Nirenberg inequality:

Lemma 2.2. *([14]) Let* $0 \le m, \alpha \le l$*, then we have*

$$\|\nabla^\alpha f\|_{L^p(\mathbb{R}^3)} \lesssim \|\nabla^m f\|_{L^q(\mathbb{R}^3)}^{1-\theta} \|\nabla^l f\|_{L^r(\mathbb{R}^3)}^{\theta}, \tag{2.2}$$

where $\theta \in [0, 1]$ *and* α *satisfies*

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta. \tag{2.3}$$

Here, when $p = \infty$*, we require that* $0 < \theta < 1$.

One also introduce the Kato-Ponce inequality which is of great importance in our paper.

Lemma 2.3. *([9]) Let* $1 < p < \infty$*,* $s > 0$*. There exists a positive constant C such that*

$$\|\nabla^s(fg)\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^3)} \|\nabla^s g\|_{L^{p_2}(\mathbb{R}^3)} + \|\nabla^s f\|_{L^{q_1}(\mathbb{R}^3)} \|g\|_{L^{q_2}(\mathbb{R}^3)}, \tag{2.4}$$

where $p_2, q_2 \in (1, \infty)$ *satisfying* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

The Hardy-Littlewood-Sobolev theorem implies the following L^p type inequality:

Lemma 2.4. *([16]) Let* $0 \le s < \frac{3}{2}$*,* $1 < p \le 2$ *and* $\frac{1}{2} + \frac{s}{3} = \frac{1}{p}$*, then*

$$\|f\|_{\dot{H}^{-s}(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{2.5}$$

In the end, we introduce the special Sobolev interpolation lemma, which will be used in the proof of Theorem 1.4.

Lemma 2.5. *([18]) Let* $s \ge 0$ *and* $l \ge 0$*, then*

$$\|\nabla^l f\|_{L^2(\mathbb{R}^3)} \le \|\nabla^{l+1} f\|_{L^2(\mathbb{R}^3)}^{1-\theta} \|f\|_{\dot{H}^{-s}(\mathbb{R}^3)}^{\theta}, \quad \text{with } \theta = \frac{1}{l+1+s}. \tag{2.6}$$

3. Proof of Theorem 1.4

Equation $(1.8)_3$ can be rewritten as

$$(d-\omega_0)_t - \Delta(d-\omega_0) = -u\cdot\nabla(d-\omega_0) - [f(d)-f(\omega_0)]. \tag{3.1}$$

In [19], the authors proved the L^2-norm estimate of $d-\omega_0$ provided that the assumptions of Lemma 1.1 hold.

Lemma 3.1. *([19]) Assume that all the assumptions of Lemma 1.1 hold. Then, the solution of (3.1) satisfies*

$$\|d-\omega_0\|_{L^2}^2 + \int_0^t \|\nabla(d-\omega_0)\|_{L^2}^2 \le \|d_0-\omega_0\|_{L^2}^2. \tag{3.2}$$

In the following, we prove the decay estimates of strong solutions for system (1.8). The case $s\in[0,\frac{1}{2}]$ was shown in Lemma 1.2, one only need to consider the case $s\in(\frac{1}{2},\frac{3}{2})$. We first derive the evolution of the negative Sobolev norms of the solution.

Lemma 3.2. *Under the assumptions of Lemma 1.1, if $s\in(\frac{1}{2},\frac{3}{2})$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(\gamma|\Lambda^{-s}\varrho|^2+|\Lambda^{-s}u|^2+|\Lambda^{-s}\nabla d|^2)dx + C\int_{\mathbb{R}^3}(|\nabla\Lambda^{-s}\nabla u|^2+|\Lambda^{-s}\nabla^2 d|^2)dx\\
\le& C\|\nabla d\|_{H^1}^2(\|\Lambda^{-s}u\|_{L^2}+\|\Lambda^{-s}\varrho\|_{L^2}+\|\Lambda^{-s}\nabla d\|_{L^2})\\
&+(\|\varrho\|_{L^2}+\|u\|_{L^2}+\|\nabla d\|_{L^2})^{s-\frac{1}{2}}(\|\nabla\varrho\|_{H^1}+\|\nabla u\|_{H^1}+\|\nabla^2 d\|_{H^1})^{\frac{5}{2}-s}\\
&\times(\|\Lambda^{-s}u\|_{L^2}+\|\Lambda^{-s}\varrho\|_{L^2}+\|\Lambda^{-s}\nabla d\|_{L^2}).
\end{aligned} \tag{3.3}$$

