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Research article

Determine unknown source problem for time fractional pseudo-parabolic equation with Atangana-Baleanu Caputo fractional derivative

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Abstract: In this paper, we consider a pseudo-parabolic equation with the Atangana-Baleanu Caputo fractional derivative. Our main tool here is using fundamental tools, namely the Fractional Tikhonov method and the generalized Tikhonov method, the error estimate is shown. Finally, we provided numerical experiments to prove the correctness of our theory.

Keywords: Caputo derivative; pseudo-parabolic equation; well-posedness; regularity estimates
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1. Introduction

The fractional diffusion equation has been of significant interest for many decades and has many important applications in practice, where mathematical formulas are abstract and can be used to provide procedures for converting outside measurements into information about inside properties thus increasingly interested in the study of inverse problems and unknown sources for partial differential equations [1–11]. So, PDEs with fractional derivatives are a generalization of equations with integer-order partial derivatives, please see in the reference [12–22]. In this paper, we consider time-fractional PDE

$$\begin{cases} {}_{0}^{ABC}D_t^{\alpha}(u(t,x) + k\mathcal{A}u(t,x)) + \mathcal{A}u(t,x) = \theta(t)f(x), & \text{in} \quad (0,T] \times \mathcal{D}, \\ u(t,x) = 0, & \text{on} \quad (0,T] \times \partial\mathcal{D}, \\ u(T,x) = g(x), & \text{in} \quad \mathcal{D}, \end{cases} \tag{1.1}$$

Here the Atangana-Baleanu Caputo derivative ${}_{0}^{\text{ABC}}D_t^{\alpha}u(t,x)$ is defined by

$${}_{0}^{\text{ABC}}D_t^{\alpha}u(t,x) = \frac{\mathcal{L}(\alpha)}{1-\alpha}\int_0^t \frac{\partial u(s,x)}{\partial s} E_{\alpha,1}\Big(\frac{-\alpha(t-s)^{\alpha}}{1-\alpha}\Big)ds, \tag{1.2}$$

where $\mathcal{L}(\alpha)$ satisfies $\mathcal{L}(\alpha) = 1-\alpha+\dfrac{\alpha}{\Gamma(\alpha)}$, and $\mathcal{L}(0) = \mathcal{L}(1) = 1$, and $E_{\alpha,1}\Big(-\frac{\alpha(t-s)^{\alpha}}{1-\alpha}\Big)$ is the Mittag-Leffler function is introduced in Section 2. The problem (1.1) can be considered in the following two cases:

The case $\alpha = 1$**:** System (1.1) is called pseudo-parabolic. There are many works on the well-posedness of the pseudo-parabolic equation with classical derivative, see [23–26], here the condition $u(T,x)$ is replaced by the initial condition $u(0,x)$. An important goal in the scientific community is to investigate the existence, uniqueness, and stability of fractional differential equations (1.1).

The case $\alpha \neq 1$**:** System (1.1) is called the fractional pseudo-parabolic equation. From the given data θ and the measured data at the final time $g \in L^2(\Omega)$, we recovered the source function $f(x)$. The problem (1.1) is severely ill-posed. The problem (1.1) is called the classical pseudo-parabolic equation. The problem (1.1) is considered with the right hand side of function $F(t,x) = \theta(t)f(x)$. In this case, we would like to take a look at some related issues as follows:

- In case $\theta(t) = 1, k = 0$ and D_t^{α} is the Caputo derivative, then (1.1) is called the fractional diffusion equation, and this type has been surveyed a lot, see in [27–29] .
- In case $\theta(t) \neq 1, k = 0$, and D_t^{α} is the Caputo derivative, there are a number of studies on this case, see in [30, 31].
- In case $\theta(t) \neq 1, k \neq 0$, with D_t^{α} is Atangana - Baleanu fractional derivative, until now, according to our understanding, we did not find papers to solve this case.

Our problem recover of a space-dependent source function $f(x)$ along with subject to the homogeneous initial condition. When exact data (θ, g) is given, observed data $(\theta_\epsilon, g_\epsilon)$ will inevitably be noisy by

$$||\theta - \theta_\epsilon||_{L^\infty(0,T)} + ||g - g_\epsilon||_{L^2(\mathcal{D})} \leq \epsilon. \tag{1.3}$$

Regarding the regularization methods, in [32], Ting Wei used a quasi boundary value method. In [33], Tuan-Long-Thinh used the Tikhonov regularization method. In [34], with a general filter method, the authors studied the problem of finding the source distribution for the linear bi-parabolic equation when we have the final observation. In [35], Zhang with the truncation regularization method. In [36], Yang and his group used fractional Tikhonov to identify the initial value problem for a time-fractional diffusion equation. In [37], Cheng and co-authors used iteration regularization to solve a time-fractional inverse diffusion problem in the frequency domain. The Landweber regularization method, see in [38,39]. In [40], Binh and co-authors studied an inverse source problem for the Rayleigh–Stokes problem using the Tikhonov method. In [41], Ma and his group identified the unknown space-dependent source term in a time-fractional diffusion equation by applying the generalized and revised generalized Tikhonov regularization methods.

However, to the best acknowledgment, the Tikhonov regularization method is very famous. Over the past decade, some modified Tikhonov methods have begun to be researched by mathematicians, such as the Fractional Tikhonov regularization method used in this article was considered by Smina Djennadi and his colleagues, see [42], and the generalized Tikhonov method can be found in reference [41]. For the reader's convenience, we would like to outline the main results:

- We give the ill-posedness of our inverse source problem.
- The first goal of this paper is to provide the fractional Tikhonov method to solve this inverse space-dependent source problem for the fractional pseudo-parabolic equation.
- The second goal of this paper is to provide the A generalized Tikhonov to solve the (1.1).
- And then finally, we provide an example of the illustration of the correctness of our theory and conclude this article.

The layout of this article is as follows. We show the preliminaries and Lemma is used. In Section 3, we have the mild solution of Problem (1.1) and the condition stability for Problem (1.1) in Theorem 3.1. In Sections 4 and 5, we have the error estimate between the exact solution and its approximation by the Fractional Tikhonov and The generalized Tikhonov method. In Section 6, we have one example numerical experiment.

2. Preliminaries

Let us recall that the spectral problem

$$\begin{cases} -\Delta e_n(x) = \lambda_n e_n(x), & \text{in } \Omega, \\ e_n(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

admits a family of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_n \le ... \nearrow \infty.$$

For all $r \ge 0$, we define by A^r the following operator

$$A^r h = \sum_{n=1}^{\infty} \langle h, \varphi_n \rangle \lambda_n^r e_n, \quad h \in \mathcal{H}(A^r) = \Big\{ h \in L^2(\mathcal{D}) : \sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \lambda_n^{2r} < \infty \Big\}. \tag{2.2}$$

The domain $\mathcal{H}(A^r)$ is the Banach space equipped with the norm

$$\|h\|_{\mathcal{H}(A^r)} = \Big(\sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \lambda_n^{2r} \Big)^{\frac{1}{2}}, \quad h \in \mathcal{H}(A^r).$$

Definition 2.1 ([43]). *The definition of the Mittag-Leffler function as follows*

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \tag{2.3}$$

where α, β are arbitrary constants.

Lemma 2.1. *([43]) Let $0 < \alpha < 1$. Then there exist positive constants $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that*

$$\frac{\mathcal{A}_1}{1+y} \le E_{\alpha,1}(-y) \le \frac{\mathcal{A}_2}{1+y}, \quad E_{\alpha,\alpha}(-y) \le \frac{\mathcal{A}_3}{1+y}, \text{ for all } y \ge 0, \ \alpha \in \mathbb{R}. \tag{2.4}$$

Lemma 2.2. *For $\lambda > 0, \alpha > 0$ and $m \in \mathbb{N}$, then*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \tag{2.5}$$

$$\frac{d}{dt}(t E_{\alpha,2}(-\lambda t^\alpha)) = E_{\alpha,1}(-\lambda t^\alpha), \tag{2.6}$$

$$\frac{d}{dt}(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = -t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha). \tag{2.7}$$

Lemma 2.3. *[33] For $t > 0$, and $\lambda > 0$, and $0 < \alpha < 1$, we get*

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha),$$

Lemma 2.4. *For $\alpha \in (0, 1)$, we put $\mathcal{D}_\alpha(\lambda_n, k) = \xi_n \alpha \mathcal{L}(\alpha)\big(\mathcal{L}(\alpha) + \lambda_n \xi_n (1-\alpha)\big)^{-2}$, it gives*

$$\mathcal{D}_\alpha(\lambda_n, k) \ge \frac{1}{\lambda_n} \frac{k\alpha\mathcal{L}(\alpha)}{\left(\frac{\mathcal{L}(\alpha)}{\lambda_1 \xi_1} + (1-\alpha)\right)^2}. \tag{2.8}$$