Proof. Applying Λ^{-s} to $(1.8)_1$, $(1.8)_2$, $\Lambda^{-s}\nabla$ to $(1.8)_3$, multiplying the resulting identities by $\gamma\Lambda^{-s}\varrho$, $\Lambda^{-s}u$ and $\Lambda^{-s}\nabla d$ respectively, summing up and integrating over $\mathbb{R}^3$ by parts, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\gamma|\Lambda^{-s}\varrho|^2+|\Lambda^{-s}u|^2+|\Lambda^{-s}\nabla d|^2)dx\\
&+\int_{\mathbb{R}^3}(\bar{\mu}|\nabla\Lambda^{-s}u|^2+(\bar{\mu}|\nabla\Lambda^{-s}u|^2+(\bar{\mu}+\bar{\lambda})|\nabla\cdot\Lambda^{-s}u|^2+|\Lambda^{-s}\nabla^2 d|^2)dx\\
=&\int_{\mathbb{R}^3}\gamma\Lambda^{-s}(-\varrho\nabla\cdot u-u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho\\
&-\Lambda^{-s}[u\cdot\nabla u+h(\varrho)(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\nabla\cdot u)+g(\varrho)\nabla\varrho+\phi(\Delta d\cdot\nabla d)]\cdot\Lambda^{-s}u\\
&-\Lambda^{-s}\nabla(u\cdot\nabla d+f(d))\cdot\Lambda^{-s}\nabla d dx\\
=&K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8.
\end{aligned} \tag{3.4}$$

Note that $s\in(\frac{1}{2},\frac{3}{2})$, it is easy to see that $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\in(2,6)$. For the terms K_1, by using Lemmas 2.2 and 2.4, Hölder's inequality, Young's inequality together with the estimates established in Lemma 1.1, we deduce that

$$\begin{aligned}
K_1=&-\int_{\mathbb{R}^3}\gamma\Lambda^{-s}(\varrho\nabla\cdot u)\cdot\Lambda^{-s}\varrho dx \le C\|\Lambda^{-s}(\varrho\nabla\cdot u)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|\varrho\nabla\cdot u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2}\le C\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned} \tag{3.5}$$