Proof. First of all, we notice that $\lambda_n \xi_n = \frac{\lambda_n}{1+k\lambda_n} = \frac{\lambda_n}{\lambda_n\left(\frac{1}{\lambda_n}+k\right)} \ge \frac{1}{\frac{1}{\lambda_1}+k} = \frac{1}{\lambda_1^{-1}+k}$, and we have

$$\begin{aligned}\mathcal{D}_\alpha(\lambda_n, k) &= \frac{\xi_n \alpha \mathcal{L}(\alpha)}{\big(\mathcal{L}(\alpha) + \lambda_n \xi_n (1-\alpha)\big)^2} \ge \frac{\alpha \mathcal{L}(\alpha)}{\frac{\lambda_n^2}{1+k\lambda_n}\left(\frac{\mathcal{L}(\alpha)}{\lambda_n \xi_n} + (1-\alpha)\right)^2} \\ &= \frac{k\alpha\mathcal{L}(\alpha)}{\lambda_n\left(\frac{\mathcal{L}(\alpha)}{\lambda_n \xi_n} + (1-\alpha)\right)^2} \ge \frac{1}{\lambda_n} \frac{k\alpha\mathcal{L}(\alpha)}{\left(\frac{\mathcal{L}(\alpha)}{\lambda_1 \xi_1} + (1-\alpha)\right)^2}.\end{aligned} \tag{2.9}$$

□

Lemma 2.5. *Let $\alpha \in (0, 1)$, by denoting*

$$\begin{aligned}\mathcal{H}_\alpha(\lambda_n, k) &= \alpha \lambda_n \xi_n \big(\mathcal{L}(\alpha) + \lambda_n \xi_n (1-\alpha)\big)^{-1}, \\ G_\alpha(\lambda_n, k, s) &= E_{\alpha,\alpha}\big(-\mathcal{H}_\alpha(\lambda_n, k)(t-s)^\alpha\big)(t-s)^{\alpha-1}.\end{aligned} \tag{2.10}$$

and $\xi_n = (1 + k\lambda_n)^{-1}$, then we get the following estimate

$$\Big(\frac{1-\alpha}{\alpha}\Big)\Big(T - \frac{\mathcal{A}_2}{\mathcal{H}_\alpha(\lambda_1, k)} \frac{T^{1-\alpha}}{1-\alpha}\Big) \le \int_0^T G_\alpha(\lambda_n, k, s) ds \le \frac{1}{\mathcal{H}_\alpha(\lambda_n, k)}. \tag{2.11}$$

Proof. For $E_{\alpha,\alpha}(-y) \ge 0$ for $0 < \alpha < 1$ and $y \ge 0$, and using Lemma 2.1, we obtain

$$a) \int_0^T G_\alpha(\lambda_n, k, s) ds \ge \frac{1}{\mathcal{H}_\alpha(\lambda_n, k)} \int_0^T \Big(1 - E_{\alpha,1}\big(-\mathcal{H}_\alpha(\lambda_n, k) t^\alpha\big)\Big) dt$$

$$\geq \Big(\frac{1-\alpha}{\alpha}\Big)\Big(\int_0^T dt - \int_0^T E_{\alpha,1}\big(-\mathcal{H}_\alpha(\lambda_n,k)t^\alpha\big)dt\Big)$$

$$\geq \Big(\frac{1-\alpha}{\alpha}\Big)\Big(T - \frac{\mathcal{A}_2}{\mathcal{H}_\alpha(\lambda_1,k)}\int_0^T t^{-\alpha}dt\Big)$$

$$\geq \Big(\frac{1-\alpha}{\alpha}\Big)\Big(T - \frac{\mathcal{A}_2}{\mathcal{H}_\alpha(\lambda_1,k)}\frac{T^{1-\alpha}}{1-\alpha}\Big), \tag{2.12}$$

$$b)\int_0^T G_\alpha(\lambda_n,k,s)ds = -\frac{1}{\mathcal{H}_\alpha(\lambda_n,k)}\int_0^T \frac{d}{ds}\Big(E_{\alpha,1}\big(-\mathcal{H}_\alpha(\lambda_n,k)(t-s)^\alpha\big)\Big)ds$$

$$= \frac{1}{\mathcal{H}_\alpha(\lambda_n,k)}\big(\mathcal{H}_\alpha(\lambda_n,k)\big)^{-1}\Big(1 - E_{\alpha,1}\big(-\mathcal{H}_\alpha(\lambda_n,k)t^\alpha\big)\Big)$$

$$\leq \frac{1}{\mathcal{H}_\alpha(\lambda_n,k)}. \tag{2.13}$$

□

Lemma 2.6 ([42]). *For constant $\mathcal{N} > 0$, $s \geq \lambda_1$, and $\frac{1}{2} \leq \sigma < 1$, then we get*

$$\frac{s}{\mathcal{N}^{2\sigma} + [\gamma(\epsilon)]s^{2\sigma}} \leq C_2[\gamma(\epsilon)]^{-\frac{1}{2\sigma}}, \tag{2.14}$$

where $C_2 = C_2(\sigma, \mathcal{N}) > 0$ is independent on s and σ.

Lemma 2.7 ([42]). *For constant $s \geq \lambda_1$, with $\frac{1}{2} \leq \sigma < 1$, we obtain*

$$\frac{[\gamma(\epsilon)]s^{(2\sigma-r)}}{\mathcal{N}^{2\sigma} + s^{2\sigma}[\gamma(\epsilon)]} \leq \begin{cases} C_4(\sigma, r, \mathcal{N})[\gamma(\epsilon)]^{\frac{r}{2\sigma}}, 0 < r \leq 2\sigma, \\ C_5(\sigma, r, \lambda_1)[\gamma(\epsilon)] \qquad r > 2\sigma. \end{cases} \tag{2.15}$$

where $C_4 = C_4(\sigma, r, \mathcal{N})$ and $C_5 = C_5(\sigma, r, \mathcal{N}, \lambda_1)$.

Lemma 2.8 ([42]). *For constant $s \geq \lambda_1 > 0$, and $\frac{1}{2} \leq \sigma < 1$, we get*

$$\frac{[\gamma(\epsilon)]s^{(2\sigma-1-r)}}{\mathcal{N}^{2\sigma} + s^{2\sigma}[\gamma(\epsilon)]} \leq \begin{cases} C_6(\sigma, r, \mathcal{N})[\gamma(\epsilon)]^{\frac{r+1}{2\sigma}}, 0 < r \leq 2\sigma - 1, \\ C_7(\sigma, r, \mathcal{N}, \lambda_1)[\gamma(\epsilon)] \qquad r > 2\sigma - 1. \end{cases} \tag{2.16}$$

Lemma 2.9. *Let θ_0, θ_1 are positive constants such that $\theta_0 < \theta < \theta_1$. By choosing $\epsilon \in \Big(0, \dfrac{\theta_1}{4}\Big)$, and $B(\theta_0, \theta_1) = \theta_1 + \dfrac{\theta_0}{4}$, we obtain*

$$\frac{\theta_0}{4} \leq |\theta_\epsilon(t)| \leq \mathcal{B}(\theta_0, \theta_1). \tag{2.17}$$

Proof. This provision was detailed in [5]. We omit it here. □

3. The inverse source problem in L^2 case

In this section, we consider a linear problem as follows

$$\begin{cases} {}_0^{ABC}D^{\alpha}(u(t,x)+k\mathcal{A}u(t,x))+\mathcal{A}u(t,x)=F(t,x), & \text{in} \quad (0,T]\times\mathcal{D}, \\ u(t,x)=0, & \text{on} \quad (0,T]\times\partial\mathcal{D}, \\ u(0,x)=u_0(x), & \text{in} \quad \mathcal{D}, \end{cases} \tag{3.1}$$

where u_0 and F are given functions. Let $u(t,x)=\sum_{n=1}^{\infty}u_n(t)e_n(x)$ be the Fourier series in $L^2(\Omega)$ with $u_n(t)=\int_{\mathcal{D}}u(t,x)e_n(x)dx$, then we have the fractional integro-differential equation involving the Atangana-Baleanu fractional derivative in the form

$${}_0^{ABC}D_t^{\alpha}(1+k\lambda_n)u_n(t)+\lambda_n u_n(t)=F_n(t), \tag{3.2}$$

along with the following condition $u_n(0)=\int_{\mathcal{D}}u(0,x)e_n(x)dx$, and $F_n(0)=\lambda_n u_n(0)$, using Lemma 3.4 in [43], the solution (3.1) is written by Fourier series as follows

$$\begin{aligned} u_n(t)=&\frac{\mathcal{L}(\alpha)}{\mathcal{L}(\alpha)+\frac{\lambda_n}{1+k\lambda_n}(1-\alpha)}E_{\alpha,1}\Big(\frac{-\alpha\frac{\lambda_n}{1+k\lambda_n}t^{\alpha}}{\mathcal{L}(\alpha)+\frac{\lambda_n}{1+k\lambda_n}(1-\alpha)}\Big)u_{0,n} \\ &+\Big(\frac{1}{1+k\lambda_n}\Big)\frac{1-\alpha}{\mathcal{L}(\alpha)+\frac{\lambda_n}{1+k\lambda_n}(1-\alpha)}F_n(t) \\ &+\Big(\frac{1}{1+k\lambda_n}\Big)\frac{\alpha\mathcal{L}(\alpha)}{\big(\mathcal{L}(\alpha)+\frac{\lambda_n}{1+k\lambda_n}(1-\alpha)\big)^2}\int_0^t E_{\alpha,\alpha}\Big(\frac{-\alpha\frac{\lambda_n}{1+k\lambda_n}(t-s)^{\alpha}}{\mathcal{L}(\alpha)+\frac{\lambda_n}{1+k\lambda_n}(1-\alpha)}\Big)(t-s)^{\alpha-1}F_n(s)ds. \end{aligned} \tag{3.3}$$

From now on, for a shorter, we put $\xi_n=(1+k\lambda_n)^{-1}$, this implies that

$$\begin{aligned} u(t,x)=&\sum_{n=1}^{\infty}\frac{\mathcal{L}(\alpha)}{\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)}E_{\alpha,1}\Big(\frac{-\alpha\lambda_n\xi_n t^{\alpha}}{\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)}\Big)\Big(\int_{\mathcal{D}}u(0,x)e_n(x)dx\Big)e_n(x), \\ &+\sum_{n=1}^{\infty}\frac{\xi_n(1-\alpha)}{\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)}\Big(\int_{\mathcal{D}}F_n(t)e_n(x)dx\Big)e_n(x) \\ &+\sum_{n=1}^{\infty}\frac{\xi_n\alpha\mathcal{L}(\alpha)}{(\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha))^2}\int_0^t E_{\alpha,\alpha}\Big(\frac{-\alpha\lambda_n\xi_n(t-s)^{\alpha}}{\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)}\Big)(t-s)^{\alpha-1}\Big(\int_{\mathcal{D}}F_n(s)e_n(x)dx\Big)dse_n(x). \end{aligned} \tag{3.4}$$

Let $t=T$ and noting that $u(0,x)=0$, $F(t,x)=\theta(t)f(x)$, and $\theta(T)=0$, we find that

$$\int_{\mathcal{D}}g(x)e_n(x)dx=\frac{\Big(\int_{\mathcal{D}}f(x)e_n(x)dx\Big)\xi_n\alpha\mathcal{L}(\alpha)}{\Big(\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)\Big)^2}\int_0^T E_{\alpha,\alpha}\Big(\frac{-\alpha\lambda_n\xi_n(T-s)^{\alpha}}{\mathcal{L}(\alpha)+\lambda_n\xi_n(1-\alpha)}\Big)(T-s)^{\alpha-1}\theta(s)ds. \tag{3.5}$$

Let us denote and (2.10), we have

$$\mathcal{D}_\alpha(\lambda_n, k) = \xi_n \alpha \mathcal{L}(\alpha)(\mathcal{L}(\alpha) + \lambda_n \xi_n (1-\alpha))^{-2}, \mathcal{H}_\alpha(\lambda_n, k) = \alpha \lambda_n \xi_n (\mathcal{L}(\alpha) + \lambda_n \xi_n (1-\alpha))^{-1}. \quad (3.6)$$

From (3.5) and (3.6), this leads to

$$\int_{\mathcal{D}} g(x) e_n(x) dx = \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) \mathcal{D}_\alpha(n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds, \quad (3.7)$$

and we can rewrite the formula (3.8) as follows:

$$\Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) = \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big) \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big)^{-1} \cdot \quad (3.8)$$

and

$$f(x) = \sum_{n=1}^{\infty} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big)^{-1} \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big) e_n(x). \quad (3.9)$$

with $\mathcal{D}_\alpha(\lambda_n, k)$ are defined as (3.6).

3.1. The ill-posedness of problem (1.1)

Theorem 3.1. *The inverse problem* (1.1) *is ill-posed.*

Proof. A linear operator $\mathcal{K} : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ as follows.

$$\mathcal{K} f(x) = \sum_{n=1}^{\infty} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big) \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) e_n(x) = \int_{\mathcal{D}} p(x, \xi) f(\xi) d\xi. \quad (3.10)$$

where

$$p(x, \xi) = \sum_{n=1}^{\infty} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big) e_n(x) e_n(\xi).$$

It is obvious that $p(x, \xi) = p(\xi, x)$, we know that $\mathcal{K}$ is self-adjoint operator. Next, its compactness is considered and the finite rank operators as follows:

$$\mathcal{K}_N f(x) = \sum_{n \le N} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big) \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) e_n(x). \quad (3.11)$$

From (3.8), we find that the problem of seeking $f(x)$ can be transformed into the following operator equation:

$$(\mathcal{K} f)(x) = g(x) = \sum_{n \le N} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s) \theta(s) ds \Big) \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) e_n(x). \quad (3.12)$$

where $\mathcal{K}_N : f \to g$ is a linear operator. To prove that $\mathcal{K}_N$ is a compact operator, defining $\mathcal{K}_N$ the finite rank operator as

$$(\mathcal{K}_N f)(x) = \sum_{n\leq N} \Big(\mathcal{D}_\alpha(\lambda_n, k) \int_0^T G_\alpha(\lambda_n, k, s)\theta(s)ds\Big)\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)e_n(x). \tag{3.13}$$

Using the estimate of Lemma 2.5, we notice that $\mathcal{D}_{\alpha,\beta}(\lambda_n, k) \int_0^T G_{\alpha,\beta}(\lambda_n, k, s)\theta(s)ds \leq \frac{\theta_1}{\lambda_n}$.
So, the $\left\|\mathcal{K}_N f - \mathcal{K} f\right\|_{L^2(\mathcal{D})}$ can be bounded as follows:

$$\left\|\mathcal{K}_N f - \mathcal{K} f\right\|_{L^2(\mathcal{D})} \leq \sum_{n\geq N} \frac{\theta_1}{\lambda_n}\Big|\int_{\mathcal{D}} f(x)e_n(x)dx\Big|^2 \leq \frac{\theta_1^2}{N^2}\left\|f\right\|^2_{L^2(\mathcal{D})}. \tag{3.14}$$

Hence, $\left\|\mathcal{K}_N f - \mathcal{K} f\right\|_{L^2(\mathcal{D})} \to 0$ in the sense of $L(L^2(\mathcal{D}), L^2(\mathcal{D}))$as $N \to \infty$ and $\mathcal{K}$ is a linear operator. Next, we show one example to illustrate the non-well posed of problem (1.1). Considering the input final data $g_m(x) = e_m(x)(\lambda_m)^{-\frac{1}{2}}$ and $g_m = 0$ and the corresponding source solution $f_m(x) = e_m(x)\left(\sqrt{\lambda_m}\mathcal{D}_\alpha(\lambda_m, k)\int_0^T G_\alpha(\lambda_m, k, s)\theta(s)ds\right)^{-1}$ and $f(x) = 0$. The error estimate in L^2 between g_m and g is

$$\|g_m - g\|_{L^2(\mathcal{D})} = \frac{1}{\sqrt{\lambda_m}} \to 0, \text{as } m \to \infty. \tag{3.15}$$

Applying Lemma 2.5, then the corresponding error between source f_m and f is

$$\left\|f_m - f\right\|_{L^2(\mathcal{D})} \geq \frac{\sqrt{\lambda_m}}{\theta_1} \to \infty, \text{as } m \to \infty. \tag{3.16}$$

From the condition (3.15) and (3.16), we conclude that the problem (3.13) is non-well posed. □

3.2. The conditional stability of the solution for problem (1.1)

Theorem 3.2. *Let $r > 0$, and $f \in \mathcal{H}(A^r(\mathcal{D}))$ such that*

$$\left\|f\right\|_{\mathcal{H}(A^r(\mathcal{D}))} \leq E, \quad E > 0. \tag{3.17}$$