similarly, for K_2–K_6, we have

$$
\begin{aligned}
K_2 =& -\int_{\mathbb{R}^3} \gamma \Lambda^{-s}(u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho dx \le C\|\Lambda^{-s}(u\cdot\nabla\varrho)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|u\cdot\nabla\varrho\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2} \le C\|u\|_{L^{\frac{3}{s}}}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
K_3 =& -\int_{\mathbb{R}^3} \gamma \Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}\varrho dx \le C\|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|u\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2} \le C\|u\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
K_4 =& -\int_{\mathbb{R}^3} \Lambda^{-s}[h(\varrho)(\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\nabla\cdot u)]\cdot\Lambda^{-s}\varrho dx\\
\le& \|\Lambda^{-s}[h(\varrho)(\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\nabla\cdot u)]\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|h(\varrho)(\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\nabla\cdot u)\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|h(\varrho)\|_{L^{\frac{3}{s}}}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \le C\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
K_5 =& -\int_{\mathbb{R}^3} \Lambda^{-s}[g(\varrho)\nabla\varrho]\cdot\Lambda^{-s}\varrho dx \le \|\Lambda^{-s}[g(\varrho)\nabla\varrho]\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|g(\varrho)\nabla\varrho\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|g(\varrho)\|_{L^{\frac{3}{s}}}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2} \le C\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2}\\
\le& C\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla\varrho\|_{L^2}\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
K_6 =& -\int_{\mathbb{R}^3} \Lambda^{-s}(\phi(\varrho)\nabla d\cdot\Delta d)\cdot\Lambda^{-s}u dx\\
\le& C\|\Lambda^{-s}(\phi(\varrho)\nabla d\cdot\Delta d)\|_{L^2}\|\Lambda^{-s}u\|_{L^2}\\
\le& C\|\phi(\varrho)\nabla d\cdot\Delta d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}u\|_{L^2}\\
\le& C\|\phi(\varrho)\|_{L^\infty}\|\nabla d\cdot\Delta d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}u\|_{L^2}\\
\le& C\|\nabla d\cdot\Delta d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}u\|_{L^2} \le C\|\nabla d\|_{L^{\frac{3}{s}}}\|\Delta d\|_{L^2}\|\Lambda^{-s}u\|_{L^2}\\
\le& C\|\nabla d\|_{L^2}^{s-\frac{1}{2}}\|\Delta d\|_{L^2}^{\frac{3}{2}-s}\|\Delta d\|_{L^2}\|\Lambda^{-s}u\|_{L^2},
\end{aligned} \tag{3.10}
$$

where we have used the fact (1.9) in (3.8)–(3.10). Next, by using Lemmas 2.2–2.4, Hölder's inequality, Young's inequality together with Lemma 1.1 on the energy estimates of the solutions, it yields that

$$
\begin{aligned}
K_7 =& -\int_{\mathbb{R}^3} \Lambda^{-s}\nabla(u\cdot\nabla d)\cdot\Lambda^{-s}\nabla d dx\\
\le& C\|\Lambda^{-s}(\nabla u\cdot\nabla d + u\cdot\nabla^2 d)\|_{L^2}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C(\|\nabla u\cdot\nabla d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} + \|u\cdot\nabla^2 d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}})\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C(\|\nabla d\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2} + \|u\|_{L^{\frac{3}{s}}}\|\nabla^2 d\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C(\|\nabla d\|_{L^2}^{s-\frac{1}{2}}\|\nabla^2 d\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2} + \|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla^2 d\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2}.
\end{aligned}
\tag{3.11}
$$

For K_8, we first consider $s\in(\frac{1}{2},1)$. Thanks to Lemma 2.2, one easily obtain

$$
\|\Lambda^{2-s}d\|_{L^2} + \|\Lambda^{2-s}(d-\omega_0)\|_{L^2} + \|\Lambda^{2-s}(d+\omega_0)\|_{L^2} \le C\|\nabla d\|_{L^2}^{s}\|\nabla^2 d\|_{L^2}^{1-s} \le C(\|\nabla d\|_{L^2} + \|\nabla^2 d\|_{L^2}).
$$

Then, by Hölder's inequality, Young's inequality, the facts $|d| < 1$, $|\omega_0| = 1$ together with Lemmas 1.1, 2.2 and 2.3, we derive that

$$
\begin{aligned}
K_8 =& -\int_{\mathbb{R}^3} \Lambda^{-s}\nabla\cdot[(d+\omega_0)(d-\omega_0)d]\cdot\Lambda^{-s}\nabla d dx\\
\le& C\|\Lambda^{-s}\nabla\cdot[(d+\omega_0)(d-\omega_0)d]\|_{L^2}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C[\|d+\omega_0\|_{L^6}\|d-\omega_0\|_{L^6}\|\Lambda^{1-s}d\|_{L^6} + \|d+\omega_0\|_{L^6}\|d\|_{L^6}\|\Lambda^{1-s}(d-\omega_0)\|_{L^6}\\
&+ \|d-\omega_0\|_{L^6}\|d\|_{L^6}\|\Lambda^{1-s}(d+\omega_0)\|_{L^6}]\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|\nabla d\|_{L^2}^2(\|\Lambda^{2-s}d\|_{L^2} + \|\Lambda^{2-s}(d-\omega_0)\|_{L^2} + \|\Lambda^{2-s}(d+\omega_0)\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|\nabla d\|_{L^2}(\|\nabla d\|_{L^2} + \|\nabla^2 d\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C(\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\Lambda^{-s}\nabla d\|_{L^2}.
\end{aligned}
\tag{3.12}
$$