Then we have

$$\|f\|_{L^2(\mathcal{D})} \leq \Big|\mathcal{V}_\alpha(\lambda_1, k, T, \mathcal{A}_2, \theta_0)\Big|^{-\frac{r}{r+1}} E^{\frac{1}{r+1}} \|g\|^{\frac{r}{r+1}}. \tag{3.18}$$

where

$$\mathcal{V}_\alpha(\lambda_1, k, T, \mathcal{A}_2, \theta_0) = \frac{\theta_0}{4}\frac{k\mathcal{L}(\alpha)}{(\frac{\mathcal{L}(\alpha)}{\lambda_1\xi_1} + (1-\alpha))^2}\Big(T(1-\alpha) - \frac{\mathcal{A}_2 T^{1-\alpha}}{\mathcal{H}_\alpha(\lambda_1, k)}\Big). \tag{3.19}$$

Proof. This theorem can refer similarly in [43]. We omit here. □

4. Fractional Tikhonov regularization method and error estimate

To get the regularized solution with the given data $g_\epsilon(x) \in L^2(\mathcal{D})$, minimizing the quantity $J^\sigma_{[\gamma(\epsilon)]}(f) \in L^2(\mathcal{D})$ such that

$$J^\sigma_{\gamma(\epsilon)}(f) = \|Kf - g\|^2_\sigma + [\gamma(\epsilon)]\|f\|^2_{L^2(\mathcal{D})}, \tag{4.1}$$

whereby $[\gamma(\epsilon)]$ is a regularization parameter, and $\|.\|_\sigma$ is a weighted seminorm defined as $\|\xi\|_\sigma = \left\|\mathcal{W}^{\frac{1}{2}}\xi\right\|_{L^2(\mathcal{D})}$ for any ξ with $\mathcal{W} = (\mathcal{K}^*\mathcal{K})^{\sigma-1}$ some of parameter $\frac{1}{2} \le \sigma < 1$ called the fractional parameter. In here, we have some reviews as follows:

- In case $\sigma = \frac{1}{2}$, it is the quasi-boundary value method, interested readers can find a view in the document [42].
- In case $\sigma = 1$, it is classical Tikhonov regularization method, this method is fully described in the following documents [33].
- For $\frac{1}{2} < \sigma < 1$, it is the new fractional Tikhonov regularization.

The minimizer f satisfy the norm equation

$$\left((\mathcal{K}^*\mathcal{K})^{[\gamma(\epsilon)]} + \sigma I\right) f_\sigma = (\mathcal{K}^*\mathcal{K})^{\sigma-1}\mathcal{K}^* g. \tag{4.2}$$

Using the singular decomposition for the compact self-adjoint operator, we get

$$f^\sigma_{\gamma(\epsilon)}(x) = \sum_{n=1}^\infty \frac{\left|\mathcal{D}_\alpha(\lambda_n, k)\int_0^T G_\alpha(\lambda_n, k, s)\theta(s)ds\right|^{2\sigma-1}}{\left|\mathcal{D}_\alpha(\lambda_n, k)\int_0^T G_\alpha(\lambda_n, k, s)\theta(s)ds\right|^{2\sigma} + [\gamma(\epsilon)]}\left(\int_{\mathcal{D}} g(x)e_n(x)dx\right)e_n(x), \quad \frac{1}{2} \le \sigma < 1. \tag{4.3}$$

For the observed data, we have

$$f^\sigma_{\epsilon,\gamma(\epsilon)}(x) = \sum_{n=1}^\infty \frac{\left|\mathcal{D}_\alpha(\lambda_n, k)\int_0^T G_\alpha(\lambda_n, k, s)\theta_\epsilon(s)ds\right|^{2\sigma-1}}{\left|\mathcal{D}_\alpha(\lambda_n, k)\int_0^T G_\alpha(\lambda_n, k, s)\theta_\epsilon(s)ds\right|^{2\sigma} + [\gamma(\epsilon)]}\left(\int_{\mathcal{D}} g_\epsilon(x)e_n(x)dx\right)e_n(x), \quad \frac{1}{2} \le \sigma < 1. \tag{4.4}$$

Next, with two formulas of regularized in (4.3) and (4.4), we consider the convergent rate of $\|f - f^\sigma_{\varepsilon,\gamma(\varepsilon)}\|_{L^2(\mathcal{D})}$, by discussing how to select the regularization parameter, with a-priori strategies.

4.1. A priori parameter choice rule

In this subpart, under a suitable choice for the regularization parameter $[\gamma(\epsilon)]$, we will give an error estimate for $\|f - f^\sigma_{\epsilon,\gamma(\epsilon)}\|_{L^2(\mathcal{D})}$.

Lemma 4.1. *Assume that the condition* (1.3) *holds, then we have*

$$\left\|f^\sigma_{\gamma(\epsilon)} - f^\sigma_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \le C_2[\gamma(\epsilon)]^{-\frac{1}{2\sigma}}\epsilon + \max\left\{\epsilon, [\gamma(\epsilon)]^{1-\frac{1}{2\sigma}}\right\}\left(\frac{32}{\theta_0^2} + \frac{2\theta_1^2}{\lambda_1^{2\beta}}C_2^2\right)^{\frac{1}{2}}\|f\|_{L^2(\mathcal{D})}. \tag{4.5}$$

Proof. To prove this Lemma, we have used two formulas (4.3), (4.4), for $\frac{1}{2} \le \sigma < 1$, we have

$$\begin{aligned}
& f^{\sigma}_{\gamma(\epsilon)}(x) - f^{\sigma}_{\epsilon,\gamma(\epsilon)}(x) \\
&= \underbrace{\sum_{n=1}^{\infty} \frac{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\Big|^{2\sigma-1}}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^{2\sigma}+[\gamma(\epsilon)]}\Big(\int_{\mathcal{D}}(g_\epsilon(x)-g(x))e_n(x)dx\Big)e_n(x)}_{E_1} \\
&+ \underbrace{\sum_{n=1}^{\infty}\Big[\frac{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\Big|^{2\sigma-1}}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\Big|^{2\sigma}+[\gamma(\epsilon)]} - \frac{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^{2\sigma-1}}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^{2\sigma}+[\gamma(\epsilon)]}\Big]\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x)}_{E_2}. \qquad (4.6)
\end{aligned}$$

Now, we need establish the upper bound of $\big\|f^{\sigma}_{\gamma(\epsilon)} - f^{\sigma}_{\varepsilon,\gamma(\varepsilon)}\big\|_{L^2(\mathcal{D})}$. For convenience, we consider the following step.

Step 1: Estimate of E_1, with $\mathcal{V}$ depends on $\alpha, \lambda_1, k, T, \mathcal{A}_2$ and $\|g_\epsilon - g\|_{L^2(\mathcal{D})} \le \epsilon$, applying Lemma 2.6 with $s = \lambda_n$, we get

$$E_1^2 \le \sup_{n\ge 1}\Big(\frac{\big|\mathcal{B}(\theta_0,\theta_1)\big|^{2\sigma-1}\lambda_n}{|\mathcal{V}_\alpha|^{2\sigma}+\lambda_n^{2\sigma}[\gamma(\epsilon)]}\Big)^2\epsilon^2 \le C_2^2[\gamma(\epsilon)]^{-\frac{1}{\sigma}}\epsilon^2. \qquad (4.7)$$

whereby $\mathcal{V}_\alpha$ is defined as (3.19) and C_2 depends on $\mathcal{V}, \sigma$ and $\mathcal{B}$.

Step 2: Estimate of E_2, applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$, after simple transformation steps, we have

$$\begin{aligned}
E_2^2 &\le 2\Big(\Big[\frac{\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)(\theta_\varepsilon(s)-\theta(s))ds}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|\,\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\Big|}\Big]\Big)^2\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)^2 \\
&+ 2\Big(\frac{[\gamma(\epsilon)]\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^{2\sigma-1}\Big|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|}{\Big(\Big|\mathcal{D}_\alpha(\lambda_n,k,s)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^{2\sigma}+[\gamma(\epsilon)]\Big)}\Big)^2\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2
\end{aligned}$$

$$\leq \Bigg(\frac{32\epsilon^2}{\theta_0^2} + \frac{2[\gamma(\epsilon)]^2\theta_1^2}{\lambda_1^{2\beta}}\Big(\frac{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^{2\sigma-1}}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^{2\sigma} + [\gamma(\epsilon)]}\Big)^2\Bigg)\Big(\int\limits_{\mathcal{D}} f(x)e_n(x)dx\Big)^2. \tag{4.8}$$

From (4.8), the right hand side can be bounded as follow:

$$E_2^2 \leq \Big(\frac{32\varepsilon^2}{\theta_0^2} + \frac{2\theta_1^2}{\lambda_1^{2\beta}}\, C_2^2[\gamma(\epsilon)]^{2-\frac{1}{\sigma}}\Big)\left\|f\right\|^2_{L^2(\mathcal{D})}. \tag{4.9}$$

Combining (4.6) to (4.9), we have

$$\left\|f^\sigma_{\gamma(\epsilon)} - f^\sigma_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \leq C_2[\gamma(\epsilon)]^{-\frac{1}{2\sigma}}\epsilon + \Big(\frac{32\varepsilon^2}{\theta_0^2} + \frac{2\theta_1^2}{\lambda_1^{2\beta}}\, C_2^2[\gamma(\epsilon)]^{2-\frac{1}{\sigma}}\Big)^{\frac{1}{2}}\left\|f\right\|_{L^2(\mathcal{D})}. \tag{4.10}$$

□

Lemma 4.2. *Suppose that $f \in \mathcal{H}^r(\mathcal{D})$, then it gives*

$$\left\|f^\epsilon_{\gamma(\epsilon)} - f\right\|_{L^2(\mathcal{D})} \leq \begin{cases} C_4(\sigma,r,\mathcal{V}_\alpha)[\gamma(\epsilon)]^{\frac{r}{2\sigma}}E, 0 < r \leq 2\sigma, \\ C_5(\sigma,r,\lambda_1)[\gamma(\epsilon)]E, \qquad r > 2\sigma. \end{cases} \tag{4.11}$$

Proof. From (3.8) and (4.3), we received

$$\begin{aligned}
&\left\|f^\sigma_{\gamma(\epsilon)} - f\right\|^2_{L^2(\mathcal{D})} \\
&\leq \Bigg\|\sum_{n=1}^\infty\Big(\frac{[\gamma(\epsilon)]}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^{2\sigma} + [\gamma(\epsilon)]}\Big)\frac{\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|}\Bigg\|^2_{L^2(\mathcal{D})} \\
&\leq \Bigg\|\sum_{n=1}^\infty\Big(\frac{[\gamma(\epsilon)]\lambda_n^{-r}}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^{2\sigma} + [\gamma(\varepsilon)]}\Big)\lambda_n^r\Big(\int\limits_{\mathcal{D}} f(x)e_n(x)dx\Big)\Bigg\|^2_{L^2(\mathcal{D})} \\
&\leq \sup_{n\geq 1}\Big(\frac{[\gamma(\epsilon)]\lambda_n^{2\sigma-r}}{\Big|4\mathcal{V}_\alpha\Big|^{2\sigma} + \lambda_n^{2\sigma}[\gamma(\epsilon)]}\Big)^2\left\|f\right\|_{\mathcal{H}^r(\mathcal{D})}.
\end{aligned} \tag{4.12}$$

Due to Lemma 4.2, the upper bound of the right hand side of (4.12) as follows:

- If $r \in (0, 2\sigma)$, then we have

$$\left\|f^\sigma_{\gamma(\epsilon)} - f\right\|_{L^2(\mathcal{D})} \leq C_4(\sigma,r,\mathcal{V})[\gamma(\epsilon)]^{\frac{r}{2\sigma}}E. \tag{4.13}$$

- If $r \geq 2\sigma$, then we have

$$\left\|f^\sigma_{\gamma(\epsilon)} - f\right\|_{L^2(\mathcal{D})} \leq C_5(\sigma,r,\lambda_1)[\gamma(\epsilon)]E. \tag{4.14}$$

□

Theorem 4.1. *Assume that the noise assumption* (1.3) *are hold, and the a-priori bound condition* (3.17). *Then we have the following estimate:*

- *If* $r \in (0, 2\sigma)$*, by choosing* $[\gamma(\epsilon)] = \left(\frac{\epsilon}{E}\right)^{\frac{2\sigma}{r+2}}$*, then we have*

$$\left\|f - f^{\sigma}_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \le \max\left\{\epsilon^{\frac{r+1}{r+2}}, \epsilon, \epsilon^{\frac{2\sigma-1}{r+2}}, \varepsilon^{\frac{r}{r+2}}\right\}\mathcal{P}_1. \tag{4.15}$$

where $\mathcal{P}_1$ *depends on* $C_2, E, \theta_0, \theta_1, \mathcal{V}_\alpha, C_4$.

- *If* $r \ge 2\sigma$*, by choosing* $[\gamma(\epsilon)] = \left(\frac{\epsilon}{E}\right)^{\frac{\sigma}{\sigma+1}}$*, then we have*

$$\left\|f - f^{\sigma}_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \le \max\left\{\epsilon^{\frac{\sigma}{\sigma+1}}, \epsilon, \epsilon^{\frac{\sigma-1/2}{\sigma+1}}, \epsilon^{\frac{\sigma}{\sigma+1}}\right\}\mathcal{P}_2. \tag{4.16}$$

and $\mathcal{P}_2$ *depends on* $C_2, E, \theta_0, \theta_1, \lambda_1, C_5$.

Proof. Using the triangle inequality, we get

$$\left\|f - f^{\sigma}_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \le \left\|f - f^{\sigma}_{\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} + \left\|f^{\sigma}_{\gamma(\epsilon)} - f^{\sigma}_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})}. \tag{4.17}$$

From (4.3), (4.4), results from Lemma 4.1 and Lemma 4.2, Theorem 4.1 is proven. □

5. A generalized Tikhonov regularization method

5.1. A priori parameter choice rule

From [41], we have a stable solution to problem (1.1) with the observed data $(g_\epsilon, \theta_\epsilon)$ by generalized Tihkhonov regularization method which minimizes the quantity

$$J(f_{\epsilon,[\gamma(\epsilon)]}) = \left\|Kf - g_\epsilon\right\|^2_{L^2(\mathcal{D})} + [\gamma(\epsilon)]\left\|f\right\|^2_{\mathcal{H}^r(\mathcal{D})}, \tag{5.1}$$

Let $f_{\epsilon,[\gamma(\epsilon)]}$ be a solution of the problem (5.1) which satisfies

$$\mathcal{K}^*\mathcal{K}f_{\epsilon,[\gamma(\epsilon)]} + [\gamma(\epsilon)]\lambda_n^{2r} f_{\epsilon,[\gamma(\epsilon)]} = \mathcal{K}^* g_\epsilon. \tag{5.2}$$

Using the SVD for compact self-adjoint operator, we get

$$f_{\epsilon,[\gamma(\epsilon)]}(x) = \sum_{n=1}^{\infty} \frac{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\right|}{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\right|^2 + [\gamma(\epsilon)]\lambda_n^{2r}}\Big(\int_{\mathcal{D}} g_\epsilon(x)e_n(x)dx\Big)e_n(x), \tag{5.3}$$

and

$$f_{[\gamma(\epsilon)]}(x) = \sum_{n=1}^{\infty} \frac{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\right|}{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\right|^2 + [\gamma(\epsilon)]\lambda_n^{2r}}\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x). \tag{5.4}$$

Lemma 5.1. *Assume that $f \in \mathcal{H}^r(\mathcal{D})$ holds, assume that the couple (g,θ) is noised by $(g_\epsilon,\theta_\epsilon)$ such that the condition* (1.3) *satisfies, then we have*

$$\left\| f_{[\gamma(\epsilon)]} - f_{\epsilon,[\gamma(\epsilon)]} \right\|_{L^2(\mathcal{D})} \le \frac{\epsilon}{2[\gamma(\epsilon)]\lambda_1^{2r}} + \frac{5\epsilon}{\theta_0}\left\| f \right\|_{L^2(\mathcal{D})}. \tag{5.5}$$

Proof. From (5.3) and (5.4), we have

$$f_{\gamma(\epsilon)}(x) - f_{\epsilon,\gamma(\epsilon)}(x) \le \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3. \tag{5.6}$$

whereby $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_3$ are defined in (5.7) as follows:

$$\begin{aligned}
\mathcal{Z}_1 &= \sum_{n=1}^{\infty} \frac{\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds}{[\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\right|^2}\Big(\int_{\mathcal{D}} (g_\epsilon(x)-g(x))e_n(x)dx\Big)e_n(x),\\
\mathcal{Z}_2 &= \sum_{n=1}^{\infty} \frac{[\gamma(\epsilon)]\lambda_n^{r}\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)(\theta_\varepsilon(s)-\theta(s))ds\right|\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x)dx}{\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\rho_n\int_0^T G_\alpha(k,\beta,\lambda_n,s)\theta(s)ds\right|^2\Big)\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\rho_n\int_0^T G_\alpha(k,\beta,\lambda_n,s)\theta_\varepsilon(s)ds\right|^2\Big)},\\
\mathcal{Z}_3 &= \sum_{n=1}^{\infty} \frac{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)(\theta_\varepsilon(s)-\theta(s))ds\right|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds}{\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\right|^2\Big)\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\right|^2\Big)}\\
&\quad\times \frac{\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\right|\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x)}{\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\right|^2\Big)\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\epsilon(s)ds\right|^2\Big)}.
\end{aligned} \tag{5.7}$$