Moreover, if $s\in(1,\frac{3}{2})$, the following inequality holds:

$$
\begin{aligned}
K_8 \le& C\|\Lambda^{-s+1}[(d+\omega_0)(d-\omega_0)d]\|_{L^2}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|(d+\omega_0)(d-\omega_0)d\|_{L^{\frac{1}{\frac{1}{2}+\frac{s-1}{3}}}}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|d+\omega_0\|_{L^\infty}\|d-\omega_0\|_{L^2}\|d\|_{L^{\frac{3}{s-1}}}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|\nabla d\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}}\|d-\omega_0\|_{L^2}\|\nabla d\|_{L^2}^{(s-1)+\frac{1}{2}}\|\nabla^2 d\|_{L^2}^{\frac{1}{2}-(s-1)}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C\|\nabla d\|_{L^2}^{s}\|\nabla^2 d\|_{L^2}^{2-s}\|\Lambda^{-s}\nabla d\|_{L^2}\\
\le& C(\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\Lambda^{-s}\nabla d\|_{L^2}.
\end{aligned}
\tag{3.13}
$$

Combining (3.4)–(3.12) together, we obtain (3.3) and complete the proof. □

Now, we give the proof of our main results.

Proof of Theorem 1.4. First of all, the sketch of proof for the decay estimate with $s\in[0,\frac{1}{2}]$ will be derived in the following. Note that this part follows more or less the lines of [19], so that we do note claim originality here. Then, by using this proved estimate, one can obtain the decay results for $s\in(\frac{1}{2},\frac{3}{2})$.

Now, consider the decay for $s \in [0, \frac{1}{2}]$. We first establish the negative Sobolev norm estimates for the strong solutions, obtain one important inequality:

$$\begin{aligned}&\frac{d}{dt}(\gamma\|\Lambda^{-s}\varrho\|_{L^2}^2 + \|\Lambda^{-s}u\|_{L^2}^2 + \|\Lambda^{-s}\nabla d\|_{L^2}^2) + C(\|\Lambda^{-s}\nabla u\|_{L^2}^2 + \|\Lambda^{-s}\nabla^2 d\|_{L^2}^2)\\ \lesssim&(\|\nabla(\varrho, u)\|_{H^1}^2 + \|\nabla d\|_{H^2}^2)(\|\Lambda^{-s}\varrho\|_{L^2}^2 + \|\Lambda^{-s}u\|_{L^2}^2 + \|\Lambda^{-s}\nabla d\|_{L^2}^2).\end{aligned}$$

Then, define

$$\mathcal{E}_{-s}(t) = \|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d(t)\|_{L^2}^2,$$

we deduce from (3.2) and (3.3) that for $s \in [0, \frac{1}{2}]$,

$$\mathcal{E}_{-s}(t) \leq \mathcal{E}_{-s}(0) + C\int_0^t (\|\nabla(\varrho, u)\|_{H^1}^2 + \|\nabla d\|_{H^2}^2)\sqrt{\mathcal{E}_{-s}(t)}d\tau \leq C(1 + \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(t)}),$$

which implies (1.12) for $s \in [0, \frac{1}{2}]$, i.e.,

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d(t)\|_{L^2}^2 \leq C_0. \tag{3.14}$$