The proof falls naturally into three parts.
Step 1: Estimate of $\mathcal{Z}_1$, we have

$$\left\|\mathcal{Z}_1\right\|_{L^2(\mathcal{D})}^2 \le \frac{1}{4[\gamma(\epsilon)]}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{2r}}\Big(\int_{\mathcal{D}}(g_\epsilon(x)-g(x))e_n(x)dx\Big)^2 \le \frac{\epsilon^2}{4[\gamma(\epsilon)]\lambda_1^{2r}}. \tag{5.8}$$

Step 2: Estimate of $\mathcal{Z}_2$, it gives

$$\left\|\mathcal{Z}_2\right\|_{L^2(\mathcal{D})}^2 \le \sum_{n=1}^{\infty}\Bigg[\frac{[\gamma(\epsilon)]\lambda_n^{2r}\left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)(\theta_\varepsilon(s)-\theta(s))ds\right|\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)}{\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\right|^2\Big)\Big([\gamma(\epsilon)]\lambda_n^{2r} + \left|\mathcal{D}_\alpha(\lambda_n,k)\int_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\right|^2\Big)}\Bigg]^2$$

$$\leq \sum_{n=1}^{\infty} \frac{\Big|[\gamma(\epsilon)]\lambda_n^{2r}\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)(\theta(s)-\theta_\varepsilon(s))ds\Big|^2}{4[\gamma(\epsilon)]\lambda_n^{2r}\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^2}\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)^2$$

$$\leq \frac{\|\theta-\theta_\epsilon\|^2_{L^\infty(0,T)}}{\theta_0^2}\sum_{n=1}^{\infty}\frac{\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)^2}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2} \leq \frac{\varepsilon^2}{\theta_0^2}\big\|f\big\|^2_{L^2(\mathcal{D})}. \tag{5.9}$$

Step 3: Estimate of $\mathcal{Z}_3$, we have

$$\big\|Z_3\big\|^2_{L^2(\mathcal{D})} \leq \sum_{n=1}^{\infty}\frac{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)(\theta(s)-\theta_\varepsilon(s))ds\Big|^2}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^2}\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)^2$$

$$\leq \sum_{n=1}^{\infty}\frac{\big\|\theta-\theta_\epsilon\big\|_{L^\infty(0,T)}\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)ds\Big|^2}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta_\varepsilon(s)ds\Big|^2}\frac{\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)^2}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2}$$

$$\leq \frac{16\epsilon^2}{\theta_0^2}\big\|f\big\|^2_{L^2(\mathcal{D})}. \tag{5.10}$$

Combining (5.8)–(5.10), we get

$$\big\|f_\epsilon - f_{\epsilon,[\gamma(\epsilon)]}\big\|_{L^2(\mathcal{D})} \leq \frac{\epsilon}{2[\gamma(\epsilon)]^{\frac{1}{2}}\lambda_1^r} + \frac{5\epsilon}{\theta_0}\big\|f\big\|_{L^2(\mathcal{D})}. \tag{5.11}$$

Lemma 5.2. *Suppose that $f \in \mathcal{H}^r(\mathcal{D})$, then we have* □

- *If $r > 1$, then we have*

$$\|f - f_{\gamma(\epsilon)}\|_{L^2(\mathcal{D})} \leq \mathcal{R}_2(\lambda_1, r, \theta_0, \mathcal{V}_\alpha, E)[\gamma(\epsilon)]^{\frac{1}{2}}. \tag{5.12}$$

- *If $r \in (0,1)$, then we have*

$$\big\|f - f_{\gamma(\epsilon)}\big\|_{L^2(\mathcal{D})} \leq (\mathcal{R}_1(\theta_0,\mathcal{V}_\alpha,E))^{\frac{1}{2}}\ [\gamma(\epsilon)]^{\frac{r}{r+1}}. \tag{5.13}$$

whereby $\mathcal{R}_1(\theta_0,\mathcal{V}_\alpha,E) = \big((2\theta_0\mathcal{V}_\alpha)^{-2}E^2 + E^2\big)$ *and* $\mathcal{R}_2(\lambda_1,r,\theta_0,\mathcal{V}_\alpha,E) = \dfrac{\lambda_1^{(1-r)}}{2\theta_0\mathcal{V}_\alpha}E.$

Proof. From (5.4) and (3.8), we get:

$$\big\|f - f_{[\gamma(\epsilon)]}\big\|_{L^2(\mathcal{D})} = \sum_{n=1}^{\infty}\frac{[\gamma(\epsilon)]\lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|\Big[[\gamma(\epsilon)]\lambda_n^{2r} + \Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2\Big]}$$

$$= \sum_{n=1}^{\infty} \frac{[\gamma(\epsilon)]\lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|\Big[[\gamma(\epsilon)]\lambda_n^{2r} + \Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(k,\beta,\lambda_n,s)\theta(s)ds\Big|^2\Big]}$$

$$\le \sup_{n\in\mathbb{N}} |\mathcal{V}(n)| \sum_{n=1}^{\infty} \frac{\lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} g(x)e_n(x)dx\Big)}{\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|} = \sup_{n\in\mathbb{N}} |\mathcal{W}(n)| \big\|f\big\|_{H^r(\mathcal{D})}, \tag{5.14}$$

Hence, $\mathcal{W}(n)$ is given by

$$\mathcal{V}(n) = \frac{[\gamma(\varepsilon)]}{[\gamma(\varepsilon)]\lambda_n^{2r} + \Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2}. \tag{5.15}$$

Next, Estimate of $\mathcal{V}(n)$, we have

$$\mathcal{W}(n) \le \frac{[\gamma(\epsilon)]}{2[\gamma(\epsilon)]^{\frac{1}{2}}\lambda_n^r\Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|} \le \frac{[\gamma(\epsilon)]^{\frac{1}{2}}\lambda_n^{1-r}}{2\theta_0\mathcal{V}_\alpha}. \tag{5.16}$$

We divide into two following cases

- If $r \ge 1$, then we have

$$\big\|f - f_{[\gamma(\epsilon)]}\big\|_{L^2(\mathcal{D})} \le [\gamma(\epsilon)]^{\frac{1}{2}} \frac{\lambda_1^{(1-r)}}{2\theta_0\mathcal{V}_\alpha} E. \tag{5.17}$$

- If $r \in (0,1)$, then we consider λ_n^{1-r} satisfies the following two cases :

$$\lambda_n^{1-r} \le [\gamma(\varepsilon)]^{-\ell} \text{ and } \lambda_n^{1-r} > [\gamma(\varepsilon)]^{-\ell}. \tag{5.18}$$

then we rewrite $\mathbb{N}$ by $\mathbb{N} = Q_1 \cup Q_2$ where

$$Q_1 = \{n \in \mathbb{N}, \lambda_n^{1-r} \le [\gamma(\varepsilon)]^{-\ell}\}, \;\; Q_2 = \{n \in \mathbb{N}, \lambda_n^{1-r} > [\gamma(\epsilon)]^{-\ell}\}. \tag{5.19}$$

From (5.15) and (5.18), we have:

$$\begin{aligned}
\|f - f_{[\gamma(\epsilon)]}\|^2_{L^2(\mathcal{D})} &= \sup_{n\in Q_1}\Big[\frac{[\gamma(\epsilon)]^{\frac{1}{2}}\lambda_n^{1-r}}{2\theta_0\mathcal{V}_\alpha}\Big]^2 \sum_{n\in Q_1} \lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} f(x)e_n(x)dx\Big)^2 \\
&\quad + \sum_{n\in Q_2}\Bigg[\frac{[\gamma(\varepsilon)]}{[\gamma(\varepsilon)]\lambda_n^{2r} + \Big|\mathcal{D}_\alpha(\lambda_n,k)\int\limits_0^T G_\alpha(\lambda_n,k,s)\theta(s)ds\Big|^2}\Bigg]^2 \lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} f(x)e_n(x)dx\Big)^2 \\
&\le (2\theta_0\mathcal{V}_\alpha)^{-2}[\gamma(\epsilon)]^{1-2\ell}\big\|f\big\|^2_{\mathcal{H}^r(\mathcal{D})} + \sup_{n\in Q_2}\lambda_n^{-4r}\sum_{n\in Q_2}\lambda_n^{2r}\Big(\int\limits_{\mathcal{D}} f(x)e_n(x)dx\Big)^2 \\
&\le (2\theta_0\mathcal{V}_\alpha)^{-2}[\gamma(\epsilon)]^{1-2\ell}\big\|f\big\|^2_{\mathcal{H}^r(\mathcal{D})} + [\gamma(\epsilon)]^{\frac{4r\ell}{1-r}}\|f\|^2_{\mathcal{H}^r(\mathcal{D})}.
\end{aligned} \tag{5.20}$$

Choose $\ell = \frac{1-r}{2(1+r)}$, we have

$$\left\|f - f_{[\gamma(\epsilon)]}\right\|^2_{L^2(\mathcal{D})} \leq [\gamma(\epsilon)]^{\frac{2r}{r+1}}\left((2\theta_0\mathcal{V}_\alpha)^{-2}E^2 + E^2\right). \tag{5.21}$$

□

Theorem 5.1. *Let $\theta_\epsilon, \theta \in L^\infty(0,T)$ such that $\|\theta_\epsilon - \theta\|_{L^\infty(0,T)} \leq \epsilon$ and $(\theta, \theta_\varepsilon)$ satisfy the condition in Lemma* (2.9)*, and the noise assumption* (3.17) *hold. By choosing the regularization parameter* $[\gamma(\epsilon)] = \left(\frac{\epsilon}{E}\right)$*, we receive*

- *If $r > 1$, then we have*

$$\|f - f_{\epsilon,\gamma(\epsilon)}\|_{L^2(\mathcal{D})} \leq \epsilon^{\frac{1}{2}}\Big(\mathcal{R}_2(\lambda_1, r, \theta_0, \mathcal{V}_\alpha)E^{\frac{1}{2}} + E^{\frac{1}{2}}\frac{1}{2\lambda_1^r} + \frac{5\varepsilon^{\frac{1}{2}}\|f\|_{L^2(\mathcal{D})}}{\theta_0}\Big). \tag{5.22}$$

- *If $r \in (0,1)$, then we have*

$$\|f - f_{\epsilon,\gamma(\epsilon)}\|_{L^2(\mathcal{D})} \leq \epsilon^{\frac{r}{r+1}}\Big(\mathcal{R}_1(\theta_0, \mathcal{V}_\alpha, E)\Big)^{\frac{1}{2}} + E^{\frac{1}{r+1}}\frac{1}{2\lambda_1^r} + \frac{5\varepsilon^{\frac{1}{r+1}}\|f\|_{L^2(\mathcal{D})}}{\theta_0}\Big). \tag{5.23}$$

whereby $\mathcal{R}_1(\theta_0, \mathcal{V}_\alpha, E) = \left((2\theta_0\mathcal{V}_\alpha)^{-2}E^2 + E^2\right)$ *and* $\mathcal{R}_2(\lambda_1, r, \theta_0, \mathcal{V}_\alpha) = \dfrac{\lambda_1^{(1-r)}}{2\theta_0\mathcal{V}_\alpha}$.

Proof. By the triangle inequality, we have

$$\left\|f - f_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} \leq \left\|f - f_{\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})} + \left\|f_{\gamma(\epsilon)} - f_{\epsilon,\gamma(\epsilon)}\right\|_{L^2(\mathcal{D})}. \tag{5.24}$$

Combining Lemma 5.1 and Lemma 5.2, we have

- If $r > 1$, then we have

$$\|f - f_{\epsilon,\gamma(\epsilon)}\|_{L^2(\mathcal{D})} \leq \text{ is of order } \varepsilon^{\frac{1}{2}}\ . \tag{5.25}$$

- If $r \in (0,1)$, then we have

$$\|f - f_{\epsilon,\gamma(\epsilon)}\|_{L^2(\mathcal{D})} \text{ is of order } \varepsilon^{\frac{r}{r+1}}\ . \tag{5.26}$$

This theorem is proven. □

6. Simulation

In this section, we present one numerical example. By choosing $\Omega = (0,\pi)$, $T = 1$, $\alpha = 0.95$, and $\sigma = 0.45$, and $r = 0.25$ $k = 0.01$ are shown in this section, respectively. For computing the generalized Mittag-Leffler function and the accuracy control in computing is 10^{-10}, we have the Matlab codes given by Podlubny [44]. To do this, we consider the problem as follows:

$$^{ABC}_{\ 0}D^\alpha_t\big(u(t,x) + k\Delta u(t,x)\big) + \Delta u(t,x) = \theta(t)f(x), \quad (x,t) \in (0,\pi) \times (0,1). \tag{6.1}$$

where $ABC_0D_t^{\alpha}u(x,t)$ is the Atangana-Baleanu fractional derivative is given by (1.2). In this calculation, we chose the operator $\Delta u = \dfrac{\partial^2}{\partial x^2}u$, we have chosen $\lambda_n = n^2,\ n = 1,2,...$ and $e_n(x) = \sqrt{\dfrac{2}{\pi}}\sin(nx)$, respectively. We have the function

$$g(x) = \sqrt{\frac{2}{\pi}}\sin(4x),\ \ \theta(t) = \Big((1-16k)\frac{\Gamma(4)t^{3-\alpha}}{\Gamma(4-\alpha)} - 16t^3\Big). \tag{6.2}$$

In general, the whole numerical procedure is summarized in the following steps:
Step 1: Finite difference to discretize the time and spatial variable for $x \in (0,\pi)$ as follows:

$$x_k = k\Delta x,\ \ 0 \le k \le N,\ \ \Delta x = \frac{\pi}{N}\cdot$$

Step 2: The input data (g,θ) is noised by observation data $(g_\varepsilon, \theta_\varepsilon)$ such that

$$\theta_\epsilon = \theta + \frac{1}{\pi}\epsilon(2\text{rand}(\cdot) - 1),\quad g_\epsilon = g + \frac{1}{\pi}\epsilon(2\text{rand}(\cdot) - 1). \tag{6.3}$$

Step 3: The relative error estimation is defined by

$$Error_1 = \left(\frac{\sum\limits_{n=1}^{N}\left\|f^{\sigma}_{\epsilon,\gamma(\epsilon)}(x_n) - f(x_n)\right\|^2_{L^2(0,\pi)}}{\sum\limits_{n=1}^{N}\left\|f(x_n)\right\|^2_{L^2(0,\pi)}}\right)^{\frac{1}{2}},\ \ Error_2 = \left(\frac{\sum\limits_{n=1}^{N}\left\|f_{\epsilon,\gamma(\epsilon)}(x_n) - f(x_n)\right\|^2_{L^2(0,\pi)}}{\sum\limits_{n=1}^{N}\left\|f(x_n)\right\|^2_{L^2(0,\pi)}}\right)^{\frac{1}{2}}, \tag{6.4}$$

where Error_1 is the error estimate between the exact solution and the regularized solution by the Fractional Tikhonov method, and Error_2 is the error estimate between the exact solution and the regularized solution by generalized Tikhonov method. In addition, taking for the priori parameter choice rule of E is large enough. By [43], we get

$$\int_0^1 x^{\gamma-1}(1-x)^{\beta-1}E_{\alpha,\beta}(z(1-x)^{\alpha})dx = \Gamma(\beta)E_{\alpha,\alpha+\beta}(z). \tag{6.5}$$

From (6.5), by replacing $\beta = \alpha$, and $z = -\frac{\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}$, we can find that

$$\int\limits_0^1 x^{\gamma-1}(1-x)^{\alpha-1}E_{\alpha,\alpha}\Big(\frac{-\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}\Big)dx = \Gamma(\gamma)E_{\alpha,\alpha+\gamma}\Big(\frac{-\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}\Big)\cdot \tag{6.6}$$

From (3.9), we have the exact solution

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{N}\Big(\mathcal{D}_\alpha(n^2,k)\int\limits_0^T G_\alpha(n^2,k,s)\theta(s)ds\Big)^{-1}\Big(\int\limits_0^\pi g(x)\sin(nx)\Big)\sin(nx). \tag{6.7}$$

From (4.4), by choosing the regularization parameter $\gamma(\epsilon) = \Big(\dfrac{\epsilon}{E}\Big)^{\frac{1}{2}}$, with N is a truncation number large enough, we have the regularized solution with Fractional Tikhonov as follows

$$f^{\sigma}_{\epsilon,\gamma(\epsilon)}(x) = \frac{2}{\pi}\sum_{n=1}^{N} \frac{\Big|\mathcal{D}_\alpha(n^2,k)\int\limits_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds\Big|^{2\sigma-1}}{\Big|\mathcal{D}_\alpha(n^2,k)\int\limits_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds\Big|^{2\sigma} + \Big(\dfrac{\epsilon}{E}\Big)^{\frac{1}{2}}}\Big(\int\limits_0^\pi g_\epsilon(x)\sin(nx)dx\Big)\sin(nx), \tag{6.8}$$