Moreover, if $l = 1, 2, \cdots, N-1$, we may use Lemma 2.4 to have

$$\|\nabla^{l+1}f\|_{L^2} \geq C\|\Lambda^{-s}f\|_{L^2}^{-\frac{1}{l+s}}\|\nabla^l f\|_{L^2}^{1+\frac{1}{l+s}}. \tag{3.15}$$

Then, by (3.14) and (3.15), it yields that

$$\|\nabla^l(\nabla\varrho, \nabla u, \nabla^2 d)\|_{L^2}^2 \geq C(\|\nabla^l(\varrho, u, \nabla d)\|_{L^2}^2)^{1+\frac{1}{l+s}}.$$

Hence, for $l = 1, 2, \cdots, N-1$,

$$\|\nabla^l(\nabla\varrho, \nabla u, \nabla^2 d)\|_{H^{N-l-1}}^2 \geq C(\|\nabla^l(\varrho, u, \nabla d)\|_{H^{N-l}}^2)^{1+\frac{1}{l+s}}. \tag{3.16}$$

Thus, we deduce from (1.11) the following inequality

$$\begin{aligned}&\frac{d}{dt}(\|\nabla^l\varrho\|_{H^{N-l}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^{l+1}d\|_{H^{N-l}}^2)\\ &+ C_0(\|\nabla^l\varrho\|_{H^{N-l}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^{l+1}d\|_{H^{N-l}}^2)^{1+\frac{1}{l+s}} \leq 0, \text{ for } l = 1, \cdots, N-1.\end{aligned}$$

Solving this inequality directly gives

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1}d\|_{H^{N-l}} \leq C(1+t)^{-\frac{l+s}{2}}, \quad \text{for } l = 1, \cdots, N-1. \tag{3.17}$$

Then, by (3.14), (3.17) and the interpolation, we obtain the following inequality holds for $s \in [0, \frac{1}{2}]$:

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1}d\|_{H^{N-l}} \leq C(1+t)^{-\frac{l+s}{2}}, \quad \text{for } l = 0, 1, \cdots, N-1. \tag{3.18}$$

Second, we consider the decay estimate for $s \in (\frac{1}{2}, \frac{3}{2})$. Notice that the arguments for $s \in [0, \frac{1}{2}]$ can not be applied to this case. However, observing that we have $\varrho_0, u_0, \nabla d_0 \in \dot{H}^{-\frac{1}{2}}$ hold since $\dot{H}^{-s} \bigcap L^2 \subset$

$\dot{H}^{-s'}$ for any $s' \in [0, s]$, we can deduce from (3.18) for (1.10) and (1.11) with $s = \frac{1}{2}$ that the following estimate holds:

$$\|\nabla^l \varrho\|^2_{H^{N-l}} + \|\nabla^l u\|^2_{H^{N-l}} + \|\nabla^l \nabla d\|^2_{H^{N-l}} \leq C_0(1+t)^{-\frac{1}{2}-l}, \quad \text{for } l = 0, 1, \cdots, N-1. \tag{3.19}$$

Therefore, we deduce from (3.3) and (3.2) that for $s \in (\frac{1}{2}, \frac{3}{2})$,

$$\begin{aligned}
&\mathcal{E}_{-s}(t)\\
\leq& \mathcal{E}_{-s}(0) + C\int_0^t \|\nabla d\|^2_{H^1}\sqrt{\mathcal{E}_{-s}(\tau)}d\tau\\
&+C\int_0^t (\|\varrho\|_{L^2} + \|u\|_{L^2} + \|\nabla d\|_{L^2})^{s-\frac{1}{2}}(\|\nabla\varrho\|_{H^1} + \|\nabla u\|_{H^1} + \|\nabla^2 d\|_{H^1})^{\frac{5}{2}-s}\sqrt{\mathcal{E}_{-s}(\tau)}d\tau\\
\leq& C_0 + C\sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)} + C\int_0^t (1+\tau)^{-\frac{7}{4}+\frac{s}{2}}d\tau \sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)}\\
\leq& C_0 + C\sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)},
\end{aligned} \tag{3.20}$$

which implies that (1.12) holds for $s \in (\frac{1}{2}, \frac{3}{2})$, i.e.,

$$\|\Lambda^{-s}\varrho(t)\|^2_{L^2} + \|\Lambda^{-s}u(t)\|^2_{L^2} + \|\Lambda^{-s}\nabla d(t)\|^2_{L^2} \leq C_0. \tag{3.21}$$