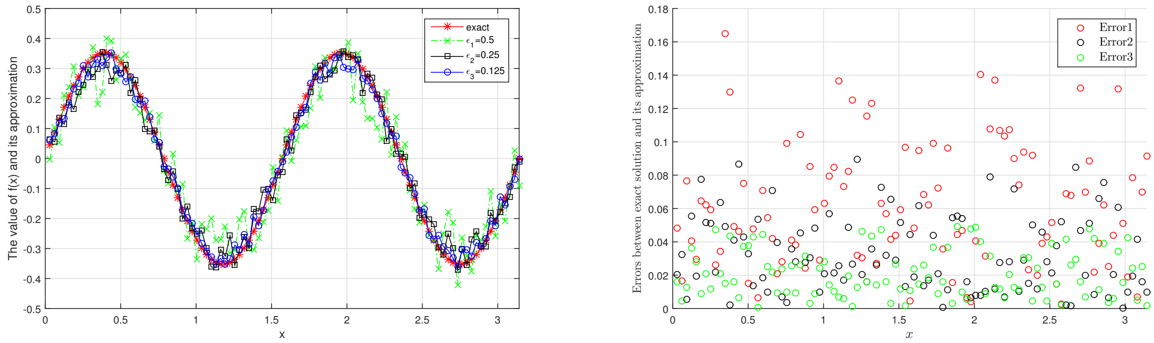
From (5.3), with the Generalized Tikhonov method, and $\gamma(\epsilon) = \frac{\epsilon}{E}$, we receive

$$f_{\epsilon,[\gamma(\epsilon)]}(x) = \frac{2}{\pi}\sum_{n=1}^{N} \frac{\Big|\mathcal{D}_\alpha(n^2,k)\int\limits_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds\Big|}{\Big|\mathcal{D}_\alpha(n^2,k)\int\limits_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds\Big|^{2} + \Big(\dfrac{\epsilon}{E}\Big)n^{4r}}\Big(\int\limits_0^\pi g_\epsilon(x)\sin(nx)dx\Big)\sin(nx), \tag{6.9}$$

and $\mathcal{D}_\alpha(n^2,k)$ is defined in Lemma 2.4. From (6.6), the integral $\int_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds$ calculated

$$\begin{aligned}
&\int\limits_0^1 G_\alpha(n^2,k,s)\theta_\epsilon(s)ds \\
&= \frac{\frac{n^2}{1+kn^2}\alpha\mathcal{L}(\alpha)}{\big(\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)\big)^2}\int\limits_0^1 E_{\alpha,\alpha}\Big(-\frac{\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}(1-s)^\alpha\Big)(1-s)^{\alpha-1}\theta_\epsilon(s)ds \\
&= (1-16k)\Gamma(4)E_{\alpha,4}\Big(-\frac{\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}\Big) - 16\Gamma(4)E_{\alpha,\alpha+4}\Big(-\frac{\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}\Big) \\
&+ \frac{1}{\pi}\epsilon(2\text{rand}(\cdot)-1)\frac{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}{\alpha\frac{n^2}{1+kn^2}}\Big(1-E_{\alpha,1}\Big(\frac{-\alpha\frac{n^2}{1+kn^2}}{\mathcal{L}(\alpha)+\frac{n^2}{1+kn^2}(1-\alpha)}\Big)\Big).
\end{aligned} \tag{6.10}$$

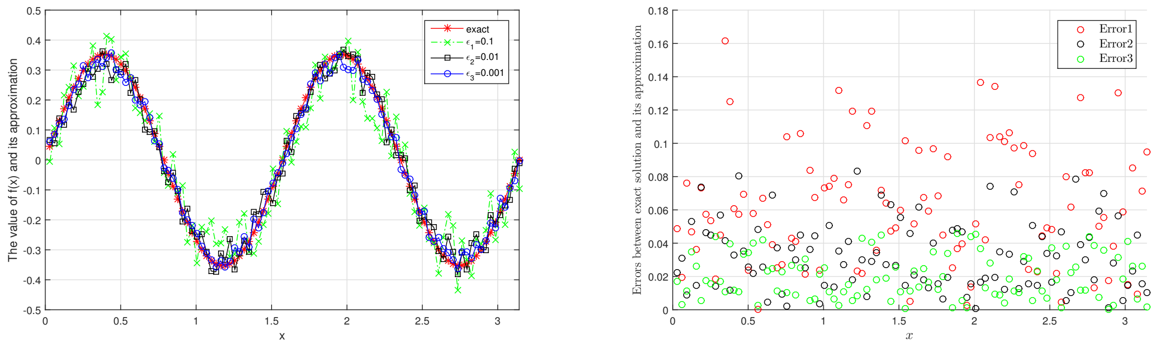
In these calculations, we choose $N = 100$ and $\epsilon = 0.5$, $\epsilon = 0.25$ and $\epsilon = 0.125$. Figure 1(a) shows the 2D graphs of the source function with the exact solution and its approximation for the case by the Fractional Tikhonov method. Figure 1(b) shows the error estimate between the exact solution and regularized solution for the case by the Fractional Tikhonov method. Figure 2(a) shows the 2D graphs of the source function with the exact solution and its approximation for the case by the generalized Tikhonov method. Figure 2(b) shows the error estimate between the exact solution and regularized solution for the case by the generalized Tikhonov method, respectively. Figure 3 shows the 2D graphs of the source function and its approximation, by two methods with $\epsilon = 0.5$, $\epsilon = 0.25$ and $\epsilon = 0.125$, respectively. The error estimates between the source function with the exact data and the measurement data are presented in Table 1. From the observations above, the approximation results are acceptable.



(a) Fractional Tikhonov method

(b) Error

Figure 1. The exact solution and the regularized solutions are shown by the left graph and the graph of error with $\epsilon = 0.5, \epsilon = 0.25$ and $\epsilon = 0.125$ for the Fractional Tikhonov method and the error estimation.



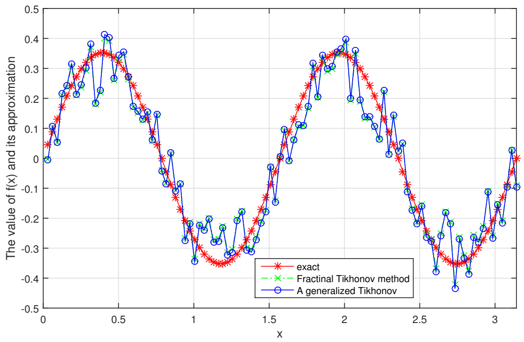
(a) Generalized Tikhonov method

(b) Error

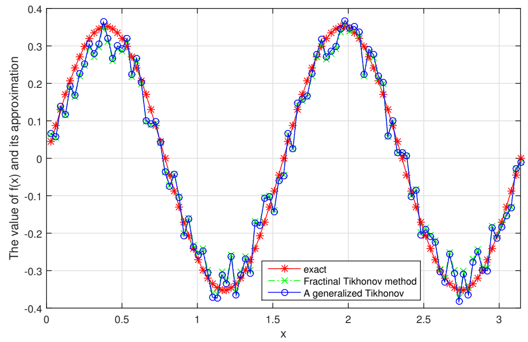
Figure 2. The exact solution and the regularized solutions are shown by the left graph and the graph of error with $\epsilon = 0.5, \epsilon = 0.25$ and $\epsilon = 0.125$ for the Generalized Tikhonov method and the error estimation.

Table 1. The error estimation.

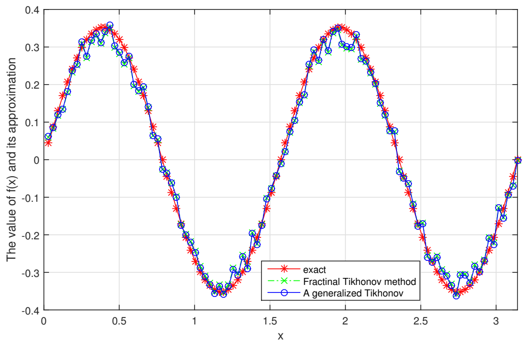
	$\alpha = 0.95$	
	Err_1	Err_2
$\epsilon = 0.5$	0.292922331	0.292928835
$\epsilon = 0.25$	0.159947533	0.152469901
$\epsilon = 0.125$	0.101037573	0.095171607



(a) $\epsilon = 0.5$



(b) $\epsilon = 0.25$



(c) $\epsilon = 0.125$

Figure 3. The exact solution and the regularized solution for two case : Fractional Tikhonov method and generalized Tikhonov method and the error estimation with $\epsilon = 0.5, \epsilon = 0.25$ and $\epsilon = 0.125$.

7. Conclusions

In this study, we applied the Fractional Tikhonov method and the Generalized Tikhonov method to regularize the inverse problem to identify an unknown source term in fractional diffusion equations involving fractional derivative with Atangana-Baleanu Caputo derivative. This problem (1.1) is ill-posed in the sense of Hadamard. So, Under *a priori* parameter choice rule, we show the result for the convergent estimate between the sought solution and the regularized solution. Finally, we show an example to illustrate our proposed regularization.

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Conflict of interest

The authors declare no conflict of interest.

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