Moreover, thanks to (1.11) and (3.16), we can also obtain the following inequality for $s \in (\frac{1}{2}, \frac{3}{2})$:

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla^l\varrho\|^2_{H^{N-l}} + \|\nabla^l u\|^2_{H^{N-l}} + \|\nabla^{l+1} d\|^2_{H^{N-l}})\\
&+C_0(\|\nabla^l\varrho\|^2_{H^{N-l}} + \|\nabla^l u\|^2_{H^{N-l}} + \|\nabla^{l+1} d\|^2_{H^{N-l}})^{1+\frac{1}{l+s}} \leq 0, \text{ for } l = 1, \cdots, N-1,
\end{aligned}$$

which implies

$$\|\nabla^l\varrho\|_{H^{N-l}} + \|\nabla^l u\|_{H^{N-l}} + \|\nabla^{l+1} d\|_{H^{N-l}} \leq C(1+t)^{-\frac{l+s}{2}}, \quad \text{for } l = 1, \cdots, N-1. \tag{3.22}$$

Next, using (3.21), (3.22), and Lemma 2.5, we easily obtain

$$\begin{aligned}
\|(\varrho, u, \nabla d)\|_{L^2} \leq& C(\|\nabla(\varrho, u, \nabla d)\|_{L^2})^{\frac{s}{1+s}}(\|\Lambda^{-s}(\varrho, u, \nabla d)\|^{\frac{1}{1+s}}_{L^2}\\
\leq& C(\|\nabla(\varrho, u, \nabla d)\|_{L^2})^{\frac{s}{1+s}}\\
\leq& C\left[(1+t)^{-\frac{1+s}{2}}\right]^{\frac{s}{1+s}} = C(1+t)^{-\frac{s}{2}}.
\end{aligned} \tag{3.23}$$

It then follows from (3.22) and (3.23) that

$$\|\varrho\|_{H^{N-l}} + \|u\|_{H^{N-l}} + \|\nabla d\|_{H^{N-l}} \leq C(1+t)^{-\frac{s}{2}}.$$

Hence, we obtain (1.15) for $s \in (\frac{1}{2}, \frac{3}{2})$ and complete the proof. □

4. Conclusions

In this paper, we consider the optimal decay estimates for the higher order derivatives of strong solutions for three-dimensional nematic liquid crystal system. We use the pure energy method, negative Sobolev norm estimates together with the classical Kato-Ponce inequality, Gagliardo-Nirenberg inequality, overcome the difficulties caused by the Ginzburg-Landau approximation and the coupling between the compressible Navier-Stokes equations and the direction equations, obtain the decay estimates. Since the result (1.16) is same to the decay of the heat equation, it is optimal. We remark that our results may attract the attentions of the researchers in the nematic liquid crystals filed.

Acknowledgments

The author would like to thank the anonymous referees and Dr. Xiaopeng Zhao for their helpful suggestions. This paper was supported by the Fundamental Research Funds of Heilongjiang Province (grant No. 145109131).

Conflict of interest

The author declares no conflict of interest.

References

1. G. Q. Chen, A. Majumdar, D. Wang, R. Zhang, Global weak solutions for the compressible active liquid crystal system, *SIAM J. Math. Anal.*, **50** (2018), 3632–3675. https://doi.org/10.1137/17M1156897
2. M. Dai, M. Schonbek, Asymptotic behavior of solutions to the Liquid crystal system in $H^M(\mathbb{R}^3)$, *SIAM J. Math. Anal.*, **46** (2014), 3131–3150. https://doi.org/10.1137/120895342
3. J. L. Ericksen, Conservation laws for liquid crystals, *Trans. Soc. Rheol.*, **5** (1961), 22–34. https://doi.org/10.1122/1.548883
4. J. L. Ericksen, Continuum theory of nematic liquid crystals, *Res. Mech.*, **21** (1987), 381–392. https://doi.org/10.1007/bf01130288
5. J. Fan, F. Li, G. Nakamura, Local well-posedness for a compressible non-isothermal model for nematic liquid crystals, *J. Math. Phys.*, **59** (2018), 031503. https://doi.org/10.1063/1.5027189
6. J. Gao, Q. Tao, Z. Yao, *Strong solutions to the densi-dependent incompressible nematic liquid crystal flows*, *J. Differ. Equations*, **260** (2016), 3691–3748. https://doi.org/10.1016/j.jde.2015.10.047
7. B. Guo, X. Xi, B. Xie, Global well-posedness and decay of smooth solutions to the non-isothermal model for compressible nematic liquid crystals, *J. Differ. Equations*, **262** (2017), 1413–1460. https://doi.org/10.1016/j.jde.2016.10.015
8. T. Huang, C. Wang, H. Wen, Strong solutions of the compressible nematic liquid crystal flow, *J. Differ. Equations*, **252** (2012), 2222–2265. https://doi.org/10.1016/j.jde.2011.07.036

9. T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Commun. Pure Appl. Math.*, **41** (1988), 891–907. https://doi.org/10.1002/cpa.3160410704
10. F. M. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.*, **28** (1968), 265–283. https://doi.org/10.1007/BF00251810
11. F. M. Leslie, Theory of flow phenomena in liquid crystals, *Adv. Liquid Cryst.*, **4** (1979), 1–81. https://doi.org/10.1016/B978-0-12-025004-2.50008-9
12. F. Lin, C. Liu, Existence of solutions for the Ericksen-Leslie system, *Arch. Ration. Mech. Anal.*, **154** (2000), 135–156. https://doi.org/10.1007/s002050000102
13. F. Lin, C. Wang, Global existence of weak solutions of the nematic liquid crystal flow in dimension three, *Commun. Pure Appl. Math.*, **69** (2016), 1532–1571. https://doi.org/10.1002/cpa.21583
14. L. Nirenberg, *On elliptic partial differential equations*, Springer, Berlin, Heidelberg, 1959. https://doi.org/10.1007/978-3-642-10926-3-1
15. J. C. Robinson, J. L. Rodrigo, W. Sadowski, *The three-dimensioanl Navier-Stokes equations*, Cambridge studies in advanced mathematic, Cambridge University Press, Cambridge, 2016. https://doi.org/10.1017/CBO9781139095143
16. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Unversity Press: Princeton, NJ 1970. https://doi.org/10.1112/blms/5.1.121
17. D. Wang, C. Yu, Global weak solution and large-time behavior for the compressible flow of liquid crystal, *Arch. Ration. Mech. Anal.*, **204** (2012), 881–915. https://doi.org/10.1007/s00205-011-0488-x
18. Y. Wang, Decay of the Navier-Stokes-Poisson equations, *J. Differ. Equations*, **253** (2012), 273–297. https://doi.org/10.1016/j.jde.2012.03.006
19. R. Wei, Y. Li, Z. Yao, Decay of the nematic liquid crystal system, *Math. Method. Appl. Sci.*, **39** (2016), 452–474. https://doi.org/10.1002/mma.3494
20. X. Zhao, Space-time decay estimates of solutions to liquid crystal system in $\mathbb{R}^3$, *Commun. Pure Appl. Anal.*, **18** (2019), 1–13. http://dx.doi.org/10.3934/cpaa.2019001

AIMS Press